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RN HYERS-ULAM-RASSIAS STABILITY OF FUNCTIONAL EQUATIONS: A DIRECT METHOD

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Abstract: In this paper we prove Hyres-Ulam-Rassias stability of an additive functional equation in random normed spaces. The concept of Hyers-Ulam-Rassias stability originated from Th.M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

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1. Preliminaries

Definition 1.1. A function $F : \mathbb{R} \to [0, 1]$ is called a distribution function if it is nondecreasing and left-continuous, with $sup_{t\in R}F(t) = 1$ and $inf_{t\in R}F(t) = 0$. The class of all distribution functions F with $F(0) = 0$ is denoted by D_+. For every $a \geq 0$, H_a is the element of D_+ defined by

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$$H_a(t) = \begin{cases} 0 & \text{if} \quad t \le a \\ 1 & \text{if} \quad t > a \end{cases}.$$

Definition 1.2. Let X be a real vector space, Ψ be a mapping from X into D_+ (for any $x \in X$, $\Psi(x)$ is denoted by Ψ_x) and T be a t-norm. The triple (X, Ψ, T) is called a random normed space (briefly RN-space) iff the following conditions are satisfied:

(i) $\Psi_x = H_0(t)$ iff $x = \theta$, the null vector;

(ii) $\Psi_{\alpha x}(t) = \Psi_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and $x \in X$.

(iii) $\Psi_{x+y}(t+s) \geq T(\Psi_x(t), \Psi_y(s))$, for all $x, y \in X$ and $t, s > 0$.

Every normed space $(X, ||.||)$ defines a random normed space (X, Ψ, T_M) where for every $t > 0$,

$$\Psi_u(t) = \frac{t}{t + ||u||}$$

and T_M is the minimum t-norm. This space is called the induced random normed space.

If the t-norm T is such that $sup_{0<a<1} T(a, a) = 1$, then every RN-space (X, Ψ, T) is a metrizable linear topological space with the topology τ (called the Ψ-topology or the (ϵ, δ)-topology) induced by the base of neighborhoods of θ, $\{U(\epsilon, \lambda) | \epsilon > 0, \lambda \in (0, 1)\}$, where

$$U(\epsilon, \lambda) = \{x \in X | \Psi_x(\epsilon) > 1 - \lambda\}$$

Definition 1.3. A sequence $\{x_n\}$ in an RN-space (X, Ψ, T) converges to $x \in X$, in the topology τ (we denote $lim x_n = x$) if $lim_{n\to\infty} \Psi_{x_n - x}(t) = 1$, $\forall t > 0$.

Definition 1.4. A sequence $\{x_n\}$ is called Cauchy sequence if for all $t > 0$, $lim_{n\to\infty} \Psi_{x_n - x_m}(t) = 1$. The RN-space (X, Ψ, T) is said to be complete if every Cauchy sequence in X is convergent.

2. Random Stability of the Functional Equation $f(mx + ny) = mf(x) + nf(y)$

Throughout this section, using direct method we prove the Hyers-Ulam-Rassias stability of functional equation $f(mx+ny) = mf(x)+nf(y)$ in random normed spaces(where $m, n \in \mathcal{N}$).

Theorem 2.1. *Let X be a vector space, (Z, Ψ, min) be an RN-space, and $\psi : X^m \to Z$ be a function such that for some $0 < \alpha < m+n$,*

$$\Psi_{\psi((m+n)x,(m+n)y)}(t) \geq \Psi_{\alpha\psi(x,y)}(t). \quad \forall x, y \in X,\ t > 0 \tag{2.1}$$

Also, for all $x, y \in X$ and $t > 0$

$$\lim_{n\to\infty} \Psi_{\psi((m+n)^p x,(m+n)^p y)}((m+n)^p t) = 1.$$

If (Y, μ, min) be a complete RN-space. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq \Psi_{\psi(x,y)}(t), \tag{2.2}$$

then there is a unique additive mapping $C(x) : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\psi(x,x)}((m+n-\alpha)t). \tag{2.3}$$

Proof. Putting $y = x$ in (4.2) we see that for all $x \in X$,

$$\mu_{\frac{f((m+n)x)}{m+n}-f(x)}(t) \geq \Psi_{\psi(x,x)}((m+n)t). \tag{2.4}$$

Replacing x by $(m+n)^p x$ in (4.4) and using (4.1), we obtain

$$\begin{aligned}\mu_{\frac{f((m+n)^{p+1}x)}{(m+n)^{p+1}}-\frac{f((m+n)^p x)}{(m+n)^p}}(t) &\geq \Psi_{\psi((m+n)^p x,(m+n)^p x)}((m+n)^{p+1}t) \\ &\geq \Psi_{\psi(x,x)}\Big(\frac{(m+n)^{p+1}t}{\alpha^p}\Big).\end{aligned} \tag{2.5}$$

