RN HYERS-ULAM-RASSIAS STABILITY OF FUNCTIONAL EQUATIONS: A DIRECT METHOD

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1. Preliminaries

Definition 1.1. A function $F : \mathbb{R} \to [0, 1]$ is called a distribution function if it is nondecreasing and left-continuous, with $\sup_{t \in \mathbb{R}} F(t) = 1$ and $\inf_{t \in \mathbb{R}} F(t) = 0$. The class of all distribution functions $F$ with $F(0) = 0$ is denoted by $D_+$. For every $a \geq 0$, $H_a$ is the element of $D_+$ defined by

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\[ H_a(t) = \begin{cases} 0 & \text{if } t \leq a \\ 1 & \text{if } t > a \end{cases} \]

**Definition 1.2.** Let \( X \) be a real vector space, \( \Psi \) be a mapping from \( X \) into \( D_+ \) (for any \( x \in X \), \( \Psi(x) \) is denoted by \( \Psi_x \)) and \( T \) be a \( t \)-norm. The triple \((X, \Psi, T)\) is called a random normed space (briefly \( RN \)-space) iff the following conditions are satisfied:

(i) \( \Psi_x = H_0(t) \) iff \( x = \theta \), the null vector;

(ii) \( \Psi_{\alpha x}(t) = \Psi_x \left( \frac{t}{|\alpha|} \right) \) for all \( \alpha \in \mathbb{R} \), \( \alpha \neq 0 \) and \( x \in X \).

(iii) \( \Psi_{x+y}(t+s) \geq T(\Psi_x(t), \Psi_y(s)) \), for all \( x,y \in X \) and \( t,s > 0 \).

Every normed space \((X, \|\|)\) defines a random normed space \((X, \Psi, T_M)\) where for every \( t > 0 \),
\[
\Psi_u(t) = \frac{t}{t + \|u\|}
\]
and \( T_M \) is the minimum \( t \)-norm. This space is called the induced random normed space.

If the \( t \)-norm \( T \) is such that \( \sup_{0<a<1} T(a,a) = 1 \), then every \( RN \)-space \((X, \Psi, T)\) is a metrizable linear topological space with the topology \( \tau \) (called the \( \Psi \)-topology or the \((\epsilon, \delta)\)-topology) induced by the base of neighborhoods of \( \theta \), \( \{U(\epsilon, \lambda) | \epsilon > 0, \lambda \in (0,1)\} \), where
\[
U(\epsilon, \lambda) = \{x \in X | \Psi_x(\epsilon) > 1 - \lambda\}
\]

**Definition 1.3.** A sequence \( \{x_n\} \) in an \( RN \)-space \((X, \Psi, T)\) converges to \( x \in X \), in the topology \( \tau \) (we denote \( \lim x_n = x \)) if \( \lim_{n \to \infty} \Psi_{x_n-x}(t) = 1 \), \( \forall t > 0 \).

**Definition 1.4.** A sequence \( \{x_n\} \) is called Cauchy sequence if for all \( t > 0 \), \( \lim_{n \to \infty} \Psi_{x_n-x_m}(t) = 1 \). The \( RN \)-space \((X, \Psi, T)\) is said to be complete if every Cauchy sequence in \( X \) is convergent.

### 2. Random Stability of the Functional Equation

\[ f(mx + ny) = mf(x) + nf(y) \]

Throughout this section, using direct method we prove the Hyers-Ulam-Rassias stability of functional equation \( f(mx + ny) = mf(x) + nf(y) \) in random normed spaces(where \( m, n \in \mathbb{N} \)).
Theorem 2.1. Let $X$ be a vector space, $(Z, \Psi, \min)$ be an RN-space, and $\psi : X^m \to Z$ be a function such that for some $0 < \alpha < m + n$,

$$\Psi_{\psi((m+n)x,(m+n)y)}(t) \geq \Psi_{\alpha \psi(x,y)}(t), \quad \forall x, y \in X, \ t > 0$$

(2.1)

Also, for all $x, y \in X$ and $t > 0$

$$\lim_{n \to \infty} \Psi_{\psi((m+n)p_x,(m+n)p_y)}((m+n)^pt) = 1.$$

If $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$

$$\mu f(mx+ny) - mf(x) - nf(y)(t) \geq \Psi_{\psi(x,y)}((m + n - \alpha)t).$$

(2.2)

then there is a unique additive mapping $C(x) : X \to Y$ such that

$$\mu f(x) - C(x)(t) \geq \Psi_{\psi(x,x)}((m + n - \alpha)t).$$

(2.3)

