CHARACTERIZATIONS OF FUZZY $\alpha$-CONNECTEDNESS IN FUZZY TOPOLOGICAL SPACES

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Abstract: In this paper we study some stronger forms of fuzzy $\alpha$-connectedness such as fuzzy super $\alpha$-connectedness and fuzzy strongly $\alpha$-connectedness are introduced and we proved that locally fuzzy $\alpha$-connectedness is a good extension of locally $\alpha$-connectedness also we get some additional results and properties for these spaces.

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1. Introduction

After Zadeh [7] introduced the concept of a fuzzy subset, Chang [4] used it to...
define fuzzy topological space. There after, several concepts of general topology have been extended to fuzzy topology and compactness is one such concept. The concept of $\alpha$-open set was introduced and studied by Njasted [6] and further this concept in fuzzy setting was defined by Bin Shahna [3] with the introduction of fuzzy $\alpha$-open sets. Lowen also defined an extension of a connectedness in a restricted family of fuzzy topologies. Fattelh and Bassam studied further the notion of fuzzy super connected and fuzzy strongly connected spaces. However they defined connectedness only for a crisp set of a fuzzy topological space. In this paper we give more results on these spaces and prove that locally fuzzy $\alpha$-connectedness is a good extension of locally $\alpha$-connectedness. Also, we investigate some more properties of this type of connectedness.

2. Preliminaries

Throughout this paper $X$ and $Y$ mean fuzzy topological spaces (fts, for short). The notations $Cl(A)$, $Int(A)$ and $\overline{A}$ will stand respectively for the fuzzy closure, fuzzy interior and complement of a fuzzy set $A$ in a fts $X$. The support of a fuzzy set $A$ in $X$ will be denoted by $S(A)$ i.e $S(A) = \{x \in X : A(x) \neq 0\}$. A fuzzy point $x_t$ in $X$ is a fuzzy set having support $x \in X$ and value $t \in (0, 1]$. The fuzzy sets in $X$ taking on respectively the constant value 0 and 1 are denoted by $0_X$ and $1_X$ respectively. For two fuzzy sets $A$ and $B$ in $X$, we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$.

The following definitions have been used to obtain the results and properties developed in this paper.

**Definition 2.1.** [2] A fuzzy set $\lambda$ in a fts $(X, T)$ is called a fuzzy $\alpha$-open if $\lambda \leq int(cl(int(\lambda)))$ and a fuzzy $\alpha$-closed set if $cl(int(cl(\lambda))) \leq \lambda$.

**Definition 2.2.** [1] A fuzzy topological space $X$ is said to be fuzzy $\alpha$-connected if it has no proper fuzzy $\alpha$-clopen set. ( A fuzzy set $\lambda$ in $X$ is proper if $\lambda \neq 0$ and $\lambda \neq 1$).

**Definition 2.3.** [2] Let $\lambda$ be a fuzzy set in a fts $X$. Then its $f_{\alpha}$-closure and $f_{\alpha}$-interior are denoted and defined by (i) $acl(\lambda) = \bigcap \{\mu : \mu$ is a fuzzy $\alpha$-closed set of $X \mbox{ and } \mu \geq \alpha\}$ (ii) $aint(\lambda) = \bigvee \{\gamma : \gamma$ is a fuzzy $\alpha$-open set of $X \mbox{ and } \lambda \geq \gamma\}$.

**Definition 2.4.** [1] A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called fuzzy regular $\alpha$-open set if $\lambda = aint(acl(\lambda))$.

**Definition 2.5.** [1] A fuzzy topological space $(X, T)$ is said to be fuzzy super $\alpha$-connected if there is no proper fuzzy regular $\alpha$-open set.
Definition 2.6. [1] A fuzzy topological space $X$ is said to be fuzzy strongly $\alpha$-connected if it has no non-zero fuzzy $\alpha$-closed sets $\lambda$ and $\mu$ such that $\lambda + \mu \leq 1$.

3. Fuzzy $\alpha$-Connectedness and its Stronger Forms

In this section we study some stronger forms of fuzzy $\alpha$-connectedness such as fuzzy super $\alpha$-connectedness and fuzzy strongly $\alpha$-connectedness are introduced and we proved that locally fuzzy $\alpha$-connectedness is a good extension of locally $\alpha$-connectedness also we get some additional results and properties for these spaces.

Definition 3.1. A fuzzy topological space $(X, T)$ is said to be fuzzy locally $\alpha$-connected at a fuzzy point $x_\alpha$ in $X$ if for every fuzzy $\alpha$-open set $\mu$ in $X$ containing $x_\alpha$, there exists a connected fuzzy $\alpha$-open set $\delta$ in $X$ such that $x_\alpha \leq \delta \leq \mu$.