So by (4.5) we obtain

$$\begin{aligned}&\mu_{\frac{f((m+n)^p x)}{(m+n)^p}-f(x)}\Big(\sum_{k=0}^{p-1}\frac{t\alpha^k}{(m+n)^{k+1}}\Big) \\ =\ &\mu_{\sum_{k=0}^{p-1}\frac{f((m+n)^{k+1}x)}{(m+n)^{k+1}}-\frac{f((m+n)^k x)}{(m+n)^k}}\Big(\sum_{k=0}^{p-1}\frac{t\alpha^k}{(m+n)^{k+1}}\Big) \\ \geq\ &T_{k=0}^{p-1}\Big(\mu_{\frac{f((m+n)^{k+1}x)}{(m+n)^{k+1}}-\frac{f((m+n)^k x)}{(m+n)^k}}\Big(\frac{t\alpha^k}{(m+n)^{k+1}}\Big)\Big) \\ \geq\ &T_{k=0}^{p-1}(\Psi_{\psi(x,x)}(t)) \\ =\ &\Psi_{\psi(x,x)}(t).\end{aligned} \tag{2.6}$$

This implies that

$$\mu_{\frac{f((m+n)^p x)}{(m+n)^p}-f(x)}(t) \geq \Psi_{\psi(x,x)}\Big(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{(m+n)^{k+1}}}\Big). \tag{2.7}$$

Replacing x by $(m+n)^q x$ in (4.7), we obtain

$$\begin{aligned}
\mu_{\frac{f((m+n)^{p+q}x)}{(m+n)^{p+q}}-\frac{f((m+n)^q x)}{(m+n)^q}}(t) &\geq \Psi_{\psi((m+n)^q x,(m+n)^q x)}\Big(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{(m+n)^{k+q+1}}}\Big)\\
&\geq \Psi_{\psi(x,x)}\Big(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^{k+q}}{(m+n)^{k+q+1}}}\Big) \qquad (2.8)\\
&= \Psi_{\psi(x,x)}\Big(\frac{t}{\sum_{k=q}^{p+q-1}\frac{\alpha^k}{(m+n)^{k+1}}}\Big).
\end{aligned}$$

As

$$\lim_{p,q\to\infty}\Psi_{\psi(x,x)}\Big(\frac{t}{\sum_{k=q}^{p+q-1}\frac{\alpha^k}{(m+n)^{k+1}}}\Big)=1,$$

then $\left\{\frac{f((m+n)^p x)}{(m+n)^p}\right\}_{n=1}^{+\infty}$ is a Cauchy sequence in complete RN-space (Y,μ,min), so there exist some point $C(x)\in Y$ such that

$$C(x)=\lim_{n\to\infty}\frac{f((m+n)^p x)}{(m+n)^p}.$$

Fix $x\in X$ and put $q=0$ in (4.8). Then we obtain

$$\mu_{\frac{f((m+n)^p x)}{(m+n)^p}-f(x)}(t) \geq \Psi_{\psi(x,x)}\Big(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{(m+n)^{k+1}}}\Big). \tag{2.9}$$

and so, for every $\epsilon>0$, we have

$$\begin{aligned}
\mu_{C(x)-f(x)}(t+\epsilon) &\geq T\Big(\mu_{C(x)-\frac{f((m+n)^p x)}{(m+n)^p}}(\epsilon),\mu_{\frac{f((m+n)^p x)}{(m+n)^p}-f(x)}(t)\Big)\\
&\geq T\Big(\mu_{C(x)-\frac{f((m+n)^p x)}{(m+n)^p}}(\epsilon),\Psi_{\psi(x,x)}\Big(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{(m+n)^{k+1}}}\Big)\Big).
\end{aligned}$$

