HYERS-ULAM-RASSIAS STABILITY OF FUNCTIONAL EQUATION IN NAB-SPACES

Dong Yun Shin¹, Hassan Azadi Kenary²§, N. Sahami³

¹Department of Mathematics
College of Natural Science
University of Seoul
KOREA

²,³Department of Mathematics
Beyza Branch
Islamic Azad University
Beyza, IRAN

Abstract: In this paper, using direct method we investigate the Hyers-Ulam-Rassias stability of an additive functional equation in non-Archimedean Banach spaces (briefly, NAB-spaces).

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1. Introduction and Preliminaries

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation $D$ must be close to an exact solution of $D$?

If the problem accepts a solution, we say that the equation $D$ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940.
We are given a group $G$ and a metric group $G'$ with metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G$, then a homomorphism $h : G \to G$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G'$?

Ulam’s problem was partially solved by Hyers [10] in 1941.

In 1978, Th. M. Rassias [18] formulated and proved the following theorem, which implies Hyers’s Theorem as a special case. Suppose that $E$ and $F$ are real normed spaces with $F$ a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f(tx)$ is continuous on $R$, and let there exist $\varepsilon \geq 0$ and $p \in [0,1)$ such that for all $x, y \in E$

$$||f(x + y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p) \quad (1.1)$$

Then there exists a unique linear mapping $T : E \to F$ such that such that for all $x \in E$

$$||f(x) - T(x)|| \leq \frac{\varepsilon||x||^p}{1 - 2^p - 1}$$

The case of the existence of a unique additive mapping had been obtained by T. Aoki [2], as it is recently noticed by Lech Maligranda. However, Aoki [2] had claimed the existence of a unique linear mapping, that is not true because he did not allow the mapping $f$ to satisfy some continuity assumption. Th.M. Rassias [18], who independently introduced the unbounded Cauchy difference was the first to prove that there exists a unique linear mapping $T$ satisfying

$$||f(x) - T(x)|| \leq \frac{\varepsilon||x||^p}{1 - 2^p - 1} \quad x \in E$$

In 1990, Th.M. Rassias [19] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [8] following the same approach as in Th. M. Rassias [24], gave an affirmative solution to this question for $p > 1$. It was proved by Z. Gajda [8], as well as by Th. M. Rassias and P. Šemrl [20] that one can not prove a Th. M. Rassias type theorem when $p = 1$. In 1994, P. Găvruta [9] provided a further generalization of Th. M. Rassias theorem in which he replaced the bound $\varepsilon(||x||^p + ||y||^p)$ by a general control function $\psi(x,y)$ for the existence of a unique linear mapping.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized
Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [25] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. In [7], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians( [1]-[5], [11]-[24]).

**Definition 1.1.** By a non-Archimedean field we mean a field $K$ equipped with a function(valuation) $|.| : K \to [0,\infty)$ such that for all $r, s \in K$, the following conditions hold:

$(i)$ $|r| = 0$ if and only if $r = 0$

$(ii)$ $|rs| = |r||s|$

$(iii)$ $|r + s| \leq \max\{|r|, |s|\}$.

**Definition 1.2.** Let $X$ be a vector space over a scalar field $K$ with a non-Archimedean non-trivial valuation $|.|$. A function $||.|| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

$(i)$ $||x|| = 0$ if and only if $x = 0$

$(ii)$ $||rx|| = |r|||x||$ ($r \in K, x \in X$)

$(iii)$ The strong triangle inequality( ultrametric); namely

$$||x + y|| \leq \max\{||x||, ||y||\}. \quad x, y \in X$$

Then $(X, ||.||)$ is called a non-Archimedean space.

Due to the fact that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n - 1\} \quad (n > m)$$

**Definition 1.3.** A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

**Example 1.1.** Fix a prime number $p$. For any nonzero rational number $x$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where $a$ and $b$ are integers not divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the $p$-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x} a_kp^k$ where $|a_k| \leq p - 1$ are integers. The addition
and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n} a_k p^k| p = p^{-n}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field.

**Definition 1.4.** Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a **generalized metric** on $X$ if $d$ satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1.1.** Let $(X, d)$ be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$  \hspace{1cm} (1.2)

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that:

(a) $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;
(b) the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
(c) $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation

$$f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y)$$  \hspace{1cm} (1.3)

in non-Archimedean normed spaces. In the rest of the paper let $|2| \neq 1$.

**2. Non-Archimedean Stability of Eq. (1.3): A Direct Method**

Throughout this section, using direct method we prove the generalized Hyers-Ulam stability of composite functional equation (1.3) in non-Archimedean spaces.

