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DECOMPOSABILITY OF NONNEGATIVE r-POTENT MATRICES

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Abstract: We consider nonnegative r-potent matrices with finite dimensions and study their decomposability. We derive the precise conditions under which an r-potent matrix is decomposable. We further determine a general structure for the r-potent matrices based on their decomposibility. Finally, we establish that semigroups of r-potent matrices are also decomposable.

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1. Introduction

A square matrix $\mathbf{A}$ is said to be idempotent [1] if and only if $\mathbf{A}^2 = \mathbf{A}$. The concept of r-potent matrices [2] is a generalization of idempotent matrices where a matrix $\mathbf{A}$ is said to be r-potent, for some natural number r, if and only if $\mathbf{A}^r = \mathbf{A}$. While every idempotent matrix is r-potent, the reverse is not necessarily true. That is, an r-potent matrix may or may not be idempotent.

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For example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{1}$$

is 3-potent (commonly known as tripotent), but not idempotent. Therefore, it makes sense to study r-potent matrices separately.

Several properties of r-potent matrices have been studied by McCloskey [2]. Decomposability of general r-potent matrices has however not been studied so far. On the other hand, decomposability of *idempotent* matrices has been studied by Marwaha [3]. The tools developed in [3] for idempotent matrices do not apply to r-potent matrices. The main focus of this paper is therefore to develop tools so that we can study decomposability of r-potent matrices.

Outline of the Paper. The rest of the paper is organized as follows. We provide an overview of the known results used in this paper in Section 2. In Section 3, we state our main result on decomposability of r-potent matrices and provide its proof. We provide our results on the structure of r-potent matrices in Section 4. In Section 5, we establish decomposability of the kronecker products of r-potent matrices. Section 6 states our main results and corresponding proofs on decomposability of semigroups of r-potent matrices. Finally, we state our results on decomposability of permutation matrices in Section 7.

Notation. Capital letters $\mathbf{A}, \mathbf{B}, \cdots$ and small letters $\mathbf{a}, \mathbf{b}, \ldots$ are used to denote matrices and vectors, respectively, over $\mathbb{R}$, where $\mathbb{R}$ stands for the space of real numbers. For any set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$, $\vee\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ denotes the (closed) linear span of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$. $\mathcal{M}_n(\mathbb{R})$ stands for the space of all $n \times n$ matrices with entries from $\mathbb{R}$. A matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ is called nonnegative if $a_{ij} \geq 0, \forall i, j = 1, 2, \ldots, n$. A nonnegative semigroup in $\mathcal{M}_n(\mathbb{R})$ is a semigroup of nonnegative matrices. Given two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \otimes \mathbf{B}$ denotes their kronecker product ([4], pp.3). Since every $n \times n$ real matrix represents a linear operator on $\mathbf{R}^n$ and vice versa (see [5], pp.276), we use capital letters $\mathbf{A}, \mathbf{B}, \cdots$ to denote finite dimensional linear transformation on $\mathbb{R}^n$ and $n \times n$ real matrices interchangeably. $R(\mathbf{A})$ and $N(\mathbf{A})$ are used to denote the range space and the null space of linear transformation $\mathbf{A}$. Further, rank($\mathbf{A}$) denotes the rank of $\mathbf{A}$ and Nullity($\mathbf{A}$) denotes the dimension of the null space of $\mathbf{A}$. Finally, $\mathbf{A}^{-1}$ stands for the inverse of square matrix $\mathbf{A}$.

2. Definitions and Overview of Known Results

Definition 1. [3] A matrix $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ is said to be decomposable if there exists a proper subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ such that $\vee\{\mathbf{Ae}_{i_1}, \mathbf{Ae}_{i_2}, \cdots, \mathbf{Ae}_{i_k}\}$ is contained in $\vee\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k}\}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard ordered basis of $\mathbb{R}^n$.

The following equivalent definition of decomposability, given as a proposition in [3], will be used throughout this paper:

Definition 2. A matrix $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ is decomposable if and only if there exists a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{2}$$

where $\mathbf{B}$ and $\mathbf{D}$ are square matrices. A matrix is said to be *indecomposable* if it is not decomposable.

The definition given above for decomposability of a single matrix is extended in the obvious manner to a semigroup in $\mathcal{M}_n(\mathbb{R})$ ([8], pp.104), where a common permutation matrix $\mathbf{P}$ decomposes every matrix in the semigroup.

Definition 3. ([6], pp.7) A linear operator $\mathbf{A}$ defined on an n-dimensional vector space $\mathcal{V}$ is *decomposable* if there exists a standard subspace (subspace spanned by a subset $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ of standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$) $\mathcal{M}$ of $\mathcal{V}$ such that $\mathcal{M}$ is invariant under the action of $\mathbf{A}$, that is, $\mathbf{A}(\mathcal{M}) \subseteq \mathcal{M}$.

2.1. Properties of Nonnegative Matrices

Our focus in this paper is on decomposability of nonnegative matrices in $\mathcal{M}_n(\mathbb{R})$. For nonnegative matrices, we have the following properties (see [1], pp.487-528, and [7], pp.661-687):

1. For every pair of indices i and j, there exists a natural number m such that $(\mathbf{A}^m)_{ij}$ is not equal to zero.