Taking the limit as $p\to\infty$, we get

$$\mu_{C(x)-f(x)}(t+\epsilon)\geq \Psi_{\psi(x,x)}((m+n-\alpha)t). \tag{2.10}$$

Since ϵ was arbitrary by taking $\epsilon\to 0$ in (4.10), we obtain

$$\mu_{C(x)-f(x)}(t)\geq \Psi_{\psi(x,x)}((m+n-\alpha)t). \tag{2.11}$$

Replacing x and y by $(m+n)^p x$ and $(m+n)^p y$ respectively, in (4.2) and using this fact that $\lim_{p\to\infty}\Psi_{\psi((m+n)^p x,(m+n)^p y)}((m+n)^p t) = 1$, we get for all $x, y \in X$ and for all $t > 0$,

$$C(mx + ny) = mC(x) + nC(y).$$

To prove the uniqueness of the additive mapping C, assume that there exist another additive mapping $D : X \to Y$ which satisfies (4.3). Since for all $p \in \mathbb{N}$ and every $x \in X$,

$$C((m+n)^p x) = (m+n)^p C(x) \ \text{ and } \ D((m+n)^p x) = (m+n)^p D(x),$$

we find that

$$\mu_{C(x)-D(x)}(t) = \lim_{n\to\infty}\mu_{\frac{C((m+n)^p x)}{(m+n)^p}-\frac{D((m+n)^p x)}{(m+n)^p}}(t). \tag{2.12}$$

So

$$\begin{aligned}
&\mu_{\frac{C((m+n)^p x)}{(m+n)^p}-\frac{D((m+n)^p x)}{(m+n)^p}}(t)\\
\geq\ & min\Big\{\mu_{\frac{C((m+n)^p x)}{(m+n)^p}-\frac{f((m+n)^p x)}{(m+n)^p}}\Big(\frac{t}{2}\Big), \mu_{\frac{D((m+n)^p x)}{(m+n)^p}-\frac{f((m+n)^p x)}{(m+n)^p}}\Big(\frac{t}{2}\Big)\Big\}\\
\geq\ & \Psi_{\psi((m+n)^p x,(m+n)^p x)}\Big(\frac{(m+n)^p(m+n-\alpha)t}{2}\Big) \qquad (2.13)\\
\geq\ & \Psi_{\psi(x,x)}\Big(\frac{(m+n)^p(m+n-\alpha)t}{2\alpha^p}\Big).
\end{aligned}$$

Since $\lim_{p\to\infty}\frac{(m+n)^p(m+n-\alpha)t}{2\alpha^p} = \infty$, we get

$$\lim_{p\to\infty}\Psi_{\psi(x,x)}\frac{(m+n)^p(m+n-\alpha)t}{2\alpha^p} = 1.$$

Therefore, it follows that for all $t > 0$, $\mu_{C(x)-D(x)}(t) = 1$ and so $C(x) = D(x)$. This completes the proof. □

Corollary 2.1. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) a complete RN-space. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$*

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq \Psi_{||x||^p z_0}(t), \tag{2.14}$$

then there is a unique additive mapping $C(x) : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\|x\|^p z_0}((m+n-(m+n)^p)t), \tag{2.15}$$

Proof. Let $\alpha = (m+n)^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = ||x||^p z_0$. □

Corollary 2.2. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) a complete RN-space. Let $p \in (0,1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$*

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq \Psi_{(||x||^p+||y||^p)z_0}(t), \tag{2.16}$$

then there is a unique additive mapping $C(x) : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{||x||^p}\Big(\frac{(m+n-(m+n)^p)t}{2}\Big). \tag{2.17}$$

Proof. Let $\alpha = (m+n)^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = (||x||^p + ||y||^p)z_0$. □

Corollary 2.3. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) a complete RN-space. Let $p, q \in \mathbb{R}^+$ where $p+q \in (0,1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$*

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq \Psi_{(||x||^{p+q}+||y||^{p+q}+||x||^p.||y||^q)z_0}(t), \tag{2.18}$$

then there is a unique additive mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{||x||^{p+q}z_0}\Big(\frac{(m+n-(m+n)^{p+q})t}{3}\Big) \tag{2.19}$$

Proof. Let $\alpha = (m+n)^{p+q}$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = (||x||^{p+q} + ||y||^{p+q} + ||x||^p.||y||^q)z_0$. □

Corollary 2.4. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) a complete RN-space. Let $z_0 \in Z$ and $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$*

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq \Psi_{\delta z_0}(t), \tag{2.20}$$

then there is a unique additive mapping $C : X \to Y$ such that for all $x \in X$ and $t > 0$

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\delta z_0}((m+n-1)t). \tag{2.21}$$

Proof. Let $\alpha = 1$ and $\psi : X^2 \to Z$ be defined by $\psi(x,y) = \delta z_0$. □

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