FIXED POINT AND APPROXIMATELY
COMPOSITE FUNCTIONAL EQUATIONS IN
NON-ARCHIMEDEAN NORMED SPACES

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Abstract: In this paper, we investigate the generalized Hyers-Ulam-Rassias (or


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1. Introduction and Preliminaries

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation $D$ must be close to an exact solution of $D$?

If the problem accepts a solution, we say that the equation $D$ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940.

We are given a group $G$ and a metric group $G'$ with metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G'$?

Ulam’s problem was partially solved by Hyers [10] in 1941.

Let $E_1$ be a normed space, $E_2$ a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon$$

for all $x, y \in E_1$, where $\varepsilon > 0$ is a constant. Then the limit

$$T(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$ and $T$ is the unique additive mapping satisfying

$$||T(x) - f(x)|| \leq \varepsilon \quad (1.1)$$

for all $x \in E_1$. Also, if for each $x$ the function $t \to f(tx)$ from $\mathbb{R}$ to $E_2$ is continuous on $\mathbb{R}$, then $T$ is linear. If $f$ is continuous at a single point of $E_1$, then $T$ is continuous everywhere in $E_1$. Moreover (1.1) is sharp.

In 1978, Th. M. Rassias [18] formulated and proved the following theorem, which implies Hyers’s Theorem as a special case. Suppose that $E$ and $F$ are real normed spaces with $F$ a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f(tx)$ is continuous on $R$, and let there exist $\varepsilon \geq 0$ and $p \in [0, 1)$ such that for all $x, y \in E$

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p) \quad (1.2)$$

Then there exists a unique linear mapping $T : E \to F$ such that such that for all $x \in E$

$$||f(x) - T(x)|| \leq \frac{\varepsilon||x||^p}{1 - 2p^{-1}}$$
The case of the existence of a unique additive mapping had been obtained by T. Aoki [2], as it is recently noticed by Lech Maligranda. However, Aoki [2] had claimed the existence of a unique linear mapping, that is not true because he did not allow the mapping \( f \) to satisfy some continuity assumption. Th.M. Rassias [18], who independently introduced the unbounded Cauchy difference was the first to prove that there exists a unique linear mapping \( T \) satisfying
\[
||f(x) - T(x)|| \leq \frac{\epsilon||x||^p}{1 - 2^{p-1}} \quad x \in E
\]

In 1990, Th.M. Rassias [19] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for \( p \geq 1 \). In 1991, Z. Gajda [8] following the same approach as in Th. M. Rassias [24], gave an affirmative solution to this question for \( p > 1 \). It was proved by Z. Gajda [8], as well as by Th. M. Rassias and P. Šemrl [20] that one can not prove a Th. M. Rassias type theorem when \( p = 1 \). In 1994, P. Gavruta [9] provided a further generalization of Th. M. Rassias theorem in which he replaced the bound \( \epsilon(||x||^p + ||y||^p) \) by a general control function \( \psi(x,y) \) for the existence of a unique linear mapping.

The functional equation \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [25] for mappings \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. In [7], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians([1]-[5], [11]-[24]).

**Definition 1.1.** By a non-Archimedean field we mean a field \( \mathbb{K} \) equipped with a function(valuation) \( |\cdot| : \mathbb{K} \to [0,\infty) \) such that for all \( r, s \in \mathbb{K} \), the following conditions hold:

\[
\begin{align*}
(i) & \quad |r| = 0 \text{ if and only if } r = 0; \\
(ii) & \quad |rs| = |r||s|; \\
(iii) & \quad |r + s| \leq \max\{|r|, |s|\}.
\end{align*}
\]

**Definition 1.2.** Let \( X \) be a vector space over a scalar field \( \mathbb{K} \) with a non-Archimedean non-trivial valuation \( |\cdot| \). A function \( ||\cdot|| : X \to \mathbb{R} \) is a non-Archimedean norm (valuation) if it satisfies the following conditions:
(i) \(||x|| = 0\) if and only if \(x = 0\);
(ii) \(||rx|| = |r|||x||\) \((r \in \mathbb{K}, x \in X)\);
(iii) The strong triangle inequality (ultrametric); namely
\[||x + y|| \leq \max\{||x||, ||y||\}. \quad x, y \in X\]

Then \((X, ||.||)\) is called a non-Archimedean space.