2. Period of an index i is the greatest common divisor of all natural numbers m such that $(\mathbf{A}^m)_{ii} > 0$. If $\mathbf{A}$ is indecomposable, then period of every index is the same and is called the *period of* $\mathbf{A}$.

3. *Primitive Matrix*: A nonnegative matrix $\mathbf{A}$ is called primitive if its m-th power, $\mathbf{A}^m$, is positive for some natural number m (that is, the same m works for all pairs of indices).

4. Primitive matrices are the same as indecomposable aperiodic nonnegative matrices.

Please note that, as a consequence of the above statements, every positive matrix is primitive and every primitive matrix is indecomposable. In other words, every positive matrix is indecomposable. Therefore, *we restrict our analysis in this paper to the study of decomposability of nonnegative matrices.*

2.2. Perron-Forbenius Theorem for Indecomposable Nonnegative Matrices

Theorem 4. *(see [1], pp.487-528, and [7], pp.661-687) Let* $\mathbf{A}$ *be an* $n \times n$ *nonnegative indecomposable matrix with period* h *and spectral radius* ρ*. Then the following statements hold:*

1. *The number* ρ *is a positive real number, is an eigenvalue of matrix* $\mathbf{A}$*, and is referred to as the Perron-Frobenius eigenvalue of* $\mathbf{A}$*.*

2. *The Perron-Frobenius eigenvalue* ρ *is simple.*

3. *Matrix* $\mathbf{A}$ *has exactly* h *complex eigenvalues with absolute value* ρ*. Each one of these eigenvalues is a simple root of the characterisitic polynomial and is the product of* ρ *with an* h*-th root of unity.*

4. *If* $\mathbf{A}$ *is a nonnegative primitive matrix in* $\mathcal{M}_n(\mathbb{R})$*, then the square matrix* $\mathbf{A}^{n^2-2n+2}$ *is positive.*

2.3. Decomposability of Nonnegative Idempotent Matrices

We now summarize the results on decomposability of idempotent matrices [3]:

1. Every nonnegative idempotent matrix of rank $k > 1$ is decomposable.

2. For an idempotent matrix $\mathbf{A}$, $\text{trace}(\mathbf{A}) = \text{rank}(\mathbf{A})$.

3. Let $\mathbf{A}$ be an $n \times n$ nonnegative idempotent matrix of rank $k > 1$. Then, the following hold:

   (a) Any maximal standard block triangularization of $\mathbf{A}$ has the following two properties:

      i. Each diagonal block is either zero or a positive idempotent matrix of rank one.

ii. There are exactly k non-zero diagonal blocks.

(b) There exists a standard block triangularization of $\mathbf{A}$ with the above stated properties 3(a)i and 3(a)ii such that no two consecutive diagonal blocks are zero (so that the total number of diagonal blocks is less than or equal to $2k+1$).

4. Suppose $\mathcal{S}$ is a band (semigroup of nonnegative idempotent matrices) in $\mathcal{M}_n(\mathbb{R})$ with nonnegative members such that $\text{rank}(\mathbf{S}) > 1$, for all $\mathbf{S} \in \mathcal{S}$. Then, $\mathcal{S}$ is decomposable.

2.4.

The following lemma by Heydar Radjavi will be repeatedly used for proving our results in this paper:

Lemma 5. *([8], pp.106) Let $\mathbf{A}$ be a nonnegative matrix such that every positive power of $\mathbf{A}$ has at least one diagonal entry equal to zero. Then, $\mathbf{A}$ is decomposable and has 0 as an eigenvalue.*

2.5. Spectral Properties of r-Potent Matrices

[2] Any r-potent matrix has $x^r - x$ as minimal polynomial. Therefore, eigenvalues of an invertible r-potent matrix are the $(r-1)$-th roots of unity. That is, the roots are $e^{\frac{2k\pi\iota}{r-1}}, k = 0, 1, \ldots, r-2$, when not counting multiplicities of the roots. For a singular r-potent matrix, zero lies in the spectrum apart from the aforesaid eigenvalues.

Lemma 6. *[2] For an r-potent matrix $\mathbf{A}$, rank$(\mathbf{A}) =$ trace$(\mathbf{A}^{r-1})$.*

Proof. We begin by noticing that $\mathbf{A}^{r-1}$ is an idempotent matrix because

$$\mathbf{A}^{2r-2} = \mathbf{A}^{r-2}\mathbf{A}^r = \mathbf{A}^{r-2}\mathbf{A} = \mathbf{A}^{r-1}. \tag{3}$$

It then follows from Section 2.3 that $\text{trace}(\mathbf{A}^{r-1}) = \text{rank}(\mathbf{A}^{r-1})$. In addition, since the range of $\mathbf{A}^r$ is contained in the range of $\mathbf{A}^{r-1}$, we get

$$\text{rank}(\mathbf{A}^r) \leq \text{rank}(\mathbf{A}^{r-1}) \leq \cdots \leq \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^r).$$

This however implies that $\text{rank}(\mathbf{A}^r) = \text{rank}(\mathbf{A}^{r-1}) = \text{rank}(\mathbf{A})$, which, in turn, yields $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^{r-1}) = \text{trace}(\mathbf{A}^{r-1})$. □

