International Journal of Pure and Applied Mathematics
Volume 95 No. 1 2014, 105-111
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v95i1.12

ijpam.eu

CHARACTERIZATIONS OF FUZZY α-CONNECTEDNESS IN FUZZY TOPOLOGICAL SPACES

R.V.M. Rangarajan[1], M. Chandrasekharan[2], A. Vadivel[3 §], M. Palanisamy[4]

[1]Department of Mathematics
K.S.R. College of Engineering
Tiruchengode, INDIA

[2]Department of Mathematics
Erode Arts and Science College
Erode, 638009, INDIA

[3]Mathematics Section (FEAT)
Annamalai University
Annamalainagar, Tamil Nadu, 608 002, INDIA

[4]Department of Mathematics
Annamalai University
Annamalainagar, 608 002, INDIA

Abstract: In this paper we study some stronger forms of fuzzy α-connectedness such as fuzzy super α-connectedness and fuzzy strongly α-connectedness are introduced and we proved that locally fuzzy α-connectedness is a good extension of locally α-connectedness also we get some additional results and properties for these spaces.

AMS Subject Classification: 54A40
Key Words: fuzzy α-connected space, fuzzy strong α-connected, fuzzy super α-connected

1. Introduction

After Zadeh [7] introduced the concept of a fuzzy subset, Chang [4] used it to

Received: May 7, 2014

§Correspondence author

define fuzzy topological space. There after, several concepts of general toplogy have been extended to fuzzy topology and compactness is one such concept. The concept of α-open set was introduced and studied by Njasted [6] and further this concept in fuzzy setting was defined by Bin Shahna [3] with the introduction of fuzzy α-open sets. Lowen also defined an extension of a connectedness in a restricted family of fuzzy topologies. Fatteh and Bassam studied further the notion of fuzzy super connected and fuzzy strongly connected spaces. However they defined connectedness only for a crisp set of a fuzzy topological space. In this paper we give more results on these spaces and prove that locally fuzzy α-connectedness is a good extension of locally α-connectedness. Also, we investigate some more properties of this type of connectedness.

2. Preliminaries

Throughout this paper X and Y mean fuzzy topological spaces (fts, for short). The notations $Cl(A)$, $Int(A)$ and $\overline{A}$ will stand respectively for the fuzzy closure, fuzzy interior and complement of a fuzzy set A in a fts X. The support of a fuzzy set A in X will be denoted by $S(A)$ i. e $S(A) = \{x \in X : A(x) \neq 0\}$. A fuzzy point x_t in X is a fuzzy set having support $x \in X$ and value $t \in (0, 1]$. The fuzzy sets in X taking on respectively the constant value 0 and 1 are denoted by 0_X and 1_X respectively. For two fuzzy sets A and B in X, we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$.

The following definitions have been used to obtain the results and properties developed in this paper.

Definition 2.1. [2] A fuzzy set λ in a fts (X, T) is called a fuzzy α-open if $\lambda \leq int(cl(int(\lambda)))$ and a fuzzy α-closed set if $cl(int(cl(\lambda))) \leq \lambda$.

Definition 2.2. [1] A fuzzy topological space X is said to be fuzzy α-connected if it has no proper fuzzy α-clopen set. (A fuzzy set λ in X is proper if $\lambda \neq 0$ and $\lambda \neq 1$).

Definition 2.3. [2] Let λ be a fuzzy set in a fts X. Then its $f\alpha$-closure and $f\alpha$-interior are denoted and defined by (i) $\alpha cl(\lambda) = \bigwedge\{\mu : \mu$ is a fuzzy α-closed set of X and $\mu \geq \alpha\}$ (ii) $\alpha int(\lambda) = \bigvee\{\gamma : \gamma$ is a fuzzy α-open set of X and $\lambda \geq \gamma\}$.

Definition 2.4. [1] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy regular α-open set if $\lambda = \alpha int(\alpha cl(\lambda))$.

Definition 2.5. [1] A fuzzy topological space (X, T) is said to be fuzzy super α-connected if there is no proper fuzzy regular α-open set.

Definition 2.6. [1] A fuzzy topological space X is said to be fuzzy strongly α-connected if it has no non- zero fuzzy α-closed sets λ and μ such that $\lambda+\mu \leq 1$.

3. Fuzzy α-Connectedness and its Stronger Forms

In this section we study some stronger forms of fuzzy α-connectedness such as fuzzy super α-connectedness and fuzzy strongly α-connectedness are introduced and we proved that locally fuzzy α-connectedness is a good extension of locally α-connectedness also we get some additional results and properties for these spaces.