Definition 3.2. A fuzzy topological space $(X, T)$ is said to be locally fuzzy super $\alpha$-connected ( locally fuzzy strong $\alpha$-connected ) at a fuzzy point $x_\alpha$ in $X$ if for every fuzzy $\alpha$-open set $\mu$ in $X$ containing $x_\alpha$ there exist a fuzzy super $\alpha$-connected ( fuzzy strong $\alpha$-connected) open set $\eta$ in $X$ such that $x_\alpha \leq \eta \leq \mu$.

Definition 3.3. A fuzzy quasi-$\alpha$-component of a fuzzy point $x_\alpha$ in a fuzzy topological space $(X, T)$ is the smallest fuzzy $\alpha$-clopen subset of $X$ containing $x_\alpha$. We denote it by $Q$.

Definition 3.4. A fuzzy path $\alpha$-component of a fuzzy point $x_\alpha$ in a fuzzy topological space $(X, T)$ is the maximal fuzzy path $\alpha$-connected in $(X, T)$ containing $x_\alpha$. We denote it by $C$.

Theorem 3.1. A fuzzy topological space $(X, \tau)$ is fuzzy locally $\alpha$-connected if and only if $(X, \omega(\tau))$ is fuzzy locally $\alpha$-connected (where $\omega(\tau)$ is set of all fuzzy lower semi-continuous functions from $(X, \tau)$ to the unit interval $I = [0, 1]$)

Proof. Let $\mu$ be a fuzzy $\alpha$-open set in $\omega(\tau)$ containing a fuzzy point $x_\alpha$. Since $\mu$ is fuzzy lower semicontinuous function, then by fuzzy local $\alpha$-connectedness of $(X, \tau)$ there exists a fuzzy $\alpha$-open connected set $U$ in $X$ containing $x$ and contained in the support of $\mu$, i.e $(x \in U \subset \text{Supp } \mu)$. Now $\chi_U$ is the characteristic function of $U$ and it is fuzzy lower semicontinuous, then $\chi_U \lor \mu$ is fuzzy $\alpha$-open set in $\omega(\tau)$. We claim $\delta = \chi_U \land \mu$ is fuzzy $\alpha$-connected set containing $x_\alpha$, if not then by [5, Theorem (3.1)], there exists a non zero fuzzy lower semicontinuous functions $\mu_1, \mu_2$ in $\omega(\tau)$ such that
$\mu_1|_\delta + \mu_2|_\delta = 1.$

Now $\text{Supp } \delta = U$ and $\text{Supp } \mu_1, \text{Supp } \mu_2$ are $\alpha$-open sets in $\tau$ such that $U \subset \text{Supp } \mu_1 \cup \text{Supp } \mu_2,$

then,

$U \cap \text{Supp } \mu_1 \neq \phi$

and

$U \cap \text{Supp } \mu_2 \neq \phi$

then

$(U \cap \text{Supp } \mu_1) \cup (U \cap \text{Supp } \mu_2) = U \cap (\text{Supp } \mu_1 \cup \text{Supp } \mu_2) = U$ is not fuzzy $\alpha$-connected. Conversely, let $U$ be a fuzzy $\alpha$-open set containing $x,$ $x_\alpha \in \chi_U,$ ($\chi_U$ is the characteristic function of $U$ ), $\chi_U$ is fuzzy open set in $\omega(\tau)$. By fuzzy $\alpha$-connectedness of $(X, \omega(\tau))$ there exists a fuzzy $\alpha$-open connected set $\mu$ in $\omega(\tau)$ such that

$x_\alpha \leq \mu \leq \chi_U$

We claim that $\text{Supp } \mu$ is fuzzy $\alpha$-connected ($x \in \text{Supp } \mu \subset U$), if not there exists two non empty $\alpha$-open sets $G_1, G_2$ such that

$\text{Supp } \mu = G_1 \cup G_2$ and $G_1 \cap G_2 = \phi.$

It is clear that

$\chi_{G_1} + \chi_{G_2} = 1_\mu,$

which is a contradiction, because $\mu$ is fuzzy $\alpha$-connected.

**Theorem 3.2.** If $G$ is a subset of a fuzzy topological space $(X, T)$ such that $\mu_G$ ($\mu_G$ is the characteristic function of a subset $G$ of $X$) is fuzzy open in $X$, then if $X$ is fuzzy super $\alpha$-connected space implies $G$ is fuzzy super $\alpha$-connected space.

