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APPROXIMATELY ADDITIVE FUNCTIONAL EQUATION IN NAB-SPACES: A DIRECT METHOD

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Abstract: In this paper we discuss on approximately additive functional equation in non-Archimedean Banach spaces(briefly, NAB-spaces).

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1. Introduction

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?.

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [3] in 1940.

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We are given a group G and a metric group G' with metric $d(.,.)$. Given $\varepsilon > 0$, dose there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G$, , then a homomorphism $h : G \to G$; exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$?.

In the next year D.H. Hyres [2], gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.

In 1897, Hensel ([1]) has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications.

2. Preliminaries

Definition 2.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function(valuation) $|.| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(i) $|r| = 0$ if and only if $r = 0$, (ii) $|rs| = |r||s|$, (iii) $|r + s| \leq max\{|r|, |s|\}$.

Definition 2.2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $||.|| : X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(i) $||x|| = 0$ if and only if $x = 0$, (ii) $||rx|| = |r|||x||$ $(r \in \mathbb{K}, x \in X)$, (iii) The strong triangle inequality(ultrametric); namely

$$||x + y|| \leq max\{||x||, ||y||\}. \quad x, y \in X$$

Then $(X, ||.||)$ is called a non-Archimedean space.

Due to the fact that $||x_n - x_m|| \leq max\{||x_{j+1} - x_j|| : m \leq j \leq m-1\}$ $(n > m)$

Definition 2.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are $p-$adic numbers. A key property of $p-$adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

3. Approximate Additive functional equation: a direct method

In this section, we prove the generalized Hyers-Ulam stability of the functional equation $f(mx+ny) = mf(x)+nf(y)$ in non-Archimedean Banach space. Throughout this section, Let H be a additive semigroup and X is a complete non-Archimedean space.

Theorem 3.1. *Let $\eta(x,y) : H^2 \to [0,+\infty)$ is a function such that*

$$\lim_{p\to\infty} \frac{\eta((m+n)^p x,(m+n)^p y)}{|m+n|^p} = 0; \qquad x,y \in H, \tag{3.1}$$

and let for each $x \in H$ the limit

$$\zeta(x) = \lim_{p\to\infty} max\Big\{\frac{\psi((m+n)^k x,(m+n)^k y)}{|m+n|^{k-1}}\ ,\ 0 \le k < p\Big\}, \tag{3.2}$$

exists. Suppose that $f : H \to X$ be a mapping satisfying

$$\Big\|f(mx+ny) - mf(x) - nf(y)\Big\|_X \le \eta(x,y). \tag{3.3}$$

Then the limit $T(x) = \lim_{p\to\infty} \frac{f((m+n)^p x)}{(m+n)^p}$, exists for all $x \in H$ and $T : H \to X$ is a mapping satisfying

$$\Big\|f(x) - T(x)\Big\|_X \le |m+n|^{-1}\zeta(x). \qquad x \in H \tag{3.4}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{p\to\infty} max\Big\{|m+n|^{-k+1}\psi((m+n)^k x,(m+n)^k y)\ ;\ j \le k < p+j\Big\} = 0, \tag{3.5}$$

then T is the unique mapping satisfying (3.4).

Proof. Putting $y = x$ in (3.3), we get

$$\Big\|\frac{f((m+n)x)}{m+n} - f(x)\Big\|_X \le \frac{1}{|m+n|}\eta(x,x). \tag{3.6}$$

Replacing x by $(m+n)^{p-1}x$ in (3.6) and dividing both sides by $(m+n)^{p-1}$, we get

$$\Big\|\frac{f((m+n)^p x)}{(m+n)^p} - \frac{f((m+n)^{p-1}x)}{(m+n)^{p-1}})\Big\|_X \le |m+n|^{-p}\eta((m+n)^{p-1}x,(m+n)^{p-1}x), \tag{3.7}$$

for all $x \in H$. It follows from (3.1) and (3.7) that the sequence $\left\{\frac{f((m+n)^{\mathsf{p}}x)}{(m+n)^{\mathsf{p}}}\right\}_{p=1}^{+\infty}$ is a Cauchy sequence. Since X is complete, so $\left\{\frac{f((m+n)^{\mathsf{p}}x)}{(m+n)^{\mathsf{p}}}\right\}_{p=1}^{+\infty}$ is convergent. Set $T(x) := \lim_{p\to\infty} \frac{f((m+n)^{\mathsf{p}}x)}{(m+n)^{\mathsf{p}}}$. Using induction we see that

$$\begin{aligned} &\left\|\frac{f((m+n)^p x)}{(m+n)^p} - f(x)\right\|_X \\ &\qquad \leq \frac{max\Big\{|m+n|^{-k+1}\eta\Big((m+n)^k x,(m+n)^k x\Big)\ ;\ 0\leq k<p\Big\}}{|m+n|}. \end{aligned} \tag{3.8}$$