3. Decomposability of r-Potent Matrices

Before we analyse decomposability of an r-potent matrix, let us look at decomposability of idempotent matrices more closely. It was shown in [3] (restated in Section 2.3 of this paper) that an idempotent matrix of rank > 1 is decomposable. An idempotent matrix of rank one, however, may or may not be decomposable. For example, the idempotent matrix

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

is an *indecomposable* idempotent matrix of rank one, while

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is a *decomposable* idempotent matrix of rank one. Let us now extend the result in [3] for idempotent matrices by stating a condition under which an idempotent matrix of rank one becomes decomposable:

Theorem 7. *Let* $\mathbf{A}$ *be a nonnegative idempotent matrix of rank one. Then,* $\mathbf{A}$ *is decomposable if and only if it has at least one diagonal entry zero.*

Proof. We first assume that $\mathbf{A}$ has at least one diagonal entry zero. Then, every positive power of $\mathbf{A}$ has at least one diagonal entry equal to zero because $\mathbf{A}^n = \mathbf{A}$, for all natural numbers n. Therefore, the result in [8] (Lemma 5 of this paper) implies that $\mathbf{A}$ must be decomposable.

We next assume that $\mathbf{A}$ is decomposable so that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \tag{4}$$

for some permutation matrix $\mathbf{P}$. In addition, $\operatorname{rank}(\mathbf{A}) = 1$ gives $\operatorname{rank}(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = 1$, which implies that either $\mathbf{B} = 0$ or $\mathbf{D} = 0$. It therefore follows that $\mathbf{A}$ has a zero diagonal entry. $\square$

Before proceeding to the statement of our main result on decomposability of nonnegative r-potent matrices in Theorem 8, two comments are in order:

First, note that an r-potent matrix of rank $\leq r - 1$ may or may not be decomposable. For example, if we define $\mathbf{A}$ as

$$\mathbf{A}\mathbf{e}_1 = \mathbf{e}_2$$

$$\begin{aligned}\mathbf{A}\mathbf{e}_2 &= \mathbf{e}_3\\ \vdots &= \vdots\\ \mathbf{A}\mathbf{e}_{r-1} &= \mathbf{e}_1,\end{aligned}$$

that is, we consider the matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 1\\ 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

then $\mathbf{A}$ is an indecomposable r-potent matrix of rank $r-1$ as we cannot find a nontrivial standard invariant subspace. On the other hand, the matrix

$$\begin{bmatrix} [1] & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \begin{bmatrix} 1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \vdots & \begin{bmatrix} 1/(r-1) & \cdots & 1/(r-1)\\ 1/(r-1) & \cdots & 1/(r-1)\\ \vdots & \ddots & \vdots\\ 1/(r-1) & \cdots & 1/(r-1)\end{bmatrix} \end{bmatrix}$$

is a decomposable r-potent matrix of rank $r-1$.

Second, it makes sense to analyse the decomposability of r-potent matrices of rank $> r-1$ (in addition to the decomposability of r-potent matrices of rank $\leq r-1$) because there exist an infinite number of such r-potent matrices. The existence of such matrices can be established using the properties of kronecker product. In particular, if $\mathbf{A}$ and $\mathbf{B}$ are two r-potent matrices of rank (say) $r-1$ each, then $\mathbf{A}\otimes\mathbf{B}$ would also be an r-potent matrix because: (see [4], pp.38)

$$(\mathbf{A}\otimes\mathbf{B})^r = \mathbf{A}^r\otimes\mathbf{B}^r = \mathbf{A}\otimes\mathbf{B}. \tag{5}$$

Moreover, ([4], pp.20)

$$\text{rank}(\mathbf{A}\otimes\mathbf{B}) = \text{rank}(\mathbf{A})\cdot\text{rank}(\mathbf{B}) = (r-1)^2 > r-1. \tag{6}$$

Since the above analysis holds for any kronecker product, there exist an infinite number of r-potent matrices of rank $> r-1$.

To summarize, the above two observations imply that a nonnegative r-potent matrix of rank $\leq r-1$ may or may not be decomposable and there exists an infinite number of r-potent matrices of rank $> r-1$. We next state our main result on decomposability of nonnegative real r-potent matrices:

Theorem 8. *For any nonnegative r-potent matrix $\mathbf{A}$,*

1. *If rank$(\mathbf{A}) > r-1$, then $\mathbf{A}$ is always decomposable.*
2. *If rank$(\mathbf{A}) \leq r-1$ such that $\mathbf{A}$ is singular and $\mathbf{A}^2, \mathbf{A}^3, \ldots, \mathbf{A}^{r-1}$ have at least one diagonal entry zero, then $\mathbf{A}$ is decomposable.*

Proof. We shall prove the two cases separately:

1. Let us assume that $\mathbf{A}$ is indecomposable. Then, the Perron-Frobenius theorem (Theorem 4) for indecomposable nonnegative matrices is applicable. This implies (substituting $\rho = 1$) that the largest real positive eigenvalue of $\mathbf{A}$ (that is, 1) is a simple root of the characteristic polynomial of $\mathbf{A}$. In addition, all other $(r-1)$-th roots of unity are simple roots of the characteristic polynomial. Moreover, $\mathbf{A}$ has no other eigenvalues. The above statements imply that rank$(\mathbf{A}) = r-1$, which is a contradiction, and hence, $\mathbf{A}$ must be decomposable.