Definition 3.1. A fuzzy topological space (X, T) is said to be fuzzy locally α-connected at a fuzzy point x_α in X if for every fuzzy α-open set μ in X containing x_α, there exists a connected fuzzy α-open set δ in X such that $x_\alpha \leq \delta \leq \mu$.

Definition 3.2. A fuzzy topological space (X, T) is said to be locally fuzzy super α-connected (locally fuzzy strong α-connected) at a fuzzy point x_α in X if for every fuzzy α-open set μ in X containing x_α there exist a fuzzy super α-connected (fuzzy strong α-connected) open set η in X such that $x_\alpha \leq \eta \leq \mu$.

Definition 3.3. A fuzzy quasi-α-component of a fuzzy point x_α in a fuzzy topological space (X, T) is the smallest fuzzy α-clopen subset of X containing x_α. We denote it by Q.

Definition 3.4. A fuzzy path α-component of a fuzzy point x_α in a fuzzy topological space (X, T) is the maximal fuzzy path α-connected in (X, T) containing x_α. We denote it by C.

Theorem 3.1. A fuzzy topological space $(X,\ \tau)$ is fuzzy locally α-connected if and only if $(X,\ \omega(\tau))$ is fuzzy locally α-connected (where $\omega(\tau)$ is set of all fuzzy lower semi-continuous functions from $(X,\ \tau)$ to the unit interval $I = [0, 1]$)

Proof. Let μ be a fuzzy α-open set in $\omega(\tau)$ containing a fuzzy point x_α. Since μ is fuzzy lower semicontinuous function, then by fuzzy local α-connectedness of $(X,\ \tau)$ there exists a fuzzy α-open connected set U in X containing x and contained in the support of μ. i.e ($x \in U \subset Supp\ \mu$). Now χ_U is the characteristic function of U and it is fuzzy lower semicontinuous, then $\chi_U \vee \mu$ is fuzzy α-open set in $\omega(\tau)$. We claim $\delta = \chi_U \wedge \mu$ is fuzzy α-connected set containing x_α, if not then by [[5], Theorem (3.1)], there exists a non zero fuzzy lower semicontinuous functions μ_1, μ_2 in $\omega(\tau)$ such that

$$\mu_1|_\delta + \mu_2|_\delta = 1.$$

Now $Supp\ \delta = U$ and $Supp\ \mu_1$, $Supp\ \mu_2$ are α-open sets in τ such that

$$U \subset Supp\ \mu_1 \cup Supp\ \mu_2,$$

then,

$$U \cap Supp\ \mu_1 \neq \phi$$

and

$$U \cap Supp\ \mu_2 \neq \phi$$

then

$(U \cap Supp\ \mu_1) \cup (U \cap Supp\ \mu_2) = U \cap (Supp\ \mu_1 \cup Supp\ \mu_2) = U$ is not fuzzy α-connected. Conversely, let U be a fuzzy α-open set containing x, $x_\alpha \in \chi_U$, (χ_U is the characteristic function of U), χ_U is fuzzy open set in $\omega(\tau)$. By fuzzy α-connectedness of $(X, \omega(\tau))$ there exists a fuzzy α-open connected set μ in $\omega(\tau)$ such that

$$x_\alpha \leq \mu \leq \chi_U$$

We claim that $Supp\ \mu$ is fuzzy α-connected ($x \in Supp\ \mu \subset U$), if not there exists two non empty α-open sets G_1, G_2 such that

$$Supp\ \mu = G_1 \cup G_2 \text{ and } G_1 \cap G_2 = \phi.$$

It is clear that

$$\chi_{G_1} + \chi_{G_2} = 1_\mu,$$

which is a contradiction, because μ is fuzzy α-connected. □

Theorem 3.2. If G is a subset of a fuzzy topological space (X, T) such that μ_G (μ_G is the characteristic function of a subset G of X) is fuzzy open in X, then if X is fuzzy super α- connected space implies G is fuzzy super α-connected space.