Indeed, (3.8) holds for $p=1$ by (3.6). Let (3.8) holds for $p-1$, then we obtain

$$\begin{aligned} \left\|\frac{f((m+n)^p x)}{(m+n)^p} - f(x)\right\|_X &= \left\|\frac{f((m+n)^p x)}{(m+n)^p} \pm \frac{f((m+n)^{p-1} x)}{(m+n)^{p-1}} - f(x)\right\|_X \\ \leq\ & max\Big\{\left\|\frac{f((m+n)^p x)}{(m+n)^p} - \frac{f((m+n)^{p-1} x)}{(m+n)^{p-1}}\right\|_X \\ ,\ & \left\|\frac{f((m+n)^{p-1} x)}{(m+n)^p} - f(x)\right\|_X\Big\} \qquad (3.9) \\ \leq\ & \frac{1}{|m+n|} max\Big\{|m+n|^{-p+1}\eta\Big((m+n)^{p-1}x,(m+n)^{p-1}x\Big) \\ ,\ & max\Big\{|m+n|^{-k+1}\eta((m+n)^k x,(m+n)^k x)\ ;\ 0\leq k<p-1\Big\}\Big\} \\ =\ & \frac{1}{|m+n|} max\Big\{|m+n|^{-k+1}\eta((m+n)^k x,(m+n)^k x)\ ;\ 0\leq k<p\Big\}. \end{aligned}$$

So for all $p \in \mathbb{N}$ and all $x \in H$, (3.8) holds. By taking p to approach infinity in (3.9) and using (3.2) one obtains (3.4). Replacing x by $(m+n)^p x$ and y by $(m+n)^p y$ respectively, in (3.3) and using (3.1), we obtain that $T(mx+ny) = mT(y)+nT(y)$. If S is another mapping satisfies (3.4), then for $x \in H$, we get

$$\begin{aligned} \Big\|T(x)-S(x)\Big\|_X &= \lim_{k\to\infty} |m+n|^{-k}\Big\|T((m+n)^k x) - S((m+n)^k x)\Big\|_X \\ &\leq \lim_{k\to\infty} |m+n|^{-k} max\Big\{\Big\|T((m+n)^k x) - f((m+n)^k x)\Big\| \\ &,\ \Big\|S((m+n)^k x) - f((m+n)^k x)\Big\|_X\Big\} \qquad (3.10) \\ &\leq \frac{1}{|m+n|}\lim_{j\to\infty}\lim_{p\to\infty} max\Big\{|m+n|^{-k+1}\eta((m+n)^k x \\ &,\ (m+n)^k x)\ ;\ j\leq k<p+j\Big\} \end{aligned}$$

$$= 0.$$

Therefore $T = S$. This completes the proof of uniqueness of T. □

Theorem 3.2. *Let $\eta(x,y) : H^2 \to [0,+\infty)$ is a function such that*

$$\lim_{n\to\infty} |m+n|^n \eta\Big(\frac{x}{(m+n)^p}, \frac{y}{(m+n)^p}\Big) = 0; \quad x, y \in H, \tag{3.11}$$

and let for each $x \in H$ the limit

$$\zeta(x) = \lim_{n\to\infty} max\Big\{|m+n|^{k-1}\eta\Big(\frac{x}{(m+n)^k}, \frac{y}{(m+n)^k}\Big) = 0\ ,\ 0 \le k < p\Big\}, \tag{3.12}$$

exists. Suppose that $f : H \to X$ be a mapping satisfying

$$\Big\|f(mx+ny) - mf(x) - nf(y)\Big\|_X \le \eta(x,y). \tag{3.13}$$

Then the limit $T(x) = \lim_{p\to\infty}(m+n)^p f\Big(\frac{x}{(m+n)^p}\Big)$, exists for all $x \in H$ and $T : H \to X$ is a mapping satisfying

$$\Big\|f(x) - T(x)\Big\|_X \le |m+n|^{-1}\zeta(x); \quad x \in H \tag{3.14}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{p\to\infty} max\Big\{|m+n|^{k-1}\eta\Big(\frac{x}{(m+n)^k}, \frac{y}{(m+n)^k}\Big)\ ;\ j \le k < n+j\Big\} = 0, \tag{3.15}$$

then T is the unique mapping satisfying (3.14).