2. Let us again assume that $\mathbf{A}$ is indecomposable and consider three jointly exhaustive cases:

   Case 1 rank$(\mathbf{A}) = r-1$

   Under the assumption of indecomposability, we can apply the Perron-Frobenius theorem (Theorem 4) so that $1, \alpha, \alpha^2, \ldots, \alpha^{r-2}$, where $\alpha = e^{\frac{2\pi\iota}{r-1}}$, are all simple roots of the characteristic polynomial of $\mathbf{A}$. Therefore,

$$\text{trace}(\mathbf{A}) = 1 + \alpha + \alpha^2 + \cdots + \alpha^{r-2} = 0. \tag{7}$$

   Since $\mathbf{A}$ is nonnegative, we must have all the diagonal entries of $\mathbf{A}$ as nonnegative. However, this non-negativity, in conjunction with Eqn. 7, implies that all the diagonal entries of $\mathbf{A}$ must be zero. Furthermore, since $\mathbf{A}$ is an r-potent matrix, this further implies that all the digaonal entries of

$$\mathbf{A} = \mathbf{A}^r = \mathbf{A}^{2r-1} = \mathbf{A}^{3r-2} = \cdots$$

must be zero. Similarly, combining our condition that $\mathbf{A}^2, \mathbf{A}^3, \ldots, \mathbf{A}^{r-1}$ have at least one diagonal entry zero with the fact that $\mathbf{A}^r = \mathbf{A}$, we get that each of the following:

$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A}^{r+1} = \mathbf{A}^{2r} = \mathbf{A}^{3r-1} = \cdots \\ \mathbf{A}^3 &= \mathbf{A}^{r+2} = \mathbf{A}^{2r+1} = \mathbf{A}^{3r} = \cdots \\ &\vdots \\ \mathbf{A}^{r-1} &= \mathbf{A}^{2r-2} = \mathbf{A}^{3r-3} = \mathbf{A}^{4r-4} = \cdots \end{aligned}$$

have at least one diagonal entry zero. This, thanks to Lemma 5, however implies that $\mathbf{A}$ must be decomposable, which is a contradiction.

Case 2 $\text{rank}(\mathbf{A}) = 1$

As $\mathbf{A}$ is indecomposable, we can again apply the Perron Frobenius theorem (Theorem 4) so that 1, being the Perron Frobenius eigenvalue, is a simple eigenvalue. Moreover, $\text{rank}(\mathbf{A}) = 1$ implies that there is no other eigenvalue with absolute value 1. Furthermore, the number of such eigenvalues is always equal to the period of the matrix. Therefore, we must have $\mathbf{A}$ to be aperiodic and hence primitive (Section 2.1). Now, for every primitive matrix $\mathbf{A}$ of order n, $\mathbf{A}^{n^2-2n+2}$ is a positive matrix (from Theorem 4). Finally, since $\mathbf{A}$ is singular and $\text{rank}(\mathbf{A}) = 1$, the matrix $\mathbf{A}$ must be a square matrix of size 2×2 or higher. We can now argue that

$$\begin{aligned} &\text{for } n = 2, \quad \mathbf{A}^{n^2-2n+2} = \mathbf{A}^2 \text{ is positive,} \\ &\text{for } n = 3, \quad \mathbf{A}^{n^2-2n+2} = \mathbf{A}^5 \text{ is positive,} \\ &\text{for } n = 4, \quad \mathbf{A}^{n^2-2n+2} = \mathbf{A}^{10} \text{ is positive,} \\ &\qquad\qquad \vdots \end{aligned}$$

which is a contradiction to the statement of this theorem that $\mathbf{A}^2, \mathbf{A}^3, \ldots,$ $\mathbf{A}^{r-1}$ have at least one diagonal entry zero.

Case 3 $1 < \text{rank}(\mathbf{A}) < r - 1$

Let $\text{rank}(\mathbf{A}) = k$, where $1 < k < r-1$. The number of eigenvalues of $\mathbf{A}$ would then be equal to the $\text{rank}(\mathbf{A}) = k$. However, the assumed indecomposability of $\mathbf{A}$ implies that the number of eigenvalues of $\mathbf{A}$ is also equal to the period of $\mathbf{A}$. Hence, the period of $\mathbf{A}$ must be equal to k. This is however a contradiction because $\mathbf{A}^k$ has at least one diagonal element as zero due to the condition of the theorem, while $\mathbf{A}^k$ should be a positive matrix according to Theorem 4.

Therefore, in each of the above jointly exhaustive cases, we have a contradiction to our assumption that $\mathbf{A}$ is indecomposable. This concludes the proof that $\mathbf{A}$ must be decomposable under the conditions stated in the theorem.