Due to the fact that
\[||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n - 1\} \quad (n > m)\]

**Definition 1.3.** A sequence \(\{x_n\}\) is Cauchy if and only if \(\{x_{n+1} - x_n\}\) converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

**Definition 1.4.** Let \(X\) be a set. A function \(d : X \times X \to [0, \infty]\) is called a generalized metric on \(X\) if \(d\) satisfies the following conditions:

(a) \(d(x, y) = 0\) if and only if \(x = y\) for all \(x, y \in X\);
(b) \(d(x, y) = d(y, x)\) for all \(x, y \in X\);
(c) \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X\).

**Theorem 1.1.** Let \((X, d)\) be a complete generalized metric space and \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then, for all \(x \in X\), either \(d(J^n x, J^{n+1} x) = \infty\) for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that:

(a) \(d(J^n x, J^{n+1} x) < \infty\) for all \(n_0 \geq n_0\);
(b) the sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J\);
(c) \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X : d(J^{n_0} x, y) < \infty\}\);
(d) \(d(y, y^*) \leq \frac{1}{1-L}d(y, Jy)\) for all \(y \in Y\).

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation
\[f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y) \quad (1.3)\]
in non-Archimedean normed spaces. In the rest of the paper let \(|2| \neq 1\).
2. Non-Archimedean Stability of Eq. (1.3): A Fixed Point Method

Throughout this section, using the fixed point method we prove the generalized Hyers-Ulam stability of the composite functional equation (1.3) in non-Archimedean spaces.

**Theorem 2.1.** Let $X$ be a non-Archimedean normed space and that $Y$ be a complete non-Archimedean normed space. Assume $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with
\[
\zeta(2x, 2y) \leq |2L\zeta(x, y) \tag{2.1}
\]
for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying
\[
\left\| f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y) \right\| \leq \zeta(x, y) \tag{2.2}
\]
for all $x, y \in X$. Then there is a unique additive mapping $A : X \to Y$ such that
\[
\| f(x) - A(x) \| \leq \frac{\zeta(x,x)}{2 - |2L|} \tag{2.3}
\]

**Proof.** Putting $y = x$ in (2.2), we have
\[
\left\| f(2x) - f(x) \right\| \leq \frac{1}{2} \zeta(x,x) \tag{2.4}
\]
for all $x \in X$. Consider the set $S := \{ g : X \to Y \}$ and the generalized metric $d$ in $S$ defined by $d(f,g) = \inf_{\mu \in (0, +\infty)} \{ \| g(x) - h(x) \| \leq \mu \zeta(x,x), \forall x \in X \}$, where $\inf \emptyset = +\infty$. It is easy to show that $(S,d)$ is complete (see [12], Lemma 2.1).

Now, we consider a linear mapping $J : (S,d) \to (S,d)$ such that $Jh(x) := \frac{1}{2} h(2x)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then $\| g(x) - h(x) \| \leq \epsilon \zeta(x,x)$ for all $x \in X$ and so
\[
\| Jg(x) - Jh(x) \| = \left\| \frac{g(2x)}{2} - \frac{h(2x)}{2} \right\| \leq \frac{1}{|2|} \epsilon \zeta(2x, 2x) \leq \frac{1}{|2|} \epsilon |2L| \zeta(x,x)
\]
for all $x \in X$. Thus $d(g,h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g,h)$ for all $g, h \in S$. It follows from (2.4) that $d(f, Jf) \leq \frac{1}{|2|} < +\infty$. By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,
\[
A(2x) = 2A(x) \tag{2.5}
\]
for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set $\Omega = \{ h \in S : d(g, h) < \infty \}$. This implies that $A$ is a unique mapping satisfying (2.5) such that there exists $\mu \in (0, \infty)$ satisfying $\| f(x) - A(x) \| \leq \mu \zeta(x, x)$ for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n \to \infty} \frac{f(2^n x)}{2^n} = A(x)$ for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,J)}{|2| - |2|L}$ with $f \in \Omega$, which implies the inequality $d(f, A) \leq \frac{1}{|2| - |2|L}$. This implies that the inequality (2.3) holds. By (2.1) and (2.2), we obtain

$$\left\| A(A(x) - A(y)) - A(x + y) - A(x - y) + A(x) + A(y) \right\|
\leq \lim_{n \to \infty} \frac{1}{|2|^n} \zeta(2^n x, 2^n y) \leq \lim_{n \to \infty} \frac{1}{|2|^n} \cdot |2|^n \zeta(x, y) = 0$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So $\left\| A(A(x) - A(y)) - A(x + y) - A(x - y) + A(x) + A(y) \right\| = 0$, for all $x, y \in X$. Thus the mapping $A : X \to Y$ satisfying in (1.3).

