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HYERS-ULAM-RASSIAS STABILITY OF FUNCTIONAL EQUATION IN NAB-SPACES

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Abstract: In this paper, using direct method we investigate the Hyers-Ulam-Rassias stability of an additive functional equation in non-Archimedean Banach spaces (briefly, NAB-spaces).

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1. Introduction and Preliminaries

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?.

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940.

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We are given a group G and a metric group G' with metric $d(.,.)$. Given $\varepsilon > 0$, dose there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G$, then a homomorphism $h : G \to G$; exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$?.

Ulam's problem was partially solved by Hyers [10] in 1941.

In 1978, Th. M. Rassias [18] formulated and proved the following theorem, which implies Hyers's Theorem as a special case. Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f(tx)$ is continuous on R, and let there exist $\varepsilon \geq 0$ and $p \in [0, 1)$ such that for all $x, y \in E$

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p) \tag{1.1}$$

Then there exists a unique linear mapping $T : E \to F$ such that such that for all $x \in E$

$$||f(x) - T(x)|| \leq \frac{\varepsilon||x||^p}{1 - 2^{p-1}}$$

The case of the existence of a unique additive mapping had been obtained by T. Aoki [2], as it is recently noticed by Lech Maligranda. However, Aoki [2] had claimed the existence of a unique linear mapping, that is not true because he did not allow the mapping f to satisfy some continuity assumption. Th.M. Rassias [18], who independently introduced the unbounded Cauchy difference was the first to prove that there exists a unique linear mapping T satisfying

$$||f(x) - T(x)|| \leq \frac{\epsilon||x||^p}{1 - 2^{p-1}} \quad x \in E$$

In 1990, Th.M. Rassias [19] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [8] following the same approach as in Th. M. Rassias [24], gave an affirmative solution to this question for $p > 1$. It was proved by Z. Gajda [8] , as well as by Th. M. Rassias and P. Šemrl [20] that one can not prove a Th. M. Rassias type theorem when $p = 1$. In 1994, P. Găvruta [9] provided a further generalization of Th. M. Rassias theorem in which he replaced the bound $\varepsilon(||x||^p + |||y||^p)$ by a general control function $\psi(x, y)$ for the existence of a unique linear mapping.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized

Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [25] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [7], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians([1]-[5], [11]-[24]).

Definition 1.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function(valuation) $|.| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(i) $|r| = 0$ if and only if $r = 0$

(ii) $|rs| = |r||s|$

(iii) $|r + s| \leq max\{|r|, |s|\}$.

Definition 1.2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $||.|| : X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(i) $||x|| = 0$ if and only if $x = 0$

(ii) $||rx|| = |r|||x|| \quad (r \in \mathbb{K}, x \in X)$

(iii) The strong triangle inequality(ultrametric); namely

$$||x + y|| \leq max\{||x||, ||y||\}. \qquad x, y \in X$$

Then $(X, ||.||)$ is called a non-Archimedean space.

Due to the fact that

$$||x_n - x_m|| \leq max\{||x_{j+1} - x_j|| : m \leq j \leq n - 1\} \qquad (n > m)$$

Definition 1.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Example 1.1. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$ which is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$ where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact filed.

Definition 1.4. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.2}$$

for all nonnegative integers n or there exists a positive integer n_0 such that:

(a) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*
(b) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(c) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*

(d) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation

$$f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y) \tag{1.3}$$

in non-Archimedean normed spaces. In the rest of the paper let $|2| \neq 1$.

2. Non-Archimedean Stability of Eq. (1.3): A Direct Method

Throughout this section, using direct method we prove the generalized Hyers-Ulam stability of composite functional equation (1.3) in non-Archimedean spaces.