□

4. Structure of an r-Potent Matrix

In this section, we study the structure of decomposable nonnegative r-potent matrices. Since a decomposable matrix can always be written in a block triangular form (by Defn. 2) via a permutation matrix, our focus will be on the properties of the diagonal blocks in such a block triangular form. However, before proceeding to our results on the properties of these diagonal blocks, we briefly digress to state a few comments (adopted from [3]) on block triangularization that would be useful in proving our results in this section.

Let $\mathcal{S}$ be a semigroup of matrices in $\mathcal{M}_n(\mathbb{R})$ and $\mathcal{L}at'\mathcal{S}$ be the lattice of all standard subspaces which are invariant under every member of $\mathcal{S}$. It can be shown by simple induction that for any semigroup $\mathcal{S}$, $\mathcal{L}at'\mathcal{S}$ has a maximal chain of such subspaces. This chain may be non-trivial or trivial according to whether $\mathcal{S}$ has a non-trivial standard subspace or not. Each nontrivial chain in $\mathcal{L}at'\mathcal{S}$ gives rise to a block triangularization for $\mathcal{S}$ and since the members in the chain are standard subspaces, we shall call it a standard block triangularization. Evidently, to say that $\mathcal{S}$ has a standard block triangularization is equivalent to saying that there exists a permutation matrix $\mathbf{P}$ such that for each $\mathbf{S} \in \mathcal{S}$, $\mathbf{P}^{-1}\mathbf{S}\mathbf{P}$ has the upper block triangular form. Suppose $\mathcal{C}$ is a chain in $\mathcal{L}at'\mathcal{S}$ and $\mathcal{N}_1$ and $\mathcal{N}_2$ are two successive elements in $\mathcal{C}$ such that $\mathcal{N}_2 \subseteq \mathcal{N}_1$, then $\mathcal{N}_1 \ominus \mathcal{N}_2$ is called a gap in the chain. If $\mathbf{P}$ is the orthogonal projection onto $\mathcal{N}_1 \ominus \mathcal{N}_2$, then the restriction of $\mathbf{P}\mathcal{S}\mathbf{P}$ to the range of $\mathbf{P}$ is called the compression of $\mathcal{S}$ to $\mathcal{N}_1 \ominus \mathcal{N}_2$. Every such compression corresponds to a diagonal block in the block triangularization of $\mathbf{S}$.

We now state our main result in this section:

Theorem 9. *Any maximal standard block triangularisation of a decomposable nonnegative r-potent matrix has the following properties:*

1. *Each diagonal block is either zero or an indecomposable r-potent matrix of rank $\leq r-1$.*
2. *If n is the number of non-zero diagonal blocks, then*

$$\frac{k}{r-1} \leq n \leq k, \tag{8}$$

where $k = \text{rank}(\mathbf{A})$.

3. *Total number of diagonal blocks (including $\mathbf{0}$ blocks) lies between $\frac{k}{r-1}$ and $2k+1$.*

Proof. We shall prove the above three statements in order:

1. Let

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \mathbf{A}_{11} & * & \cdots & * \\ \mathbf{0} & \mathbf{A}_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{A}_{nn} \end{pmatrix} \tag{9}$$

be a maximal standard block triangularization of $\mathbf{A}$ via a permutation matrix $\mathbf{P}$. As $\mathbf{A}^r = \mathbf{A}$, we have $(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^r = \mathbf{P}^{-1}\mathbf{A}^r\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ so that $\mathbf{A}_{ii}^r = \mathbf{A}_{ii}$, for all $i = 1, 2, \ldots, n$. Thus, each diagonal block is itself an r-potent matrix. Further, each such diagonal block $\mathbf{A}_{ii}$ would be indecomposable and would have $\text{rank}(\mathbf{A}_{ii}) \leq r-1$. This can be seen as follows: Suppose some $\mathbf{A}_{jj}$ is decomposable with standard subspace $\mathcal{K}$. Now, $\mathbf{A}_{jj}$ corresponds to some gap, say $\mathcal{N}_1 \ominus \mathcal{N}_2$, in the maximal chain of invariant subspaces for the aforesaid block triangularization of $\mathbf{A}$. Then, $\mathcal{N}_2 \oplus \mathcal{K}$ is a standard subspace, invariant under $\mathbf{A}$ which lies strictly between $\mathcal{N}_1$ and $\mathcal{N}_2$, thus contradicting the maximality of the above triangularization. In other words, $\mathbf{A}_{jj}$ must be indecomposable. Finally, since all r-potent matrices of rank $> r-1$ are decomposable by Theorem 8, we get $\text{rank}(\mathbf{A}_{ii}) \leq r-1, \forall i$.

