International Journal of Pure and Applied Mathematics
Volume 95 No. 1 2014, 99-103
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v95i1.11

ijpam.eu

NEGATIVE BINOMIAL APPROXIMATION TO THE GENERALIZED HYPERGEOMETRIC DISTRIBUTION

K. Teerapabolarn

Department of Mathematics
Faculty of Science
Burapha University
Chonburi, 20131, THAILAND

Abstract: This paper uses Stein's method and the w-function associated with the generalized hypergeometric random variable to determine a bound for the total variation distance between the generalized hypergeometric distribution with parameters α, β and N and the negative binomial distribution with parameters $r = \beta + 1$ and $p = 1 - q = \frac{\alpha+\beta+2}{\alpha+\beta+N+1}$. In view of this bound, it is observed that the desired result gives a good negative binomial approximation when α is large.

AMS Subject Classification: 62E17, 60F05
Key Words: generalized hypergeometric distribution, negative binomial approximation, total variation distance, Stein's method

1. Introduction

A non-negative integer-valued random variable X is said to have the generalized hypergeometric distribution with parameters α, β and N, $\mathbb{GH}_{\alpha,\beta,N}$, if its probability mass function is as follows [3]:

$$p_X(x) = \binom{N-1}{x} \frac{\Gamma(N+\alpha-x)\Gamma(\beta+1+x)\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)},\ x = 0, ..., N-1,$$

where $N \in \mathbb{N} \setminus \{1\}, \alpha \geq 0$ and $\beta > -1$ and the mean and variance of X are

Received: April 23, 2014

$\mu = \frac{(N-1)(\beta+1)}{\alpha+\beta+2}$ and $\sigma^2 = \frac{(N-1)(\beta+1)(\alpha+1)(\alpha+\beta+N+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}$, respectively. We know that this distribution can be approximated by some appropriate discrete distributions if some conditions of their parameters are satisfied. In this case, Crosu [3] used Stein's method and the w-function associated with the generalized hypergeometric random variable to obtain a bound for the total variation distance between $\mathbb{GH}_{\alpha,\beta,N}$ and a Poisson distribution, $\mathbb{P}_\mu$, with mean $\mu = \frac{(N-1)(\beta+1)}{\alpha+\beta+2}$ when $\beta + 2 \geq N$ as follows:

$$\begin{aligned} d_{TV}(\mathbb{GH}_{\alpha,\beta,N}, \mathbb{P}_\mu) &= \sup_{A \subseteq \mathbb{N}\cup\{0\}} |\mathbb{GH}_{\alpha,\beta,N}\{A\} - \mathbb{P}_\mu\{A\}| \\ &\leq (1 - e^{-\mu}) \frac{(\alpha+1)(\beta+3-N) + (\beta+1)(\beta+2)}{(\alpha+\beta+2)(\alpha+\beta+3)}. \end{aligned} \tag{1}$$

Later, Teerapabolarn [4] used the same tools to obtain a bound for the total variation distance between $\mathbb{GH}_{\alpha,\beta,N}$ and a binomial distribution, $\mathbb{B}_{n,p}$, with parameters $n = N - 1$ and $p = \frac{\beta+1}{\alpha+\beta+2}$ as follows:

$$\begin{aligned} d_{TV}(\mathbb{GH}_{\alpha,\beta,N}, \mathbb{B}_{n,p}) &= \sup_{A \subseteq \{0,\dots,n\}} |\mathbb{GH}_{\alpha,\beta,N}\{A\} - \mathbb{B}_{n,p}\{A\}| \\ &\leq \left(1 - p^{n+1} - q^{n+1}\right) \frac{(N-1)(N-2)}{N(\alpha+\beta+3)}. \end{aligned} \tag{2}$$

In this paper, we are interested to determine a bound for $d_{TV}(\mathbb{GH}_{\alpha,\beta,N}, \mathbb{NB}_{r,p})$, where $\mathbb{NB}_{r,p}$ is a negative binomial distribution with parameters r and p.

2. Method

The tools for giving the desired result consist of Stein's method for the negative binomial distribution and the w-function associated with the generalized hypergeometric random variable. Following [1], Stein's equation for negative binomial distribution with parameters $r > 0$ and $p = 1 - q \in (0, 1)$ is, for given h, of the form

$$h(x) - \mathcal{NB}_{r,p}(h) = q(r + x)f(x + 1) - xf(x), \tag{3}$$

where $\mathcal{NB}_{r,p}(h) = \sum_{k=0}^{\infty} h(k)\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k$ and f and h are bounded real-valued functions defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let $h_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{4}$$

Let $f_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ satisfy (3), and let $x \in \mathbb{N}$ and $\Delta f_A(x) = f_A(x+1) - f_A(x)$. Brown and Phillips [1] showed that

$$\sup_A |\Delta f_A(x)| \le \frac{1-p^r}{rq}. \tag{5}$$

For the w-function associated with the generalized hypergeometric random variable, following [2], if $E|w(X)\Delta f_A(X)| < \infty$ then

$$Cov(X, f_A(X)) = \sigma^2 E[w(X)\Delta f_A(X)] \tag{6}$$

and we have $E[w(X)] = 1$. The following lemma presents this w-function, which obtained from [3].

Lemma 2.1. *Let $w(X)$ be the w-function associated with the generalized hypergeometric random variable X. Then, we have the following:*

$$w(x) = \frac{(\beta+x+1)(N-x-1)}{(\alpha+\beta+2)\sigma^2}\ , \ x = 0, ..., N-1. \tag{7}$$

3. Result

The following theorem presents a bound for the total variation distance between $\mathbb{GH}_{\alpha,\beta,N}$ and $\mathbb{NB}_{r,p}$.

