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DIRECT METHOD AND RN-APPROXIMATELY FUNCTIONAL EQUATIONS

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Abstract: In this paper, we prove the generalized Hyers-Ulam stability of an additive functional equation in random normed spaces.

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1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [4] in 1940. In the next year, Hyers [1] gave a positive answer to the above question for additive groups

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under the assumption that the groups are Banach spaces. In 1978, Rassias [2] proved a generalization of Hyers's theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations.

2. Preliminaries

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [3].

Throughout this paper, let Γ^+ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Gamma^+ : l^- F(-\infty) = 1\}$, where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0 & \text{if} \quad t \leq a, \\ 1 & \text{if} \quad t > a. \end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(x,1) = x$ for all $x \in [0,1]$;

(d) $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$ for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are as follows: $T(x,y) = xy$, $T(x,y) = \max\{a+b-1, 0\}$, $T(x,y) = \min(a,b)$. Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $T_{i=n}^\infty x_i$ is defined by $T_{i=1}^\infty x_{n+i}$.

Definition 2.2. A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:

(a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;

(b) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;

(c) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.

(3) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 2.1. *([3]) If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Throughout this paper, using direct method we proved the generalized Hyers-Ulam stability of the following functional equation:

$$f(mx + ny) = mf(x) + nf(y) \tag{2.1}$$

for all $x, y \in X$ in random normed spaces.

3. Random Stability of Functional Equation (2.1)

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam-Rassias stability of functional equation (2.1) in random normed spaces.

Theorem 3.1. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{3.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;

(b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(c) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$*;*

(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

Theorem 3.2. *Let* X *be a linear space,* (Y, μ, T_M) *be a complete RN-space and* Φ *be a mapping from* X^2 *to* D^+ ($\Phi(x, y)$ *is denoted by* $\Phi_{x,y}$) *such that there exists* $0 < \alpha < \frac{1}{m+n}$ *such that*

$$\Phi_{(m+n)x,(m+n)y}(t) \leq \Phi_{x,y}(\alpha t) \tag{3.2}$$

for all $x, y \in X$ *and* $t > 0$*. Let* $f : X \to Y$ *be a mapping satisfying*

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq \Phi_{x,y}(t) \tag{3.3}$$

for all $x, y \in X$ *and* $t > 0$*. Then, for all* $x \in X$*,* $A(x) := \lim_{n\to\infty} (m + n)^p f\left(\frac{x}{(m+n)^p}\right)$ *exists and* $A : X \to Y$ *is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0}\left(\frac{(1-(m+n)\alpha)t}{\alpha}\right) \tag{3.4}$$

for all $x \in X$ *and* $t > 0$*.*

Proof. Putting $y = x$ and replacing x by $\frac{x}{m+n}$ in (3.3), we have

$$\mu_{(m+n)f(\frac{x}{m+n})-f(x)}(t) \geq \Phi_{\frac{x}{m+n},\frac{x}{m+n}}(t) \geq \Phi_{x,x}\left(\frac{t}{\alpha}\right) \tag{3.5}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric d in S defined by $d(f, g) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,x}(t), \forall x \in X, t > 0\}$, where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete. Now, we consider a linear mapping $J : S \to S$ such that $Jh(x) := (m + n)h\Big(\frac{x}{m+n}\Big)$ for all $x \in X$.

First, we prove that J is a strictly contractive mapping with the Lipschitz constant $(m+n)\alpha$. In fact, let $g, h \in S$ be such that $d(g, h) < \epsilon$. Then we have $\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,x}(t)$ for all $x \in X$ and $t > 0$ and so

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}((m+n)\alpha\epsilon t) &= \mu_{(m+n)g(\frac{x}{m+n})-(m+n)h(\frac{x}{m+n})}((m+n)\alpha\epsilon t) \\ &= \mu_{g(\frac{x}{m+n})-h(\frac{x}{m+n})}(\alpha\epsilon t) \geq \Phi_{\frac{x}{m+n},\frac{x}{m+n}}(\alpha t) \geq \Phi_{x,x}(t)\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) < \epsilon$ implies that $d(Jg, Jh) < (m + n)\alpha\epsilon$. This means that $d(Jg, Jh) \leq (m + n)\alpha d(g, h)$ for all $g, h \in S$. It follows from (3.5) that $d(f, Jf) \leq \alpha$.

By Theorem 3.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{m+n}\right) = \frac{1}{m+n}A(x) \tag{3.6}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (3.6) such that there exists $u \in (0,\infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,x}(t)$ for all $x \in X$ and $t > 0$.

(2) $d(J^p f, A) \to 0$ as $p \to \infty$. This implies the equality

$$\lim_{p\to\infty} (m+n)^p f\left(\frac{x}{(m+n)^p}\right) = A(x)$$

for all $x \in X$.

(3) $d(f,A) \leq \frac{d(f,Jf)}{1-(m+n)\alpha} \leq \frac{\alpha}{1-(m+n)\alpha}$ with $f \in \Omega$ and so

$$\mu_{f(x)-A(x)}\left(\frac{\alpha t}{1-(m+n)\alpha}\right) \geq \Phi_{x,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.4) holds. On the other hand

$$\mu_{(m+n)^p f\left(\frac{mx+ny}{(m+n)^p}\right) - m(m+n)^p f\left(\frac{x}{(m+n)^p}\right) - n(m+n)^p f\left(\frac{y}{(m+n)^p}\right)}(t) \geq \Phi_{\frac{x}{(m+n)^p},\frac{y}{(m+n)^p}}\left(\frac{t}{(m+n)^p}\right)$$

for all $x, y \in X$, $t > 0$ and $n \geq 1$ and so, from (3.2), it follows that

$$\Phi_{\frac{x}{(m+n)^p},\frac{y}{(m+n)^p}}\left(\frac{t}{(m+n)^p}\right) \geq \Phi_{x,y}\left(\frac{t}{((m+n)\alpha)^p}\right)$$

. Since $\lim_{p\to\infty} \Phi_{x,y}\left(\frac{t}{((m+n)\alpha)^p}\right) = 1$ for all $x, y \in X$ and $t > 0$, we have $\mu_{A(mx+ny)-mA(x)-nA(y)}(t) = 1$ for all $x, y \in X$ and $t > 0$. This completes the proof. □

Corollary 3.1. *Let X be a real normed space, $\theta \geq 0$ and p be a real number with $p > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying*

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq t(t + \theta(\|x\|^p + \|y\|^p))^{-1}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$, $A(x) = \lim_{p\to\infty} (m+n)^p f\left(\frac{x}{(m+n)^p}\right)$ exists and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq (((m+n)^p - (m+n))t)(((m+n)^p - (m+n))t + \theta\|x\|^p)^{-1}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.2 if we take $\Phi_{x,y}(t) = t(t + \theta(\|x\|^p + \|y\|^p))^{-1}$ for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = (m+n)^{-p}$, then we get the desired result. □

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References

[1] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA 27 (1941), 222-224.

[2] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[3] B. Schewizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland, New York, USA, 1983.

[4] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons, 1964.