Proof. Suppose that G is not fuzzy super α-connected space then by [[1] Proposition 2.21 (4)], exists fuzzy α-open sets λ_1, λ_2 in X such that

$$\lambda_1|_G \neq 0,\ \lambda_2|_G \neq 0$$

and

$$\lambda_1|_G + \lambda_2|_G \leq 1,$$
$$\lambda_1 \wedge \mu_G + \lambda_2 \wedge \mu_G \leq 1.$$

Then X is not fuzzy super α-connected space and we get contradiction. □

Theorem 3.3. If A and B are fuzzy strong α-connected subsets of a fuzzy topological space (X, T) and $\overline{\mu_B}|_A \neq 0$ or $\overline{\mu_A}|_B \neq 0$, then $A \vee B$ is a fuzzy strong α connected subset of X where μ_A, μ_B are the characteristic function of a subset A and B respectively.

Proof. Suppose $Y = A \vee B$ is not fuzzy strong α-connected subset of X. Then there exist fuzzy α-closed sets δ and λ such that $\delta|_Y \neq 0$ and $\lambda|_Y \neq 0$ and $\delta|_Y + \lambda|_Y \leq 1$. Since A is fuzzy strong α-connected subset of X, then either

$\delta|_A = 0$ or $\lambda|_A = 0$. Without loss the generality assume that $\delta|_A = 0$. In this case since B is also fuzzy strong α-connected, we have

$$\delta|_A = 0,\ \lambda|_A \neq 0,\ \delta|_B \neq 0,\ \lambda|_B = 0$$

and therefore

$$\lambda|_A + \overline{\mu_B}|_A \leq 1. \tag{1}$$

If $\overline{\mu_B}|_A \neq 0$, then $\lambda|_A \neq 0$ with (1) imply that A is not fuzzy α-connected subset of X. In the same way if $\overline{\mu_A}|_B \neq 0$ then $\delta|_B \neq 0$ and $\lambda|_B + \overline{\mu_A}|_B \leq 0$ imply that B is not a fuzzy strong α-connected subset of X, we get a contradiction. □

Theorem 3.4. If A and B are subsets of a fuzzy topological space (X, T) and $\mu_A \leq \mu_B \leq \overline{\mu}_A$, if A is fuzzy strong α-connected subset of X then B is also a fuzzy strong α-connected.

Proof. Let B is not fuzzy strong α-connected, then there exist two non zero fuzzy α-closed sets $f|_B$ and $k|_B$ such that

$$f|_B + k|_B \leq 1 \tag{1}$$

If $f|_A = 0$ then $f + \mu_A \leq 1$ and this implies

$$f + \mu_A \leq f + \mu_B \leq f + \overline{\mu_A} \tag{2}$$

then $f + \mu_B \leq 1$, thus $f|_B = 0$, a contradiction, and therefore $f|_A \neq 0$, similarly we can show that $k|_A \neq 0$. By (1) with the relation $\mu|_A \leq \mu_B$ imply

$$f|_A + k|_A \leq 1$$

so A is not fuzzy strong α-connected which is contradiction also. □

Theorem 3.5. A fuzzy topological space $(X,\ T)$ is locally fuzzy α-connected iff every fuzzy open subspace of X is fuzzy locally α-connected.

Proof. Let A be a fuzzy open subspace of X and let η be a fuzzy α-open set in X. To prove A is fuzzy α-connected, let x^a_α be a fuzzy point in A and let $\eta|_A$ be a fuzzy α-open set in A containing x_α, it must prove that there exist a α-connected fuzzy open set $\mu|_A$ in A such that

$$x^a_\alpha \leq \mu|_A \leq \eta|_A.$$

Clearly, the fuzzy point x_α in X lies in η. Since X is locally fuzzy x_α-connected, then there exists an open fuzzy α-connected μ such that

$$x_\alpha \leq \mu \leq \eta \text{ and } \mu \leq \eta \wedge \chi_A.$$

It is easy to prove that

$$x^a_\alpha \leq \mu|_A \leq \eta|_A.$$

If $\mu|_A$ is not fuzzy α-connected, there exist a proper fuzzy α-clopen $\lambda|_A$ in $\mu|_A$ (λ is proper fuzzy α-clopen in μ). This is a contradiction with the fact that μ is fuzzy α-connected and hence A is fuzzy α-connected. □

In the same way we can prove an analogue of Theorem 3.5 provided (X, T) is a fuzzy strong α-connected or a fuzzy super α-connected space.

Theorem 3.6. Let X be a fuzzy locally super α-connected and Y be a fuzzy topological space, let F be a fuzzy continuous from X onto Y, then Y is fuzzy locally super α-connected.