Theorem 2.1. *Let G is an additive semigroup and X is a complete non-Archimedean space. Assume that $\varphi : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0 \tag{2.1}$$

for all $x, y \in G$. Let for all $x \in G$

$$\Phi(x) = Sup_{k\geq 0}\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k};\ k \in \mathbb{N}\cup\{0\}\right\} \tag{2.2}$$

exists. Suppose that $f : G \to X$ be a mapping satisfying the inequality

$$\Big\| f(f(x) - f(y)) - f(x+y) - f(x-y) + f(x) + f(y) \Big\| \leq \varphi(x, y) \tag{2.3}$$

for all $x, y \in G$. Then the limit

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exist for all $x \in G$ and $A : G \to X$ is an additive mapping satisfying

$$||f(x) - A(x)|| \leq \frac{1}{|2|}\Phi(x) \tag{2.4}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} max\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k}; j \leq k < n+j\right\} = 0$$

Then A is the unique mapping satisfying (2.4).

Proof. Putting $y = x$ in (2.3), we have

$$\left\|\frac{f(2x)}{2} - f(x)\right\|_X \leq \frac{1}{|2|}\varphi(x, x) \tag{2.5}$$

Replacing x by $2^n x$ in (2.5), we get

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\right\|_X \quad \leq \quad \frac{\varphi(2^n x, 2^n x)}{|2|^{n+1}} \tag{2.6}$$

It follows from (2.1) and (2.6) that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, so $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ is convergent. Set

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

Using induction we see that

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\|_X \quad \leq \quad \frac{1}{|2|} max\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k}; 0 \leq k < n\right\}. \tag{2.7}$$

Indeed, (2.7) holds for $n = 1$ by (2.5). Let, (2.7) holds for n, so by (2.6), we obtain

$$\begin{aligned}\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - f(x)\right\|_X &= \left\|\frac{f(2^{n+1}x)}{2^{n+1}} \pm \frac{f(2^n x)}{2^n} - f(x)\right\|_X \qquad (2.8)\\ \leq\ & max\left\{\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\right\|_X , \left\|\frac{f(2^n x)}{2^n} - f(x)\right\|_X\right\}\\ \leq\ & \frac{1}{|2|}max\left\{\frac{\varphi(2^n x, 2^n x)}{|2|^n}, max\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k}; 0 \leq k < n\right\}\right\}\\ =\ & \frac{1}{|2|}max\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k}; 0 \leq k < n+1\right\}.\end{aligned}$$

So for all $n \in \mathbb{N}$ and all $x \in G$, (2.7) holds. By taking n to approach infinity in (2.8), one obtains (2.4).

If L is another mapping satisfies (2.4) , then for $x \in G$, we get

$$\begin{aligned}\|A(x) - L(x)\|_X &= \lim_{k\to\infty}\left\|\frac{A(2^k x)}{2^k} - \frac{L(2^k x)}{2^k}\right\|_X\\ &= \lim_{k\to\infty}\left\|\frac{A(2^k x)}{2^k} \pm \frac{f(2^k x)}{2^k} - \frac{L(2^k x)}{2^k}\right\|_X\\ &\leq \lim_{k\to\infty} max\left\{\left\|\frac{A(2^k x) - f(2^k x)}{2^k}\right\|_X , \left\|\frac{f(2^k x) - L(2^k x)}{2^k}\right\|_X\right\}\\ &\leq \lim_{j\to\infty}\lim_{n\to\infty} max\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k}; j \leq k < n+j\right\}\\ &= 0\end{aligned}$$

Therefore $A = L$. This completes the proof. □

Corollary 2.1. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\lambda(t), \qquad \xi(|2|) < |2|$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ is a mapping satisfying the inequality

$$\left\|f(f(x)-f(y))-f(x+y)-f(x-y)+f(x)+f(y)\right\| \leq \delta\left(\xi(||x||) + \xi(||y||)\right) \quad (2.9)$$

for all $x, y \in G$. Then the limit $A(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ exists for all $x \in G$ and $A : G \to X$ is a unique additive mapping such that

$$||f(x) - A(x)|| \leq \frac{2\delta\xi(||x||)}{|2|}$$

for all $x \in G$.

