DIRECT METHOD AND 
RN-APPROXIMATELY FUNCTIONAL EQUATIONS

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\textbf{Abstract:} In this paper, we prove the generalized Hyers-Ulam stability of an additive functional equation in random normed spaces.

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1. Introduction

A classical question in the theory of functional equations is the following: “When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?”. If the problem accepts a solution, we say that the equation is \textit{stable}. The first stability problem concerning group homomorphisms was raised by Ulam \cite{4} in 1940. In the next year, Hyers \cite{1} gave a positive answer to the above question for additive groups.
under the assumption that the groups are Banach spaces. In 1978, Rassias [2] proved a generalization of Hyers’s theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations.

2. Preliminaries

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [3].

Throughout this paper, let $\Gamma^+$ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that $F$ is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Gamma^+ : l^-F(-\infty) = 1\}$, where $l^-f(x) = \lim_{t \to x^-} f(t)$, is a subset of $\Gamma^+$.

The set $\Gamma^+$ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of $D^+$ is defined by

$$H_a(t) = \begin{cases} 0 & \text{if } t \leq a, \\ 1 & \text{if } t > a. \end{cases}$$

We can easily show that the maximal element in $\Gamma^+$ is the distribution function $H_0(t)$.

**Definition 2.1.** A function $T : [0, 1]^2 \to [0, 1]$ is a continuous triangular norm (briefly, a $t$-norm) if $T$ satisfies the following conditions:

(a) $T$ is commutative and associative;

(b) $T$ is continuous;

(c) $T(x, 1) = x$ for all $x \in [0, 1]$;

(d) $T(x, y) \leq T(z, w)$ whenever $x \leq z$ and $y \leq w$ for all $x, y, z, w \in [0, 1]$.

Three typical examples of continuous $t$-norms are as follows: $T(x, y) = xy, \ T(x, y) = \max\{a + b - 1, 0\}, \ T(x, y) = \min(a, b).$ Recall that, if $T$ is a $t$-norm and $\{x_n\}$ is a sequence in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $T_{i=1}^\infty x_n$ is defined by $T_{i=1}^\infty x_{n+i}$. 
Definition 2.2. A random normed space (briefly, RN-space) is a triple \((X, \mu, T)\), where \(X\) is a vector space, \(T\) is a continuous \(t\)-norm and \(\mu : X \to D^+\) is a mapping such that the following conditions hold:

\[(a) \mu_x(t) = H_0(t)\] for all \(x \in X\) and \(t > 0\) if and only if \(x = 0\);

\[(b) \mu_\alpha x(t) = \mu_x(\frac{t}{|\alpha|})\] for all \(\alpha \in \mathbb{R}\) with \(\alpha \neq 0\), \(x \in X\) and \(t \geq 0\);

\[(c) \mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))\] for all \(x, y \in X\) and \(t, s \geq 0\).

Definition 2.3. Let \((X, \mu, T)\) be an RN-space.

(1) A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x \in X\) (write \(x_n \to x\) as \(n \to \infty\)) if \(\lim_{n \to \infty} \mu_{x_n-x}(t) = 1\) for all \(t > 0\).

(2) A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence in \(X\) if \(\lim_{n \to \infty} \mu_{x_n-x_m}(t) = 1\) for all \(t > 0\).

(3) The RN-space \((X, \mu, T)\) is said to be complete if every Cauchy sequence in \(X\) is convergent.

Theorem 2.1. ([3]) If \((X, \mu, T)\) is an RN-space and \(\{x_n\}\) is a sequence such that \(x_n \to x\), then \(\lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t)\).

Throughout this paper, using direct method we proved the generalized Hyers-Ulam stability of the following functional equation:

\[f(mx + ny) = mf(x) + nf(y)\] (2.1)

for all \(x, y \in X\) in random normed spaces.

3. Random Stability of Functional Equation (2.1)

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam-Rassias stability of functional equation (2.1) in random normed spaces.

Theorem 3.1. Let \((X, d)\) be a complete generalized metric space and \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then, for all \(x \in X\), either

\[d(J^n x, J^{n+1} x) = \infty\] (3.1)

for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that

(a) \(d(J^n x, J^{n+1} x) < \infty\) for all \(n_0 \geq n_0\);

(b) the sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J\);
(c) $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;
(d) $d(y, y^*) \leq \frac{1}{1-L}d(y, Jy)$ for all $y \in Y$.

Theorem 3.2. Let $X$ be a linear space, $(Y, \mu, T_M)$ be a complete RN-space and $\Phi$ be a mapping from $X^2$ to $D^+$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that there exists $0 < \alpha < \frac{1}{m+n}$ such that
\begin{equation}
\Phi_{(m+n)x,(m+n)y}(t) \leq \Phi_{x,y}(\alpha t) \tag{3.2}
\end{equation}
for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying
\begin{equation}
\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq \Phi_{x,y}(t) \tag{3.3}
\end{equation}
for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$, $A(x) := \lim_{n \to \infty} (m+n)^p f\left(\frac{x}{(m+n)^p}\right)$ exists and $A : X \to Y$ is a unique additive mapping such that
\begin{equation}
\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0}\left(\frac{(1-(m+n)\alpha)t}{\alpha}\right) \tag{3.4}
\end{equation}
for all $x \in X$ and $t > 0$.