Proof. Let y_λ be a fuzzy point of Y. To prove Y is locally fuzzy super α-connected to show that for every fuzzy open set μ in Y containing y_λ ($y_\lambda \leq \mu$) there exist a super α-connected fuzzy open set η such that $y_\lambda \leq \eta \leq \mu$. Let $F : X \to Y$ be fuzzy continuous, then there exist a fuzzy point x_δ of X such that $F(x_\delta) = y_\lambda$, $F^{-1}(\mu)$ is fuzzy α-open set in X then

$$F^{-1}(\mu)(x_\delta) = \mu(F(x_\delta)) = \mu(y_\lambda),\ F(x_\delta) \leq \mu$$

and thus $x_\delta \leq F^{-1}(\mu)$. Since X is locally fuzzy super α-connected there exist a fuzzy super α-connected open set η such that

$$x_\delta \leq \eta \leq F^{-1}(\mu),$$

then

$$F(x_\delta) \leq F(\eta) \leq \mu$$

and then $F(\mu)$ is fuzzy super α-connected. □

In the same way we can prove an analogue of Theorem 3.6 the case for locally fuzzy strong α-connected space.

Definition 3.5. Let A be a subspace of a fuzzy topological space (X, T) and let $\{u_s\}_{s\in S}$ be a family of fuzzy α-open subsets of X such that $A \leq \bigvee_{s\in S} u_s$. If A is fuzzy α-compact then there exist a finite set $\{s_1, s_2, ..., s_k\}$ such that $A \leq \bigvee_{i=1}^{k} u_{s_i}$.

Theorem 3.7. In a fuzzy topological space (X, T) a fuzzy path-α-component C is smaller than the fuzzy quasi-α-component Q for every point x_1.

Proof. Let x_1 be a fuzzy point in (X, T), suppose $C \nleq Q$, let μ be any fuzzy α-clopen subset of X contain x_1, let us consider $C \wedge \mu$ and $C - \mu$. It is clear that $c \wedge \mu \neq 0$,

$$(C - \mu)(x) = \begin{cases} C(x), & \text{if } C(x) > \mu(x), \\ 0, & \text{otherwise.} \end{cases}$$

If $C - \mu = C$ this mean

$$(C \wedge \mu)(x) + (C - \mu)(x) = 1_c,$$

a contradiction, it must be $C - \mu = 0$, then $C \leq \mu$ since μ is arbitrary, thus $C \leq Q$. □

Lemma 3.1. Let μ be a fuzzy α-open subset of a topological space (X, T). If a family $\{F_s\}_{s\in S}$ of closed subset of X contains at least one fuzzy α-compact set, in particular if X is fuzzy α-compact and if $\bigwedge_{s\in S} F_s < \mu$, there exists a finite set $\{s_1, s_2, ..., s_k\}$ such that $\bigwedge_{i=1}^{k} F_{s_i} < \mu$.

Proof. let μ be a fuzzy α-open set, then $1-\mu = \mu^c$ is fuzzy α-closed and

$$(\bigwedge_{s\in S} F_s < \mu)^c = \bigvee_{s\in S} F_s^c > \mu^c = 1-\mu$$

which is fuzzy α-compact [every fuzzy α-closed subset of fuzzy α-compact set is fuzzy α-compact]. Then we have

$$1-\mu < \bigvee_{s\in S} F_s^c.$$

Therefore

$$1-\mu < \bigvee_{i=1}^{k} F_{s_i}^c$$

and then

$$\bigwedge_{i=1}^{k} F_{s_i} < \mu.$$

□

References

[1] G. Balasubramanian and V. Chandrasekar, Fuzzy α-connectedness and fuzzy α-disconnectedness in fuzzy topological spaces, Matematnukn Bechnk, 56 (2004), 47-56.

[2] A. S. Bin Shahna, On fuzzy strong semi continuity and fuzzy pre-continuity, Fuzzy sets and systems, 44 (1991), 303-308.

[3] A. S. Bin Shahna, Fuzzy connectedness and weaker forms of fuzzy continuity, Bull. Cal. Math. Soc., 86 (1994), 271-274.

[4] C. L. Chang, Fuzzy Topological Spaces, J. Math. Anal. Appl. 24, (1968) 182-190.

[5] R. Lowen, Connectedness in fuzzy topological spaces, Rocky Mt. J. Math., 11(1981), 427-433.

[6] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (3) (1965), 961-970.

[7] L. A. Zadeh, Fuzzy sets, Inform and control, 8, (1965) 338-353.