2. We start by noticing that

$$\text{trace}(\mathbf{A}^{r-1}) = \text{trace}(\mathbf{A}_{11}^{r-1}) + \cdots + \text{trace}(\mathbf{A}_{nn}^{r-1}). \tag{10}$$

Therefore,

$$k = \text{rank}(\mathbf{A}) \tag{11}$$

$$
\begin{aligned}
&= \mathrm{trace}(\mathbf{A}^{r-1}) \qquad \text{(from Lemma 6)} && (12)\\
&= \mathrm{trace}(\mathbf{A}_{11}^{r-1}) + \cdots + \mathrm{trace}(\mathbf{A}_{nn}^{r-1}) && (13)\\
&= \mathrm{rank}(\mathbf{A}_{11}) + \cdots + \mathrm{rank}(\mathbf{A}_{nn}) \qquad \text{(from Lemma 6)} && (14)\\
&\le (r-1) + \cdots + (r-1) && (15)\\
&= n(r-1), && (16)
\end{aligned}
$$

and therefore, we get the lower bound:

$$n \ge \frac{k}{r-1}. \tag{17}$$

On the other hand, let us consider an extreme case that each non-zero diagonal block has rank 1. Then, the maximum number of non-zero diagonal blocks is k, so that we get the upper bound $n \le k$. Combining the two bounds, we can write

$$\frac{k}{r-1} \le n \le k. \tag{18}$$

3. We claim that two consecutive diagonal blocks cannot be zero. Suppose that two consecutive diagonal blocks are zero. Then, a 2×2 block matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

would be an r-potent if and only if it is zero. Therefore, the total number of diagonal blocks (including $\mathbf{0}$ blocks) lies between $\frac{k}{r-1}$ and $2k+1$.

□

5. Decomposability of Kronecker Product of r-potent matrices

In this section, we shall discuss the decomposability of kronecker products of r-potent matrices. We start by reiterating that the kronecker product of two r-potent matrices is itself an r-potent matrix because

$$
\begin{aligned}
(\mathbf{A} \otimes \mathbf{B})^r &= \mathbf{A}^r \otimes \mathbf{B}^r && (19)\\
&= \mathbf{A} \otimes \mathbf{B}. && (20)
\end{aligned}
$$

We can now state the two main results in this section as follows:

Theorem 10. *Let* $\mathbf{A}$ *be a nonnegative* r*-potent matrix of rank* $> r-1$ *and* $\mathbf{B}$ *be any non-zero nonnegative* r*-potent matrix. Then,* $\mathbf{A}\otimes\mathbf{B}$ *is a decomposable* r*-potent of rank* $> r-1$.

Proof. Since

$$\begin{aligned}\text{rank}(\mathbf{A}\otimes\mathbf{B}) &= \text{rank}(\mathbf{A})\text{rank}(\mathbf{B}) &(21)\\ &> (r-1)\text{rank}(\mathbf{B}), &(22)\end{aligned}$$

which implies that

$$\text{rank}(\mathbf{A}\otimes\mathbf{B}) > r-1. \qquad (23)$$

Therefore, from Theorem 8 of this paper, $\mathbf{A}\otimes\mathbf{B}$ is a decomposable r-potent matrix. □

Theorem 11. *Let* $\mathbf{A}$ *be a nonnegative* r*-potent matrix of rank* $> r-1$ *and* $\mathbf{B}$ *be a non-zero nonnegative idempotent matrix. Then,* $\mathbf{A}\otimes\mathbf{B}$ *is a decomposable* r*-potent matrix.*

Proof. Since an idempotent matrix is, by definition, also an r-potent matrix, we have

$$(\mathbf{A}\otimes\mathbf{B})^r = \mathbf{A}^r\otimes\mathbf{B}^r = \mathbf{A}\otimes\mathbf{B}, \qquad (24)$$

and therefore, $\mathbf{A}\otimes\mathbf{B}$ is also an r-potent matrix. In addition,

$$\begin{aligned}\text{rank}(\mathbf{A}\otimes\mathbf{B}) &= \text{rank}(\mathbf{A})\text{rank}(\mathbf{B}) &(25)\\ &> (r-1)\text{rank}(\mathbf{B}), &(26)\end{aligned}$$

which implies that

$$\text{rank}(\mathbf{A}\otimes\mathbf{B}) > r-1, \qquad (27)$$

and hence, $\mathbf{A}\otimes\mathbf{B}$ is decomposable by Theorem 8 of this paper. □

6. Decomposibility of Semi-Groups of r-Potent Matrices

Let us first analyse the decomposability of a semigroup generated by a single r-potent matrix.

Theorem 12. *Let* $\mathbf{A}$ *be a nonnegative* r*-potent matrix. Consider the semigroup* $\mathcal{S} = \{\mathbf{A}, \mathbf{A}^2, \cdots, \mathbf{A}^{r-1}\}$ *generated by* $\mathbf{A}$*.*

1. *If rank*$(\mathbf{A}) > r-1$*, then* $\mathcal{S}$ *is decomposable.*
2. *If rank*$(\mathbf{A}) \leq r-1$*,* $\mathbf{A}$ *is singular and* $\mathbf{A}^2, \mathbf{A}^3, \cdots, \mathbf{A}^{r-1}$ *have a zero diagonal entry, then* $\mathcal{S}$ *is decomposable.*

Proof. We shall prove the two parts seperately.

1. Since rank$(\mathbf{A}) > r-1$, $\mathbf{A}$ is decomposable by Theorem 8. Let
$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \tag{28}$$
be a block-triangular decomposition of $\mathbf{A}$. Then
$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^k = \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P} = \begin{pmatrix} \mathbf{B}^k & * \\ \mathbf{0} & \mathbf{D}^k \end{pmatrix} \tag{29}$$
for all $k = 2, 3, \ldots, r-1$. Therefore, $\mathcal{S}$ is decomposable via permutation matrix $\mathbf{P}$.
2. Under the stated conditions, $\mathbf{A}$ is decomposable. Decomposability of $\mathcal{S}$ follows from an argument similar to (1) above.

