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FIXED POINT AND APPROXIMATELY COMPOSITE FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN NORMED SPACES

Mohammad Hadi Eghtesadifard[1], Dong Yun Shin[2 §], Hassan Azadi Kenary[3], Najmeh Sahami[4]

[1]Instructor of Fars Education Department
Shiraz, IRAN

[2]Department of Mathematics
College of Natural Science
University of Seoul
KOREA

[3,4]Department of Mathematics
Beyza Branch
Islamic Azad University
Beyza, IRAN

Abstract: In this paper, we investigate the generalized Hyers-Ulam-Rassias(or Hyers-Ulam) stability of a composite additive functional equation in non-Archimedean normed spaces.

The concept of Hyers-Ulam-Rassias or Hyers-Ulam stability originated from the Th.M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

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§Correspondence author

1. Introduction and Preliminaries

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?.

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940.

We are given a group G and a metric group G' with metric $d(.,.)$. Given $\varepsilon > 0$, dose there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G$, then a homomorphism $h : G \to G$; exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$?.

Ulam's problem was partially solved by Hyers [10] in 1941.

Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon$$

for all $x, y \in E_1$, where $\varepsilon > 0$ is a constant. Then the limit

$$T(x) = lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$ and T is the unique additive mapping satisfying

$$||T(x) - f(x)|| \leq \varepsilon \tag{1.1}$$

for all $x \in E_1$. Also, if for each x the function $t \to f(tx)$ from $\mathbb{R}$ to E_2 is continuous on $\mathbb{R}$, then T is linear. If f is continuous at a single point of E_1, then T is continuous everywhere in E_1. Moreover (1.1) is sharp.

In 1978, Th. M. Rassias [18] formulated and proved the following theorem, which implies Hyers's Theorem as a special case. Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f(tx)$ is continuous on R, and let there exist $\varepsilon \geq 0$ and $p \in [0, 1)$ such that for all $x, y \in E$

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p) \tag{1.2}$$

Then there exists a unique linear mapping $T : E \to F$ such that such that for all $x \in E$

$$||f(x) - T(x)|| \leq \frac{\varepsilon||x||^p}{1 - 2^{p-1}}$$

The case of the existence of a unique additive mapping had been obtained by T. Aoki [2], as it is recently noticed by Lech Maligranda. However, Aoki [2] had claimed the existence of a unique linear mapping, that is not true because he did not allow the mapping f to satisfy some continuity assumption. Th.M. Rassias [18], who independently introduced the unbounded Cauchy difference was the first to prove that there exists a unique linear mapping T satisfying

$$||f(x) - T(x)|| \leq \frac{\epsilon||x||^p}{1 - 2^{p-1}} \qquad x \in E$$

In 1990, Th.M. Rassias [19] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [8] following the same approach as in Th. M. Rassias [24], gave an affirmative solution to this question for $p > 1$. It was proved by Z. Gajda [8] , as well as by Th. M. Rassias and P. Šemrl [20] that one can not prove a Th. M. Rassias type theorem when $p = 1$. In 1994, P. Găvruta [9] provided a further generalization of Th. M. Rassias theorem in which he replaced the bound $\varepsilon(||x||^p + |||y||^p)$ by a general control function $\psi(x, y)$ for the existence of a unique linear mapping.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [25] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [7], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians([1]-[5], [11]-[24]).

Definition 1.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function(valuation) $|.| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(*i*) $|r| = 0$ if and only if $r = 0$;

(*ii*) $|rs| = |r||s|$;

(*iii*) $|r + s| \leq max\{|r|, |s|\}$.

Definition 1.2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $||.|| : X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(i) $||x|| = 0$ if and only if $x = 0$;

(ii) $||rx|| = |r|||x||$ $(r \in \mathbb{K}, x \in X)$;

(iii) The strong triangle inequality(ultrametric); namely

$$||x + y|| \leq max\{||x||, ||y||\}. \quad x, y \in X$$

Then $(X, ||.||)$ is called a non-Archimedean space.

Due to the fact that

$$||x_n - x_m|| \leq max\{||x_{j+1} - x_j|| : m \leq j \leq n - 1\} \quad (n > m)$$

Definition 1.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Definition 1.4. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that:*

(a) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*

(b) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(c) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*

(d) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation

$$f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y) \qquad (1.3)$$

in non-Archimedean normed spaces. In the rest of the paper let $|2| \neq 1$.

2. Non-Archimedean Stability of Eq. (1.3): A Fixed Point Method

Throughout this section, using the fixed point method we prove the generalized Hyers-Ulam stability of the composite functional equation (1.3) in non-Archimedean spaces.