Proof. Defining $\varphi : G^2 \to [0, \infty)$ by $\varphi(x, y) := \delta\left(\xi(||x||) + \xi(||y||)\right)$. Since $\frac{\xi(|2|)}{|2|} < 1$, we have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} \leq \lim_{n\to\infty} \left(\frac{\xi(|2|)}{|2|}\right)^n \varphi(x, y) = 0$$

for all $x, y \in G$. Also for all $x \in G$

$$\Phi(x) = Sup_{k\geq 0}\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k}; k \in \mathbb{N} \cup \{0\}\right\} = \varphi(x, x) = 2\delta\xi(||x||)$$

exists for all $x \in G$. On the other hand

$$\lim_{j\to\infty} \lim_{n\to\infty} max\left\{\frac{\varphi(2^k x, 2^k x)}{|2|^k}; j \leq k < n + j\right\} = \lim_{j\to\infty} \frac{\varphi(2^j x, 2^j x)}{|2|^j} = 0.$$

Applying Theorem 2.1, then we get the desired result. □

Theorem 2.2. *Let G is an additive semigroup and X is a complete non-Archimedean space. Assume that $\varphi : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{2.10}$$

for all $x, y \in G$. Let for each $x \in G$

$$\Phi(x) = Sup_{k\geq 0}\left\{|2|^k \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right);\ k \in \mathbb{N} \cup \{0\}\right\} \tag{2.11}$$

exists. Suppose that $f : G \to X$ is a mapping satisfying the inequality (2.3). Then the limit

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exist for all $x \in G$ and $A : G \to X$ is an additive mapping satisfying

$$||f(x) - A(x)|| \leq \Phi(x) \tag{2.12}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} max\left\{|2|^k \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right); j \leq k < n + j\right\} = 0 \tag{2.13}$$

Then A is the unique mapping satisfying (2.12).

Proof. By (**??**), we have

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_X \leq \zeta\left(\frac{x}{2}, \frac{x}{2}\right) \tag{2.14}$$

for all $x \in G$. Replacing x by $\frac{x}{2^n}$ in (2.14) , we get

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\|_X \leq |2|^n \varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right) \tag{2.15}$$

It follows from (2.10) and (2.15) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, so $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is convergent. It follows from (2.15) that

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^p f\left(\frac{x}{2^p}\right)\right\|_X = \left\|\sum_{k=p}^{n-1} 2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right)\right\|_X \tag{2.16}$$

$$\leq max\left\{\left\|2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right)\right\|_X ;\ p \leq k < n-1\right\}$$

$$\leq max\left\{|2|^{k+1} \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right);\ p \leq k < n-1\right\}$$

for all $x \in G$ all non-negative integer n, p with $n > p \geq 0$. Letting $p = 0$ and passing the limit $n \to \infty$ in the last inequality, we obtain (2.12). The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.2. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|^{-1}t) \leq \xi(|2|^{-1})\lambda(t), \quad \xi(|2|^{-1}) < |2|^{-1}$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ is a mapping satisfying the inequality (2.9). Then the limit $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in G$ and $A : G \to X$ is a unique additive mapping such that

$$||f(x) - A(x)|| \leq \frac{2\xi(\|x\|)}{|2|} \tag{2.17}$$

for all $x \in G$.

Proof. Defining $\varphi : G^2 \to [0, \infty)$ by $\varphi(x, y) := \delta\left(\xi(||x||) + \xi(||y||)\right)$. Proceeding as in the proof of the Corollary 2.1, we have $\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0$ for all $x, y \in G$. Also for all $x \in G$

$$\Phi(x) = Sup_{k\geq 0}\left\{|2|^k \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right);\ k \in \mathbb{N} \cup \{0\}\right\} = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) = \frac{2\xi(\|x\|)}{|2|}$$

exists for all $x \in G$. Applying Theorem 2.2, then we get the desired result. □

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