□

The above theorem motivates us to study decomposability of a general semigroup of r-potent matrices. To this end, we require the following equivalent conditions for decomposability of nonnegative semigroups in $\mathcal{M}_n(\mathbb{R})$.

Lemma 13. *[3] For a semigroup* $\mathcal{S}$ *in* $\mathcal{M}_n(\mathbb{R})$ *with nonnegative matrices, the following are equivalent:*

1. $\mathcal{S}$ *is decomposable.*
2. *There exists a non-zero nonnegative functional on* $\mathcal{M}_n(\mathbb{R})$ *whose restriction to* $\mathcal{S}$ *is zero.*
3. $\mathcal{S}$ *has a common zero entry.*

4. *$\mathcal{S}$ has a common non-diagonal zero entry.*

5. *There exist $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n(\mathbb{R})$, both non-zero and nonnegative such that $\mathbf{A}\mathcal{S}\mathbf{B} = \{0\}$.*

We now proceed to the two main results of this section and detail their respective proofs:

Theorem 14. *Let $\mathcal{S}$ be a semigroup of nonnegative r-potent matrices of rank $> r-1$. Then, $\mathcal{S}$ is decomposable.*

Proof. We start by noticing that a semigroup of r-potent matrices always contains an idempotent matrix. For example, if $\mathbf{A}$ is an r-potent matrix in $\mathcal{S}$, then $\mathbf{A}^{r-1}$ is an idempotent matrix in $\mathcal{S}$ (see Eqn. 3). Consider now a minimal rank idempotent matrix $\mathbf{P}$ in $\mathcal{S}$. We can always choose such a minimal rank idempotent matrix because if we choose any minimal rank matrix $\mathbf{B}$ in $\mathcal{S}$, then the rank of the corresponding idempotent matrix $\mathbf{B}^{r-1}$ is upperbounded by $\operatorname{rank}(\mathbf{B})$ because ([4], pp.61)

$$\operatorname{rank}(\mathbf{B}^{r-1}) \leq \min\{\operatorname{rank}(\mathbf{B}^{r-2}), \operatorname{rank}(\mathbf{B})\} \tag{30}$$

$$\leq \operatorname{rank}(\mathbf{B}). \tag{31}$$

Now, since $\operatorname{rank}(\mathbf{P}) > r-1 > 1$, $\mathbf{P}$ must be decomposable. Let $\mathbf{P}$ have the form

$$\begin{pmatrix} \mathbf{P}_1 & \mathbf{K} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix}$$

with respect to some permutation of basis where both $\mathbf{P}_1$ and $\mathbf{P}_2$ are non-zero.

Let us now consider an arbitrary $\mathbf{S} \in \mathcal{S}$. Then, $(\mathbf{PSP})^{r-1}$ will be an idempotent matrix, also in $\mathcal{S}$. Further, the range of $(\mathbf{PSP})^{r-1}$ is contained in the range of $\mathbf{P}$ and the null space of $(\mathbf{PSP})^{r-1}$ contains the null space of $\mathbf{P}$. Therefore,

$$\operatorname{rank}((\mathbf{PSP})^{r-1}) \leq \operatorname{rank}(\mathbf{P}). \tag{32}$$

Considering that $\mathbf{P}$ is minimal rank in $\mathcal{S}$, we get

$$\operatorname{rank}((\mathbf{PSP})^{r-1}) = \operatorname{rank}(\mathbf{P}). \tag{33}$$

Using rank-nullity theorem of linear algebra ([7], pp.199), we have nullity of $(\mathbf{PSP})^{r-1}$ is equal to the nullity of $\mathbf{P}$, which, in turn, yields

$$(\mathbf{PSP})^{r-1} = \mathbf{P}. \tag{34}$$

This, however, implies that

$$\mathbf{PSP} = \mathbf{P}^{\frac{1}{r-1}} = \mathbf{P} \tag{35}$$

due to the following lemma:

Lemma 15. *In the given semigroup $\mathcal{S}$, the only nonnegative $(r-1)$-th root of minimal rank idempotent $\mathbf{P}$ is $\mathbf{P}$ itself.*

Proof. Since

$$\mathbf{P}^2 = \mathbf{P} \tag{36}$$

$$\Rightarrow \mathbf{P}^{r-1} = \mathbf{P}, \tag{37}$$

$\mathbf{P}$ is itself a nonnegative $(r-1)$-th root of $\mathbf{P}$ in $\mathcal{S}$. Let $\mathbf{P}'$ be another nonnegative $(r-1)$-th root of $\mathbf{P}$ in $\mathcal{S}$, that is,

$$(\mathbf{P}')^{r-1} = \mathbf{P}. \tag{38}$$

As $\mathbf{P}'$ belongs to $\mathcal{S}$, we also have

$$(\mathbf{P}')^{r} = \mathbf{P}'. \tag{39}$$

It follows from Eqn. 38 and Eqn. 39 that

$$\mathbf{PP}' = \mathbf{P}'\mathbf{P} = \mathbf{P}'. \tag{40}$$

Using the fact that $\mathbf{P}$ is of minimal rank in $\mathcal{S}$, it follows from rank-nullity theorem that $R(\mathbf{P}) = R(\mathbf{P}')$ and $N(\mathbf{P}) = N(\mathbf{P}')$. Moreover, Eqn. 36 implies that