Proof. Letting $y = x$ in (3.13), we get

$$\Big\|f((m+n)x) - (m+n)f(x)\Big\|_X \le \eta(x,x), \tag{3.16}$$

for all $x \in H$. If we replace x by $\frac{x}{(m+n)^p}$ in (3.16), then we have

$$\Big\|(m+n)^{p-1} f\Big(\frac{x}{(m+n)^{p-1}}\Big) - (m+n)^p f\Big(\frac{x}{(m+n)^p}\Big)\Big)\Big\|_X$$
$$\le |m+n|^{p-1}\psi\Big(\frac{x}{(m+n)^p}, \frac{x}{(m+n)^{p-1}}\Big), \tag{3.17}$$

for all $x \in H$ and all non-negative integer n. It follows from (3.11) and (3.17) that the sequence $\Big\{(m+n)^p f\Big(\frac{x}{(m+n)^p}\Big)\Big\}_{p=1}^{\infty}$ is a Cauchy sequence in X for all

$x \in H$. Since X is complete, the sequence $\Big\{(m+n)^p f\Big(\frac{x}{(m+n)^p}\Big)\Big\}_{n=1}^{\infty}$ converges for all $x \in H$. On the other hand, it follows from (3.17) that

$$\begin{aligned}
&\Big\|(m+n)^p f\Big(\frac{x}{(m+n)^p}\Big) - (m+n)^q f\Big(\frac{x}{(m+n)^q}\Big)\Big\|_X \\
&= \Big\|\sum_{k=p}^{q-1}(m+n)^{k+1} f\Big(\frac{x}{(m+n)^{k+1}}\Big) \\
&-(m+n)^k f\Big(\frac{x}{(m+n)^k}\Big)\Big\|_X \\
\leq\ & max\Big\{\Big\|(m+n)^{k+1} f\Big(\frac{x}{(m+n)^{k+1}}\Big) \\
&-(m+n)^k f\Big(\frac{x}{(m+n)^k}\Big)\Big\|_X \ ; \ p \leq k < q\Big\} \\
\leq\ & \frac{1}{|m+n|} max\Big\{|m+n|^{k-1}\eta\Big(\frac{x}{(m+n)^k}, \frac{x}{(m+n)^k}\Big) \\
;\ & \ p \leq k < q\Big\},
\end{aligned}$$

for all $x \in H$ and all non-negative integers p, q with $q > p \geq 0$. Letting $p = 0$ and passing the limit $q \to \infty$ in the last inequality and using (3.12), we obtain (3.14).

The rest of the proof is similar to the proof of Theorem (3.1). □

Corollary 3.1. *Let $\gamma : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\gamma\Big(\frac{t}{|m+n|}\Big) \leq \gamma\Big(\frac{1}{|m+n|}\Big)\gamma(t) \quad (t \geq 0), \quad \gamma\Big(\frac{1}{|m+n|}\Big) < \frac{1}{|m+n|}. \tag{3.18}$$

Let $\delta > 0$ and $f : H \to X$ is a mapping satisfying

$$\Big\|f(mx+ny) - mf(x) - nf(y)\Big\|_X \leq \delta\Big(\gamma(|x|) + \gamma(|y|)\Big); \ \ x, y \in H. \tag{3.19}$$

Then there exists a unique mapping $T : H \to X$ such that

$$\Big\|f(x) - T(x)\Big\|_X \leq 2|m+n|^{-1}\delta\gamma(|x|); \quad x \in H, \tag{3.20}$$

Proof. Using induction one can show that for all $p \in \mathbb{N}$,

$$\gamma\Big(\frac{t}{|m+n|^p}\Big) \leq \gamma^p\Big(\frac{1}{|m+n|}\Big)\gamma(t) \leq \frac{1}{|m+n|^p}\gamma(t). \tag{3.21}$$