Theorem 3.1. *Let $r = \beta + 1$ and $q = 1 - p = \frac{N-1}{\alpha+\beta+N+1}$. Then we have*

$$d_{TV}(\mathbb{GH}_{\alpha,\beta,N}, \mathbb{NB}_{r,p}) \le (1-p^r)\frac{(\beta+1)(\alpha+\beta+N+1)}{(\alpha+\beta+2)(\alpha+\beta+3)}. \tag{8}$$

Proof. For $A \subseteq \mathbb{N} \cup \{0\}$, substituting h, x by h_A, X respectively and taking expectation in (3), we obtain

$$\mathbb{GH}_{\alpha,\beta,N}\{A\} - \mathbb{NB}_{r,p}\{A\} = E[q(r+X)f(X+1) - Xf(X)], \tag{9}$$

where $f = f_A$ is defined as mentioned above.

Let $\delta(\mathbb{G}, \mathbb{N}) = \mathbb{GH}_{\alpha,\beta,N}\{A\} - \mathbb{NB}_{r,p}\{A\}$, then we obtain

$$\begin{aligned}\delta(\mathbb{G}, \mathbb{N}) &= E[rqf(X+1) + qX\Delta f(X) - pXf(X)] \\ &= rqE[f(X+1)] + qE[X\Delta f(X)] - pE[Xf(X)]\end{aligned}$$

$$= rqE[f(X+1)] + qE[X\Delta f(X)] - p\{E[(X-\mu)f(X)] + \mu E[f(X)]\}$$
$$= rqE[\Delta f(X)] + qE[X\Delta f(X)] - pCov(X, f(X)).$$

Using Lemma 2.1 and (6), we have

$$E|w(X)\Delta f(X)| \leq \frac{1-p^r}{rq} E[w(X)] < \infty.$$

From which it follows that

$$\begin{aligned}
\delta(\mathbb{G},\mathbb{N}) &= rqE[\Delta f(X)] + qE[X\Delta f(X)] - pE[\sigma^2 w(X)\Delta f(X)] \\
&= E\{[(r+X)q - \sigma^2 w(X)p]\Delta f(X)\} \\
&= E\left\{\left[\frac{(\beta+X+1)(N-1)}{\alpha+\beta+N+1} - \frac{(\beta+X+1)(N-1-X)}{\alpha+\beta+N+1}\right]\Delta f(X)\right\} \\
&= E\left\{\left[\frac{(\beta+X+1)X}{\alpha+\beta+N+1}\right]\Delta f(X)\right\}.
\end{aligned}$$

Therefore, it follows from (9) and (2), we obtain

$$\begin{aligned}
d_{TV}(\mathbb{GH}_{\alpha,\beta,N}, \mathbb{NB}_{r,p}) &\leq E\{|(r+X)q - \sigma^2 w(X)p||\Delta f(X)|\} \\
&\leq \frac{1-p^r}{rq} E|(r+X)q - \sigma^2 w(X)p| \\
&= \frac{1-p^r}{rq} E[(r+X)q - \sigma^2 w(X)p] \\
&= \frac{1-p^r}{rq} (\mu - \sigma^2 p) \\
&= (1-p^r)\frac{(\beta+1)(\alpha+\beta+N+1)}{(\alpha+\beta+2)(\alpha+\beta+3)},
\end{aligned}$$

which completes the proof. □

If $\beta = 0$, then $r = 1$ and $p = \frac{\alpha+2}{\alpha+N+1}$. Thus the approximation in Theorem 3.1 is an approximation of the generalized hypergeometric distribution with parameters α and N by a geometric distribution, $\mathbb{G}_p$, with parameter $p = \frac{\alpha+2}{\alpha+N+1}$.

Corollary 3.1. *If $\beta = 0$, then we have the following geometric approximation:*

$$d_{TV}(\mathbb{GH}_{\alpha,N}, \mathbb{G}_p) \leq \frac{N-1}{(\alpha+2)(\alpha+3)}. \tag{10}$$

4. Concluding Remarks

In the present study, a bound for the total variation distance between the generalized hypergeometric and negative binomial distributions is obtained by using Stein's method and the w-function associated with the generalized hypergeometric random variable. In view of this bound, it is observed that if $\frac{\beta}{\alpha}$ and $\frac{N}{\alpha}$ are small, or α is large, then the result in Theorem 3.1 gives a good negative binomial approximation, that is, the binomial distribution with parameters $r = \beta + 1$ and $p = \frac{\alpha+\beta+2}{\alpha+\beta+N+1}$ can be used as an approximation of the generalized hypergeometric distribution with parameters α, β and N when α is sufficiently large. Moreover, it has no any conditions on their parameters in this approximation, which is similar to the binomial approximation in (2) and is different from the Poisson approximation as mentioned in (1).

References

[1] T.C. Brown, M.J. Phillips, Negative binomial approximation with Stein's method, *Meth. Comp. Appl. Probab.*, **1** (1999), 407–421.

[2] T. Cacoullos, V. Papathanasion, Characterization of distributions by variance bounds, *Statist. Probab. Lett.*, **7** (1989), 351–356.

[3] C. Crosu, Some application of Chernoff bounds, *Scientific Journal of The Technical University of Civil Engineering*, **6** (2010), 66–71.

[4] K. Teerapabolarn, Binomial approximation to the generalized hypergeometric distribution, *Int. J. Pure Appl. Math.*, **83** (2013), 559–563.