**Theorem 2.1.** Let $G$ be an additive semigroup and $X$ is a complete non-Archimedean space. Assume that $\varphi : G^2 \to [0, +\infty)$ be a function such that

$$\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0$$  \hspace{1cm} (2.1)
for all $x, y \in G$. Let for all $x \in G$

$$\Phi(x) = \text{Sup}_{k \geq 0} \left\{ \frac{\varphi(2^k x, 2^k x)}{|2|^k}; \ k \in \mathbb{N} \cup \{0\} \right\}$$  \hspace{1cm} (2.2)

exists. Suppose that $f : G \to X$ be a mapping satisfying the inequality

$$\left\| f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y) \right\| \leq \varphi(x, y)$$  \hspace{1cm} (2.3)

for all $x, y \in G$. Then the limit

$$A(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exist for all $x \in G$ and $A : G \to X$ is an additive mapping satisfying

$$\left\| f(x) - A(x) \right\| \leq \frac{1}{|2|} \Phi(x)$$  \hspace{1cm} (2.4)

for all $x \in G$. Moreover, if

$$\lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ \frac{\varphi(2^k x, 2^k x)}{|2|^k}; \ j \leq k < n + j \right\} = 0$$

Then $A$ is the unique mapping satisfying (2.4).

Proof. Putting $y = x$ in (2.3), we have

$$\left\| \frac{f(2x)}{2} - f(x) \right\| \leq \frac{1}{|2|} \varphi(x, x)$$  \hspace{1cm} (2.5)

Replacing $x$ by $2^n x$ in (2.5), we get

$$\left\| \frac{f(2^{n+1} x)}{2^{n+1}} - \frac{f(2^n x)}{2^n} \right\| \leq \frac{\varphi(2^n x, 2^n x)}{|2|^{n+1}}$$  \hspace{1cm} (2.6)

It follows from (2.1) and (2.6) that the sequence $\left\{ \frac{f(2^n x)}{2^n} \right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $X$ is complete, so $\left\{ \frac{f(2^n x)}{2^n} \right\}_{n=1}^{\infty}$ is convergent. Set

$$A(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n}.$$ 

Using induction we see that

$$\left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \leq \frac{1}{|2|} \max \left\{ \frac{\varphi(2^k x, 2^k x)}{|2|^k}; 0 \leq k < n \right\}.$$  \hspace{1cm} (2.7)
Indeed, (2.7) holds for \( n = 1 \) by (2.5). Let, (2.7) holds for \( n \), so by (2.6), we obtain

\[
\left\| \frac{f(2^{n+1}x)}{2^{n+1}} - f(x) \right\|_X = \left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^nx)}{2^n} - f(x) \right\|_X \tag{2.8}
\]

\[
\leq \max \left\{ \left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^nx)}{2^n} \right\|_X, \left\| \frac{f(2^nx)}{2^n} - f(x) \right\|_X \right\}
\]

\[
\leq \frac{1}{|2|} \max \left\{ \frac{\varphi(2^nx, 2^nx)}{|2|^n}, \max \left\{ \frac{\varphi(2^kx, 2^kx)}{|2|^k}; 0 \leq k < n \right\} \right\}
\]

\[
= \frac{1}{|2|} \max \left\{ \frac{\varphi(2^kx, 2^kx)}{|2|^k}; 0 \leq k < n + 1 \right\}.
\]

So for all \( n \in \mathbb{N} \) and all \( x \in G \), (2.7) holds. By taking \( n \) to approach infinity in (2.8), one obtains (2.4).

If \( L \) is another mapping satisfies (2.4), then for \( x \in G \), we get

\[
\|A(x) - L(x)\|_X = \lim_{k \to \infty} \left\| \frac{A(2^kx)}{2^k} - \frac{L(2^kx)}{2^k} \right\|_X
\]

\[
= \lim_{k \to \infty} \left\| \frac{A(2^kx)}{2^k} + \frac{f(2^kx)}{2^k} - \frac{L(2^kx)}{2^k} \right\|_X
\]

\[
\leq \lim_{k \to \infty} \max \left\{ \left\| \frac{A(2^kx) - f(2^kx)}{2^k} \right\|_X, \left\| \frac{f(2^kx) - L(2^kx)}{2^k} \right\|_X \right\}
\]

\[
\leq \lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ \frac{\varphi(2^kx, 2^kx)}{|2|^k}; j \leq k < n + j \right\}
\]

\[
= 0.
\]

Therefore \( A = L \). This completes the proof. \( \square \)

**Corollary 2.1.** Let \( \xi : [0, \infty) \to [0, \infty) \) be a function satisfying

\[
\xi(|2|) \leq \xi(|2|)\lambda(t), \quad \xi(|2|) < |2|
\]

for all \( t \geq 0 \). Let \( \delta > 0 \) and \( f : G \to X \) is a mapping satisfying the inequality

\[
\left\| f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y) \right\| \leq \delta \left( \xi(||x||) + \xi(||y||) \right) \tag{2.9}
\]

for all \( x, y \in G \). Then the limit \( A(x) = \lim_{n \to \infty} \frac{f(2^nx)}{2^n} \) exists for all \( x \in G \) and \( A : G \to X \) is a unique additive mapping such that

\[
\left\| f(x) - A(x) \right\| \leq \frac{2\delta\xi(||x||)}{|2|}
\]