Defining $\eta : H^2 \to [0, \infty)$ by $\eta(x, y) := \delta(\gamma(|x|) + \gamma(|y|))$. Since

$$|m+n|\gamma\Big(\frac{1}{|m+n|}\Big) < 1,$$

then we obtain that for all $x, y \in H$

$$\lim_{p\to\infty} |m+n|^p \eta\Big(\frac{x}{(m+n)^p}, \frac{y}{(m+n)^p}\Big) \leq \lim_{n\to\infty} \Big(|m+n|\gamma\Big(\frac{1}{|m+n|}\Big)\Big)^p \eta(x, y) = 0. \tag{3.22}$$

Also

$$\zeta(x) = \lim_{p\to\infty} max\Big\{|m+n|^{k-1}\eta\Big(\frac{x}{(m+n)^k}, \frac{x}{(m+n)^k}\Big)\ ;\ 0 \leq k < p\Big\} = \eta(x, x), \tag{3.23}$$

and

$$\lim_{j\to\infty}\lim_{p\to\infty} max\Big\{|m+n|^{k-1}\eta\Big(\frac{x}{(m+n)^k}, \frac{y}{(m+n)^k}\Big); j \leq k < n+j\Big\} = \lim_{j\to\infty} |m+n|^j \eta\Big(\frac{x}{(m+n)^j}, \frac{y}{(m+n)^j}\Big) = 0. \tag{3.24}$$

Hence the result follows by Theorem (3.2). □

Corollary 3.2. *Let $\gamma : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\gamma(|m+n|t) \leq \gamma(|m+n|)\gamma(t) \quad (t \geq 0), \quad \gamma(|m+n|) < |m+n| \tag{3.25}$$

Let $\delta > 0$ and $f : H \to X$ is a mapping satisfying

$$\Big\|f(mx+ny) - mf(x) - nf(y)\Big\|_X \leq \delta\Big(\gamma(|x| + \gamma(|y|)\Big);\ \ x, y \in H \tag{3.26}$$

Then there exists a unique mapping $T : H \to X$ such that

$$\Big\|f(x) - T(x)\Big\|_X \leq 2|m+n|^{-1}\delta\gamma(|x|); \quad x \in H, \tag{3.27}$$

Proof. Let $\eta : H^2 \to [0, \infty)$ be defined by $\eta(x, y) := \delta\Big(\gamma(|x| + \gamma(|y|)\Big)$. □

Corollary 3.3. *Let $\gamma : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\gamma\Big(\frac{t}{|m+n|}\Big) \leq \gamma\Big(\frac{1}{|m+n|}\Big)\gamma(t) \quad (t \geq 0), \quad \gamma\Big(\frac{1}{|m+n|}\Big) < \frac{1}{|m+n|} \tag{3.28}$$

Let $\delta > 0$ and $f : H \to X$ is a mapping satisfying

$$\Big\| f(mx + ny) - mf(x) - nf(y) \Big\|_X \leq \delta\Big(\gamma(|x|).\gamma(|y|)\Big); \;\; x, y \in H. \tag{3.29}$$

Then there exists a unique mapping $T : H \to X$ such that

$$\Big\| f(x) - T(x) \Big\|_X \leq |m + n|^{-1}\delta\gamma^2(|x|); \quad x \in H, \tag{3.30}$$

Proof. Define $\psi : H^2 \to [0, \infty)$ by $\psi(x, y) = \delta\Big(\gamma(|x|).\gamma(|y|)\Big)$. □

Corollary 3.4. *Let $\gamma : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\gamma(|m + n|t) \leq \gamma(|m + n|)\gamma(t) \quad (t \geq 0), \quad \gamma(|m + n|) < |m + n| \tag{3.31}$$

Let $\delta > 0$ and $f : H \to X$ is a mapping satisfying

$$\Big\| f(mx + ny) - mf(x) - nf(y) \Big\| \leq \delta\Big(\gamma(|x|).\gamma(|y|)\Big); \;\; x, y \in H. \tag{3.32}$$

Then there exists a unique mapping $T : H \to X$ such that

$$\Big\| f(x) - T(x) \Big\|_X \leq |m + n|^{-1}\delta\gamma^2(|x|); \quad x \in H, \tag{3.33}$$

Proof. Define $\psi : H^2 \to [0, \infty)$ by $\psi(x, y) = \delta\Big(\gamma(|x|).\gamma(|y|)\Big)$. □

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References

[1] K. Hensel, *Ubereine news Begrundung der Theorie der algebraischen Zahlen*, Jahresber. Deutsch. Math. Verein 6 (1897), 83-88.

[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA 27 (1941), 222-224.

[3] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons, 1964.