$$\mathbf{P} = \mathbf{I} \qquad \text{on} \quad R(\mathbf{P}). \tag{41}$$

which, due to Eqn. 38, implies that

$$(\mathbf{P}')^{r-1} = \mathbf{I} \qquad \text{on} \quad R(\mathbf{P}) \tag{42}$$

where $\mathbf{I}$ is the identity operator. In other words, $\mathbf{P}'$ is a nonnegative $(r-1)$-th root of $\mathbf{I}$ on $R(\mathbf{P})$. However, the only nonnegative $(r-1)$-th root of the identity operator is the identity operator itself. Therefore

$$\mathbf{P}' = \mathbf{I} = \mathbf{P} \qquad \text{on} \quad R(\mathbf{P}) = R(\mathbf{P}') \tag{43}$$

and

$$\mathbf{P}' = \mathbf{P} = \mathbf{0} \qquad \text{on} \quad N(\mathbf{P}) = N(\mathbf{P}'). \tag{44}$$

Therefore

$$\mathbf{P}' = \mathbf{P}. \tag{45}$$

□

Finally, let

$$\begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix}$$

be the representation of an arbitrary $\mathbf{S} \in \mathcal{S}$ with respect to this permuted basis. Then, $\mathbf{PSP} = \mathbf{P}$ implies that $\mathbf{P}_2\mathbf{S}_{21}\mathbf{P}_1 = \mathbf{0}$. If p_{ij} and r_{kl} are non-zero entries in $\mathbf{P}_2$ and $\mathbf{P}_1$, respectively, then it is easy to see that the (j,k)-th entry in each $\mathbf{S} \in \mathcal{S}$ is zero. This makes use of the fact that $\mathbf{P}_2, \mathbf{S}_{21}$, and $\mathbf{P}_1$ are all nonnegative matrices. By Lemma 13, $\mathcal{S}$ is decomposable. □

Theorem 16. *Let $\mathcal{S}$ be a semigroup of nonnegative r-potent matrices of rank $> r-1$. Then, any maximal standard block triangularization of $\mathcal{S}$ has the property that each non-zero diagonal block is a semigroup of nonnegative r-potent matrices with at least one element of rank $\leq r-1$.*

Proof. By Theorem 14, $\mathcal{S}$ is decomposabable. Consider any maximal chain in $\mathcal{L}at'\mathcal{S}$ resulting in a standard block triangularization of $\mathcal{S}$. Consider any two subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ in this chain such that $\mathcal{N}_1 \subseteq \mathcal{N}_2$ and $\mathcal{N}_2 \ominus \mathcal{N}_1$ is a gap. If the compression of $\mathcal{S}$ to $\mathcal{N}_2 \ominus \mathcal{N}_1$ is non-zero, it forms a semigroup of nonnegative r-potents. Further, it must be indecomposable, for otherwise, if it has a standard invariant subspace $\mathcal{K}$, then $\mathcal{N}_1 \oplus \mathcal{K}$ is in $\mathcal{L}at'\mathcal{S}$ and lies strictly between $\mathcal{N}_1$ and $\mathcal{N}_2$, contradicting the maximality of this chain. Thus, every non-zero compression (or diagonal-block) constitutes an indecomposable semigroup of r-potent matrices. By Theorem 14, it must contain at least one element of rank $\leq r-1$. □

7. Decomposability of permutation matrices

In this section, we shall study decomposability (or rather, indecomposability) of permutation matrices. We start by noticing that an $n \times n$ circulant matrix generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ (standard basis vectors), given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n},$$

is indecomposable as we cannot find a standard invariant subspace. To generalize this observation as a formal result, we require the following lemma:

Lemma 17. *([8], pp.106) For a semigroup $\mathcal{S}$ of nonnegative matrices, the following are equivalent:*

1. *$\mathcal{S}$ is decomposable.*
2. *Every sum of members of $\mathcal{S}$ has a zero entry.*

We now state the main result of this section.

Theorem 18. *Group $\mathcal{G}$ of $n \times n$ permutation matrices generated by the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, comprises of idempotent matrices, tripotent matrices, quadripotent matrices, pentapotent matrices,$\ldots,(n+1)$-potent matrices and is indecomposable.*

Proof. The group comprises the following:

Idempotent Matrices

$$\mathbf{I}_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

Tripotent Matrices are formed by permuting two rows at a time. For example

$$\mathbf{A}_{21} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

Quadripotent Matrices are formed by permuting three rows at a time. For example

$$\mathbf{A}_{231} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

(n+1)-potent Matrices are formed by permuting all n rows at a time. For example

$$\mathbf{A}_{234\ldots(n+1)1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}.$$

Then, $\mathbf{I}_{n\times n} + \sum \mathbf{A}_{234\ldots(n+1)1}$ does not have any zero entry. Therefore, by Lemma 17, $\mathcal{G}$ is indecomposable. □

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