for all \( x \in G \).
Proof. Defining $\varphi : G^2 \to [0, \infty)$ by $\varphi(x, y) := \delta (\xi(||x||) + \xi(||y||))$. Since $\frac{\xi(||2||)}{2} < 1$, we have
\[
\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} \leq \lim_{n \to \infty} \left( \frac{\xi(||2||)}{|2|} \right)^n \varphi(x, y) = 0
\]
for all $x, y \in G$. Also for all $x \in G$
\[
\Phi(x) = \sup_{k \geq 0} \left\{ \frac{\varphi(2^k x, 2^k x)}{|2|^k}; k \in \mathbb{N} \cup \{0\} \right\} = \varphi(x, x) = 2\delta \xi(||x||)
\]
exists for all $x \in G$. On the other hand
\[
\lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ \frac{\varphi(2^k x, 2^k x)}{|2|^k}; j \leq k < n + j \right\} = \lim_{j \to \infty} \frac{\varphi(2^j x, 2^j x)}{|2|^j} = 0.
\]
Applying Theorem 2.1, then we get the desired result.
\[
\square
\]

**Theorem 2.2.** Let $G$ is an additive semigroup and $X$ is a complete non-Archimedean space. Assume that $\varphi : G^2 \to [0, +\infty)$ be a function such that
\[
\lim_{n \to \infty} |2|^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0
\]
for all $x, y \in G$. Let for each $x \in G$
\[
\Phi(x) = \sup_{k \geq 0} \left\{ |2|^k \varphi \left( \frac{x}{2^k+1}, \frac{x}{2^k+1} \right); k \in \mathbb{N} \cup \{0\} \right\}
\]
exists. Suppose that $f : G \to X$ is a mapping satisfying the inequality (2.3). Then the limit
\[
A(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
\]
exist for all $x \in G$ and $A : G \to X$ is an additive mapping satisfying
\[
||f(x) - A(x)|| \leq \Phi(x)
\]
for all $x \in G$. Moreover, if
\[
\lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ |2|^k \varphi \left( \frac{x}{2^k+1}, \frac{x}{2^k+1} \right); j \leq k < n + j \right\} = 0
\]
Then $A$ is the unique mapping satisfying (2.12).
Proof. By (??), we have

$$\left\| f(x) - 2f \left( \frac{x}{2} \right) \right\|_X \leq \zeta \left( \frac{x}{2}, \frac{x}{2} \right)$$

(2.14)

for all $x \in G$. Replacing $x$ by $\frac{x}{2^n}$ in (2.14), we get

$$\left\| 2^n f \left( \frac{x}{2^n} \right) - 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) \right\|_X \leq |2|^n \phi \left( \frac{x}{2^{n+1}}, \frac{x}{2^{n+1}} \right)$$

(2.15)

It follows from (2.10) and (2.15) that the sequence $\{2^n f \left( \frac{x}{2^n} \right)\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $X$ is complete, so $\{2^n f \left( \frac{x}{2^n} \right)\}_{n=1}^{\infty}$ is convergent. It follows from (2.15) that

$$\left\| 2^n f \left( \frac{x}{2^n} \right) - 2^p f \left( \frac{x}{2^p} \right) \right\|_X = \left\| \sum_{k=p}^{n-1} 2^{k+1} f \left( \frac{x}{2^{k+1}} \right) - 2^k f \left( \frac{x}{2^k} \right) \right\|_X$$

(2.16)

$$\leq \max \left\{ \left\| 2^{k+1} f \left( \frac{x}{2^{k+1}} \right) - 2^k f \left( \frac{x}{2^k} \right) \right\|_X \mid ; p \leq k < n-1 \right\}$$

$$\leq \max \left\{ \left| 2^{k+1} \varphi \left( \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right) \right| ; p \leq k < n-1 \right\}$$

for all $x \in G$ all non-negative integer $n, p$ with $n > p \geq 0$. Letting $p = 0$ and passing the limit $n \to \infty$ in the last inequality, we obtain (2.12). The rest of the proof is similar to the proof of Theorem 2.1.

Corollary 2.2. Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying

$$\xi(|2|^{-1}t) \leq \xi(|2|^{-1}) \lambda(t), \quad \xi(|2|^{-1}) < |2|^{-1}$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ is a mapping satisfying the inequality (2.9). Then the limit $A(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)$ exists for all $x \in G$ and $A : G \to X$ is a unique additive mapping such that

$$\left\| f(x) - A(x) \right\| \leq \frac{2\xi(||x||)}{|2|}$$

(2.17)

for all $x \in G$.

Proof. Defining $\varphi : G^2 \to [0, \infty)$ by $\varphi(x, y) := \delta (\xi(||x||) + \xi(||y||))$. Proceeding as in the proof of the Corollary 2.1, we have $\lim_{n \to \infty} \varphi \left( \frac{2^n x, 2^n y}{|2|^n} \right) = 0$ for all $x, y \in G$. Also for all $x \in G$

$$\Phi(x) = \Sup_{k \geq 0} \left\{ |2|^k \varphi \left( \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right) ; k \in \mathbb{N} \cup \{0\} \right\} = \varphi \left( \frac{x}{2}, \frac{x}{2} \right) = \frac{2\xi(||x||)}{|2|}$$

exists for all $x \in G$. Applying Theorem 2.2, then we get the desired result. \qed
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References