On the other hand

$$2A \left( \frac{x}{2} \right) - A(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} - \lim_{n \to \infty} \frac{f(2^n x)}{2^n} = 0.$$

So, $A$ is an additive mapping, as desired. This completes the proof. \qed

**Corollary 2.1.** Let $\theta \geq 0$ and $r$ be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying

$$\left\| f(f(x)) - f(x + y) - f(x - y) + f(x) + f(y) \right\| \leq \theta(\|x\|^r + \|y\|^r) \quad (2.6)$$

for all $x, y \in X$. Then the limit $A(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\| f(x) - A(x) \| \leq \frac{2\theta \|x\|^r}{|2| - |2|^{r+1}}$$

for all $x \in X$.

**Proof.** The proof follows from Theorem 2.1 by taking $\zeta(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. In fact, if we choose $L = |2|^r$, then we get the desired result. \qed
**Theorem 2.2.** Let \( X \) be a non-Archimedean normed space and that \( Y \) be a complete non-Archimedean normed space. Assume \( \zeta : X^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with
\[
\zeta\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{|2|} \zeta(x, y)
\]
for all \( x, y \in X \). Let \( f : X \to Y \) be a mapping satisfying (2.2). Then the limit \( A(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \) exists for all \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that
\[
\|f(x) - A(x)\| \leq \frac{L \zeta(x, x)}{|2| - |2|L}
\]

**Proof.** Substituting \( y = x \) in (2.2) and then replacing \( x \) by \( \frac{x}{2} \), we get
\[
\left\| f\left(\frac{x}{2}\right) - 2f\left(\frac{x}{4}\right) \right\| \leq \zeta\left(\frac{x}{2}, \frac{x}{2}\right)
\]
for all \( x \in X \). Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 2.1. Consider a linear mapping \( J : (S, d) \to (S, d) \) such that 
\[
Jh\left(\frac{x}{2}\right) := 2h\left(\frac{x}{2}\right)
\]
for all \( x \in X \). Let \( g, h \in S \) be such that \( d(g, h) = \epsilon \). Then
\[
\|g(x) - h(x)\| \leq \epsilon \zeta(x, x)
\]
for all \( x \in X \). Thus \( d(g, h) = \epsilon \) implies that \( d(Jg, Jh) \leq L\epsilon \). This means that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for all \( g, h \in S \). It follows from (2.8) that \( d(f, Jf) \leq \frac{L}{|2|} \).

By Theorem 1.1, there exists a mapping \( A : X \to Y \) satisfying the following:

1. \( A \) is a fixed point of \( J \), that is,
\[
A\left(\frac{x}{2}\right) = \frac{1}{2} A(x)
\]
for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( J \) in the set \( \Omega = \{ h \in S : d(g, h) < \infty \} \). This implies that \( A \) is a unique mapping satisfying (2.9) such that there exists \( \mu \in (0, \infty) \) satisfying \( \|f(x) - A(x)\| \leq \mu \zeta(x, x) \) for all \( x \in X \).

2. \( d(J^n f, A) \to 0 \) as \( n \to \infty \). This implies the equality \( \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) = A(x) \) for all \( x \in X \).

3. \( d(f, A) \leq \frac{d(f, Jf)}{1 - \frac{L}{|2|}} \) with \( f \in \Omega \), which implies the inequality \( d(f, A) \leq \frac{L}{|2| - |2|L} \). The rest of the proof is similar to the proof of Theorem 2.1. □
**Corollary 2.2.** Let $\theta \geq 0$ and $r$ be a real number with $r > 1$. Let $f : X \to Y$ be a mapping satisfying (2.6). Then the limit $A(x) = \lim_{n \to \infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$
\|f(x) - A(x)\| \leq \frac{2|2|^{-1}\theta\|x\|^r}{|2| - |2|^r}
$$

for all $x \in X$.

**Proof.** The proof follows from Theorem 2.2 by taking $\zeta(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. In fact, if we choose $L = |2|^{-1}$, then we get the desired result. \qed

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