**Proof.** Suppose that $G$ is not fuzzy super $\alpha$-connected space then by [[1] Proposition 2.21 (4)], exists fuzzy $\alpha$-open sets $\lambda_1, \lambda_2$ in $X$ such that

$\lambda_1|_G \neq 0, \lambda_2|_G \neq 0$

and

$\lambda_1|_G + \lambda_2|_G \leq 1,$

$\lambda_1 \land \mu_G + \lambda_2 \land \mu_G \leq 1.$

Then $X$ is not fuzzy super $\alpha$-connected space and we get contradiction. □

**Theorem 3.3.** If $A$ and $B$ are fuzzy strong $\alpha$-connected subsets of a fuzzy topological space $(X, T)$ and $\mu_B|_A \neq 0$ or $\mu_A|_B \neq 0,$ then $A \lor B$ is a fuzzy strong $\alpha$ connected subset of $X$ where $\mu_A, \mu_B$ are the characteristic function of a subset $A$ and $B$ respectively.

**Proof.** Suppose $Y = A \lor B$ is not fuzzy strong $\alpha$-connected subset of $X.$ Then there exist fuzzy $\alpha$-closed sets $\delta$ and $\lambda$ such that $\delta|_Y \neq 0$ and $\lambda|_Y \neq 0$ and $\delta|_Y + \lambda|_Y \leq 1.$ Since $A$ is fuzzy strong $\alpha$-connected subset of $X,$ then either
\[ \delta|_A = 0 \text{ or } \lambda|_A = 0. \] Without loss the generality assume that \( \delta|_A = 0 \). In this case since \( B \) is also fuzzy strong \( \alpha \)-connected, we have
\[ \delta|_A = 0, \lambda|_A \neq 0, \delta|_B \neq 0, \lambda|_B = 0 \]
and therefore
\[ \lambda|_A + \mu|_B|_A \leq 1. \tag{1} \]
If \( \mu|_A \neq 0 \), then \( \lambda|_A \neq 0 \) with (1) imply that \( A \) is not fuzzy \( \alpha \)-connected subset of \( X \). In the same way if \( \delta|_A|_B \neq 0 \) then \( \delta|_B \neq 0 \) and \( \lambda|_B + \mu|_A|_B \leq 0 \) imply that \( B \) is not a fuzzy strong \( \alpha \)-connected subset of \( X \), we get a contradiction. 

\[ \text{Theorem 3.4.} \quad \text{If } A \text{ and } B \text{ are subsets of a fuzzy topological space } (X, T) \text{ and } \mu_A \leq \mu_B \leq \mu_A, \text{ if } A \text{ is fuzzy strong } \alpha \text{-connected subset of } X \text{ then } B \text{ is also a fuzzy strong } \alpha \text{-connected.} \]

Proof. Let \( B \) be not fuzzy strong \( \alpha \)-connected, then there exist two non zero fuzzy \( \alpha \)-closed sets \( f|_B \) and \( k|_B \) such that
\[ f|_B + k|_B \leq 1 \tag{1} \]
If \( f|_A = 0 \) then \( f + \mu_A \leq 1 \) and this implies
\[ f + \mu_A \leq f + \mu_B \leq f + \mu_A \tag{2} \]
then \( f + \mu_B \leq 1 \), thus \( f|_B = 0 \), a contradiction, and therefore \( f|_A \neq 0 \). By (1) with the relation \( \mu|_A \leq \mu_B \) imply
\[ f|_A + k|_A \leq 1 \]
so \( A \) is not fuzzy strong \( \alpha \)-connected which is contradiction also. 

\[ \text{Theorem 3.5.} \quad \text{A fuzzy topological space } (X, T) \text{ is locally fuzzy } \alpha \text{-connected iff every fuzzy open subspace of } X \text{ is fuzzy locally } \alpha \text{-connected.} \]

Proof. Let \( A \) be a fuzzy open subspace of \( X \) and let \( \eta \) be a fuzzy \( \alpha \)-open set in \( X \). To prove \( A \) is fuzzy \( \alpha \)-connected, let \( x^\alpha \) be a fuzzy point in \( A \) and let \( \eta|_A \) be a fuzzy \( \alpha \)-open set in \( A \) containing \( x^\alpha \), it must prove that there exist a \( \alpha \)-connected fuzzy open set \( \mu|_A \) in \( A \) such that
\[ x^\alpha \leq \mu|_A \leq \eta|_A. \]
Clearly, the fuzzy point \( x^\alpha \) in \( X \) lies in \( \eta \). Since \( X \) is locally fuzzy \( x^\alpha \)-connected, then there exists an open fuzzy \( \alpha \)-connected \( \mu \) such that
\[ x^\alpha \leq \mu \leq \eta \text{ and } \mu \leq \eta \wedge \chi_A. \]
It is easy to prove that
\[ x^\alpha \leq \mu|_A \leq \eta|_A. \]
If \( \mu|_A \) is not fuzzy \( \alpha \)-connected, there exist a proper fuzzy \( \alpha \)-clopen \( \lambda|_A \) in \( \mu|_A \) (\( \lambda \) is proper fuzzy \( \alpha \)-clopen in \( \mu \)). This is a contradiction with the fact that \( \mu \) is fuzzy \( \alpha \)-connected and hence \( A \) is fuzzy \( \alpha \)-connected. 