Proof. Putting $y = x$ in (4.2) we see that for all $x \in X$,

$$\mu f((m+n)x) - f(x)(t) \geq \Psi_{\psi(x,x)}((m + n)t).$$

(2.4)

Replacing $x$ by $(m + n)p_x$ in (4.4) and using (4.1), we obtain

$$\mu \frac{f((m+n)p_x)}{(m+n)^p} - f(x)(t) \geq \Psi_{\psi((m+n)p_x,(m+n)p_x)}((m+n)^{p+1}t) \geq \Psi_{\psi(x,x)}((m + n)^{p+1}t).$$

(2.5)

So by (4.5) we obtain

$$\mu \frac{f((m+n)p_x)}{(m+n)^p} - f(x) \left( \sum_{k=0}^{p-1} \frac{t\alpha^k}{(m+n)^{k+1}} \right) \geq \mu \frac{f((m+n)p_x)}{(m+n)^p} - f(x) \left( \sum_{k=0}^{p-1} \frac{t\alpha^k}{(m+n)^{k+1}} \right) \geq \Psi_{\psi(x,x)}((m + n)^{p+1}t).$$

(2.6)
This implies that

\[ \mu \frac{f((m+n)^p x)}{(m+n)^p} - f(x) (t) \geq \Psi (x,x) \left( \frac{t}{\sum_{k=0}^{p-1} \frac{\alpha^k}{(m+n)^{k+1}}} \right). \quad (2.7) \]

Replacing \( x \) by \( (m+n)^q x \) in (4.7), we obtain

\[ \mu \frac{f((m+n)^p + q x)}{(m+n)^p + q} \frac{f((m+n)^q x)}{(m+n)^q} (t) \geq \Psi (x,x) \left( \frac{t}{\sum_{k=0}^{p-1} \frac{\alpha^k}{(m+n)^{k+1}}} \right) \]

\[ \geq \Psi (x,x) \left( \frac{t}{\sum_{k=0}^{p+q-1} \frac{\alpha^k}{(m+n)^{k+q+1}}} \right) \]

\[ = \Psi (x,x) \left( \frac{t}{\sum_{k=q}^{p+q-1} \frac{\alpha^k}{(m+n)^{k+1}}} \right). \quad (2.8) \]

As

\[ \lim_{p,q \to \infty} \Psi (x,x) \left( \frac{t}{\sum_{k=q}^{p+q-1} \frac{\alpha^k}{(m+n)^{k+1}}} \right) = 1, \]

then \( \left\{ \frac{f((m+n)^p x)}{(m+n)^p} \right\}_{n=1}^{+\infty} \) is a Cauchy sequence in complete RN-space \((Y, \mu, \min)\), so there exist some point \( C(x) \in Y \) such that

\[ C(x) = \lim_{n \to \infty} \frac{f((m+n)^p x)}{(m+n)^p}. \]

Fix \( x \in X \) and put \( q = 0 \) in (4.8). Then we obtain

\[ \mu \frac{f((m+n)^p x)}{(m+n)^p} - f(x) (t) \geq \Psi (x,x) \left( \frac{t}{\sum_{k=0}^{p-1} \frac{\alpha^k}{(m+n)^{k+1}}} \right). \quad (2.9) \]

and so, for every \( \epsilon > 0 \), we have

\[ \mu C(x) - f(x) (t + \epsilon) \geq T \left( \mu C(x) - f((m+n)^p x) (\epsilon), \frac{f((m+n)^p x)}{(m+n)^p} - f(x) (t) \right) \]

\[ \geq T \left( \mu C(x) - f((m+n)^p x) (\epsilon), \Psi (x,x) \left( \frac{t}{\sum_{k=0}^{p-1} \frac{\alpha^k}{(m+n)^{k+1}}} \right) \right). \]

Taking the limit as \( p \to \infty \), we get

\[ \mu C(x) - f(x) (t + \epsilon) \geq \Psi (x,x) \left( (m + n - \alpha)t \right). \quad (2.10) \]

Since \( \epsilon \) was arbitrary by taking \( \epsilon \to 0 \) in (4.10), we obtain

\[ \mu C(x) - f(x) (t) \geq \Psi (x,x) \left( (m + n - \alpha)t \right). \quad (2.11) \]
Replacing $x$ and $y$ by $(m + n)^p x$ and $(m + n)^p y$ respectively, in (4.2) and using this fact that $\lim_{p \to \infty} \psi((m + n)^p x, (m + n)^p y)((m + n)^p t) = 1$, we get for all $x, y \in X$ and for all $t > 0$,

$$C(mx + ny) = mC(x) + nC(y).$$

To prove the uniqueness of the additive mapping $C$, assume that there exist another additive mapping $D : X \to Y$ which satisfies (4.3). Since for all $p \in \mathbb{N}$ and every $x \in X$,

$$C((m + n)^p x) = (m + n)^p C(x) \quad \text{and} \quad D((m + n)^p x) = (m + n)^p D(x),$$

we find that

$$\mu_{C(x) - D(x)}(t) = \lim_{n \to \infty} \frac{\mu C((m + n)^p x)}{(m + n)^p} - \frac{D((m + n)^p x)}{(m + n)^p} (t). \tag{2.12}$$