Theorem 2.1. *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean normed space. Assume $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\zeta(2x, 2y) \leq |2|L\zeta(x, y) \tag{2.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\Big\| f(f(x) - f(y)) - f(x+y) - f(x-y) + f(x) + f(y) \Big\| \leq \zeta(x, y) \tag{2.2}$$

for all $x, y \in X$. Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{\zeta(x, x)}{|2| - |2|L} \tag{2.3}$$

Proof. Putting $y = x$ in (2.2), we have

$$\left\| \frac{f(2x)}{2} - f(x) \right\| \leq \frac{1}{|2|}\zeta(x, x) \tag{2.4}$$

for all $x \in X$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric d in S defined by $d(f, g) = \inf_{\mu\in(0,+\infty)} \{\|g(x) - h(x)\| \leq \mu\zeta(x, x), \forall x \in X\}$, where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [12], Lemma 2.1).

Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that $Jh(x) := \frac{1}{2}h(2x)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then $\|g(x) - h(x)\| \leq \epsilon\zeta(x, x)$ for all $x \in X$ and so

$$\|Jg(x) - Jh(x)\| = \left\| \frac{g(2x)}{2} - \frac{h(2x)}{2} \right\| \leq \frac{1}{|2|}\epsilon\zeta(2x, 2x) \leq \frac{1}{|2|}\epsilon|2|L\zeta(x, x)$$

for all $x \in X$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. It follows from (2.4) that $d(f, Jf) \leq \frac{1}{|2|} < +\infty$. By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A(2x) = 2A(x) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (2.5) such that there exists $\mu \in (0, \infty)$ satisfying $\|f(x) - A(x)\| \leq \mu\zeta(x, x)$ for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x)$ for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, A) \leq \frac{1}{|2|-|2|L}$. This implies that the inequality (2.3) holds. By (2.1) and (2.2), we obtain

$$\begin{aligned}
&\Big\|A(A(x) - A(y)) - A(x+y) - A(x-y) + A(x) + A(y)\Big\| \\
&\lim_{n\to\infty} \left\| \frac{f(f(2^n x) - f(2^n y))}{2^n} - \frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x - 2^n y)}{2^n} + \frac{f(2^n x)}{2^n} + \frac{f(2^n y)}{2^n} \right\| \\
&\leq \lim_{n\to\infty} \frac{1}{|2|^n}\zeta(2^n x, 2^n y) \leq \lim_{n\to\infty} \frac{1}{|2|^n}.L^n.|2|^n\zeta(x, y) = 0
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So $\Big\|A(A(x) - A(y)) - A(x+y) - A(x-y) + A(x) + A(y)\Big\| = 0$, for all $x, y \in X$. Thus the mapping $A : X \to Y$ satisfying in (1.3).

On the other hand

$$2A\left(\frac{x}{2}\right) - A(x) = \lim_{n\to\infty} \frac{f(2^{n-1}x)}{2^{n-1}} - \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = 0.$$

So, A is an additive mapping, as desired. This completes the proof. □

Corollary 2.1. *Let $\theta \geq 0$ and r be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\Big\|f(f(x) - f(y)) - f(x+y) - f(x-y) + f(x) + f(y)\Big\| \leq \theta(\|x\|^r + \|y\|^r) \quad (2.6)$$

for all $x, y \in X$. Then the limit $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\|f(x) - A(x)\| \leq \frac{2\theta\|x\|^r}{|2| - |2|^{r+1}}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\zeta(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. In fact, if we choose $L = |2|^r$, then we get the desired result. □

Theorem 2.2. *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean normed space. Assume $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\zeta\Big(\frac{x}{2}, \frac{y}{2}\Big) \le \frac{L}{|2|}\zeta(x, y) \tag{2.7}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2). Then the limit $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L\zeta(x, x)}{|2| - |2|L}$$

Proof. Substituting $y = x$ in (2.2) and then replacing x by $\frac{x}{2}$, we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \zeta\left(\frac{x}{2}, \frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Consider a linear mapping $J : (S, d) \to (S, d)$ such that $Jh(x) := 2h\left(\frac{x}{2}\right)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then $\|g(x) - h(x)\| \le \epsilon\zeta(x, x)$ for all $x \in X$ and so

$$\|Jg(x) - Jh(x)\| = \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le |2|\epsilon\zeta\left(\frac{x}{2}, \frac{x}{2}\right) \le |2|\epsilon.\frac{L}{|2|}\zeta(x, x)$$

for all $x \in X$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \le L\epsilon$. This means that $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in S$. It follows from (2.8) that $d(f, Jf) \le \frac{L}{|2|}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\Big(\frac{x}{2}\Big) = \frac{1}{2}A(x) \tag{2.9}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (2.9) such that there exists $\mu \in (0, \infty)$ satisfying $\|f(x) - A(x)\| \le \mu\zeta(x, x)$ for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$ for all $x \in X$.

(3) $d(f, A) \le \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, A) \le \frac{L}{|2|-|2|L}$. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.2. *Let $\theta \geq 0$ and r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping satisfying (2.6). Then the limit $A(x) = \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\|f(x) - A(x)\| \leq \frac{2|2|^{r-1}\theta\|x\|^r}{|2| - |2|^r} \tag{2.10}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\zeta(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. In fact, if we choose $L = |2|^{r-1}$, then we get the desired result. □

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