In the same way we can prove an analogue of Theorem 3.5 provided \((X, T)\) is a fuzzy strong \( \alpha \)-connected or a fuzzy super \( \alpha \)-connected space.
Theorem 3.6. Let $X$ be a fuzzy locally super $\alpha$-connected and $Y$ be a fuzzy topological space, let $F$ be a fuzzy continuous from $X$ onto $Y$, then $Y$ is fuzzy locally super $\alpha$-connected.

Proof. Let $y_{\lambda}$ be a fuzzy point of $Y$. To prove $Y$ is locally fuzzy super $\alpha$-connected to show that for every fuzzy open set $\mu$ in $Y$ containing $y_{\lambda}$ ($y_{\lambda} \leq \mu$) there exist a super $\alpha$-connected fuzzy open set $\eta$ such that $y_{\lambda} \leq \eta \leq \mu$. Let $F : X \to Y$ be fuzzy continuous, then there exist a fuzzy point $x_{\delta}$ of $X$ such that $F(x_{\delta}) = y_{\lambda}, F^{-1}(\mu)$ is fuzzy $\alpha$-open set in $X$ then

$$F^{-1}(\mu)(x_{\delta}) = \mu(F(x_{\delta})) = \mu(y_{\lambda}), F(x_{\delta}) \leq \mu$$

and thus $x_{\delta} \leq F^{-1}(\mu)$. Since $X$ is locally fuzzy super $\alpha$-connected there exist a fuzzy super $\alpha$-connected open set $\eta$ such that $x_{\delta} \leq \eta \leq F^{-1}(\mu)$,

then

$$F(x_{\delta}) \leq F(\eta) \leq \mu$$

and then $F(\mu)$ is fuzzy super $\alpha$-connected. 

In the same way we can prove an analogue of Theorem 3.6 the case for locally fuzzy strong $\alpha$-connected space.

Definition 3.5. Let $A$ be a subspace of a fuzzy topological space $(X, T)$ and let $\{u_s\}_{s \in S}$ be a family of fuzzy $\alpha$-open subsets of $X$ such that $A \leq \bigvee_{s \in S} u_s$.

If $A$ is fuzzy $\alpha$-compact then there exist a finite set $\{s_1, s_2, ..., s_k\}$ such that $A \leq \bigvee_{i=1}^{k} u_{s_i}$.

Theorem 3.7. In a fuzzy topological space $(X, T)$ a fuzzy path-$\alpha$-component $C$ is smaller than the fuzzy quasi-$\alpha$-component $Q$ for every point $x_1$.

Proof. Let $x_1$ be a fuzzy point in $(X, T)$, suppose $C \nleq Q$, let $\mu$ be any fuzzy $\alpha$-clopen subset of $X$ contain $x_1$, let us consider $C \land \mu$ and $C - \mu$. It is clear that $c \land \mu \neq 0$,

$$(C - \mu)(x) = \begin{cases} C(x), & \text{if } C(x) > \mu(x), \\ 0, & \text{otherwise.} \end{cases}$$

If $C - \mu = C$ this mean

$$(C \land \mu)(x) + (C - \mu)(x) = 1_c,$$

a contradiction, it must be $C - \mu = 0$, then $C \leq \mu$ since $\mu$ is arbitrary, thus $C \leq Q$. 

Lemma 3.1. Let $\mu$ be a fuzzy $\alpha$-open subset of a topological space $(X, T)$. If a family $\{F_s\}_{s \in S}$ of closed subset of $X$ contains at least one fuzzy $\alpha$-compact
set, in particular if \( X \) is fuzzy \( \alpha \)-compact and if \( \bigwedge_{s \in S} F_s < \mu \), there exists a finite set \( \{s_1, s_2, ..., s_k\} \) such that \( \bigwedge_{i=1}^{k} F_{s_i} < \mu \).

**Proof.** Let \( \mu \) be a fuzzy \( \alpha \)-open set, then \( 1 - \mu = \mu^c \) is fuzzy \( \alpha \)-closed and

\[
(\bigwedge_{s \in S} F_s < \mu)^c = \bigvee_{s \in S} F_s^c > \mu^c = 1 - \mu
\]

which is fuzzy \( \alpha \)-compact [every fuzzy \( \alpha \)-closed subset of fuzzy \( \alpha \)-compact set is fuzzy \( \alpha \)-compact]. Then we have

\[
1 - \mu < \bigvee_{s \in S} F_s^c.
\]

Therefore

\[
1 - \mu < \bigvee_{i=1}^{k} F_{s_i}^c
\]

and then

\[
\bigwedge_{i=1}^{k} F_{s_i} < \mu. \quad \Box
\]

**References**