So

$$\frac{\mu C((m + n)^p x)}{(m + n)^p} - \frac{D((m + n)^p x)}{(m + n)^p} (t) \geq \min \left\{ \mu C((m + n)^p x) - f((m + n)^p x) \left( \frac{t}{2} \right), \mu D((m + n)^p x) - f((m + n)^p x) \left( \frac{t}{2} \right) \right\} \tag{2.13}$$

$$\geq \Psi \psi((m + n)^p x, (m + n)^p x) \left( \frac{(m + n)^p (m + n - \alpha) t}{2} \right).$$

Since $\lim_{p \to \infty} \frac{(m + n)^p (m + n - \alpha) t}{2\alpha^p} = \infty$, we get

$$\lim_{p \to \infty} \Psi \psi(x,x) \frac{(m + n)^p (m + n - \alpha) t}{2\alpha^p} = 1.$$

Therefore, it follows that for all $t > 0$, $\mu_{C(x) - D(x)}(t) = 1$ and so $C(x) = D(x)$. This completes the proof. \[\square\]

**Corollary 2.1.** Let $X$ be a real linear space, $(Z, \Psi, \min)$ be an RN-space and $(Y, \mu, \min)$ a complete RN-space. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$

$$\mu_{f(mx + ny) - f(x) - f(y)}(t) \geq \Psi ||x||^{p_{z_0}}(t), \tag{2.14}$$

then there is a unique additive mapping $C(x) : X \to Y$ such that

$$\mu_{f(x) - C(x)}(t) \geq \Psi ||x||^{p_{z_0}}((m + n - (m + n)^p)t), \tag{2.15}$$
Proof. Let $\alpha = (m+n)^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = ||x||^p z_0$.

Corollary 2.2. Let $X$ be a real linear space, $(Z, \Psi, \min)$ be an RN-space and $(Y, \mu, \min)$ a complete RN-space. Let $p \in (0,1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x,y \in X$ and $t > 0$

$$\mu_f(mx+ny)-mf(x)-nf(y)(t) \geq \Psi(||x||^p+||y||^p)z_0(t),$$  \hspace{1cm} (2.16)

then there is a unique additive mapping $C(x) : X \to Y$ such that

$$\mu_f(x-y)(t) \geq \Psi(||x||^p \frac{(m+n-(m+n)^p)t}{2}).$$  \hspace{1cm} (2.17)

Proof. Let $\alpha = (m+n)^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = (||x||^p + ||y||^p)z_0$.

Corollary 2.3. Let $X$ be a real linear space, $(Z, \Psi, \min)$ be an RN-space and $(Y, \mu, \min)$ a complete RN-space. Let $p,q \in \mathbb{R}^+$ where $p+q \in (0,1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x,y \in X$ and $t > 0$

$$\mu_f(mx+ny)-mf(x)-nf(y)(t) \geq \Psi(||x||^{p+q}+||y||^{p+q}+||x||^p,||y||^q)z_0(t),$$  \hspace{1cm} (2.18)

then there is a unique additive mapping $C : X \to Y$ such that

$$\mu_f(x-y)(t) \geq \Psi(||x||^{p+q} \frac{(m+n-(m+n)^{p+q})t}{3}).$$  \hspace{1cm} (2.19)

Proof. Let $\alpha = (m+n)^{p+q}$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = (||x||^{p+q} + ||y||^{p+q} + ||x||^p,||y||^q)z_0$.

Corollary 2.4. Let $X$ be a real linear space, $(Z, \Psi, \min)$ be an RN-space and $(Y, \mu, \min)$ a complete RN-space. Let $z_0 \in Z$ and $f : X \to Y$ is a mapping such that for all $x,y \in X$ and $t > 0$

$$\mu_f(mx+ny)-mf(x)-nf(y)(t) \geq \Psi z_0(t),$$  \hspace{1cm} (2.20)

then there is a unique additive mapping $C : X \to Y$ such that for all $x \in X$ and $t > 0$

$$\mu_f(x-y)(t) \geq \Psi \delta z_0((m+n-1)t).$$  \hspace{1cm} (2.21)

Proof. Let $\alpha = 1$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = \delta z_0$.

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