Proof. Putting $y = x$ and replacing $x$ by $\frac{x}{m+n}$ in (3.3), we have
\begin{equation}
\mu_{(m+n)f(\frac{x}{m+n})-f(x)}(t) \geq \Phi_{\frac{x}{m+n}, \frac{x}{m+n}}(t) \geq \Phi_{x,x}\left(\frac{t}{\alpha}\right) \tag{3.5}
\end{equation}
for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric $d$ in $S$ defined by $d(f, g) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,x}(t), \forall x \in X, t > 0\}$, where $\inf \emptyset = +\infty$. It is easy to show that $(S, d)$ is complete. Now, we consider a linear mapping $J : S \to S$ such that $Jh(x) := (m+n)h\left(\frac{x}{m+n}\right)$ for all $x \in X$.

First, we prove that $J$ is a strictly contractive mapping with the Lipschitz constant $(m+n)\alpha$. In fact, let $g, h \in S$ be such that $d(g, h) < \epsilon$. Then we have $\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,x}(t)$ for all $x \in X$ and $t > 0$ and so
\begin{align*}
\mu_{Jg(x)-Jh(x)}((m+n)\alpha t) &= \mu_{(m+n)g(\frac{x}{m+n})-(m+n)h(\frac{x}{m+n})}((m+n)\alpha t) \\
&= \mu_{g(\frac{x}{m+n})-h(\frac{x}{m+n})}(\alpha t) \geq \Phi_{\frac{x}{m+n}, \frac{x}{m+n}}(\alpha t) \geq \Phi_{x,x}(t)
\end{align*}
for all $x \in X$ and $t > 0$. Thus $d(g, h) < \epsilon$ implies that $d(Jg, Jh) < (m+n)\alpha \epsilon$. This means that $d(Jg, Jh) \leq (m+n)\alpha d(g, h)$ for all $g, h \in S$. It follows from (3.5) that $d(f, Jf) \leq \alpha$. 

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By Theorem 3.1, there exists a mapping \( A : X \to Y \) satisfying the following:

1. \( A \) is a fixed point of \( J \), that is,
   \[
   A \left( \frac{x}{m+n} \right) = \frac{1}{m+n} A(x)
   \] (3.6)
   for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( J \) in the set \( \Omega = \{ h \in S : d(g,h) < \infty \} \). This implies that \( A \) is a unique mapping satisfying (3.6) such that there exists \( u \in (0, \infty) \) satisfying \( \mu f(x) - A(x) \geq \Phi_x(t) \) for all \( x \in X \) and \( t > 0 \).

2. \( d(J^p f, A) \to 0 \) as \( p \to \infty \). This implies the equality
   \[
   \lim_{p \to \infty} (m+n)^p f \left( \frac{x}{(m+n)^p} \right) = A(x)
   \]
   for all \( x \in X \).

3. \( d(f, A) \leq \frac{d(f,Jf)}{1-(m+n)\alpha} \leq \frac{\alpha t}{1-(m+n)\alpha} \) with \( f \in \Omega \) and so
   \[
   \mu f(x) - A(x) \left( \frac{\alpha t}{1-(m+n)\alpha} \right) \geq \Phi_{x,x}(t)
   \]
   for all \( x \in X \) and \( t > 0 \). This implies that the inequality (3.4) holds. On the other hand

   \[
   \mu (m+n)^p f \left( \frac{mx+ny}{(m+n)^p} \right) - m(m+n)^p f \left( \frac{x}{(m+n)^p} \right) - n(m+n)^p f \left( \frac{y}{(m+n)^p} \right) \geq \Phi_{x,y} \left( \frac{t}{(m+n)^p} \right)
   \]
   for all \( x, y \in X, t > 0 \) and \( n \geq 1 \) and so, from (3.2), it follows that

   \[
   \Phi \left( \frac{x}{(m+n)^p} \right) \left( \frac{t}{(m+n)^p} \right) \geq \Phi_{x,y} \left( \frac{t}{((m+n)\alpha)^p} \right)
   \]. Since \( \lim_{p \to \infty} \Phi_{x,y} \left( \frac{t}{((m+n)\alpha)^p} \right) = 1 \) for all \( x, y \in X \) and \( t > 0 \), we have

   \[
   \mu A(mx+ny) - mA(x) - nA(y)(t) = 1 \] for all \( x, y \in X \) and \( t > 0 \). This completes the proof. \( \square \)
Corollary 3.1. Let $X$ be a real normed space, $\theta \geq 0$ and $p$ be a real number with $p > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying

$$\mu_{f(mx+ny)-mf(x)-nf(y)}(t) \geq t(t + \theta(\|x\|^p + \|y\|^p))^{-1}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$, $A(x) = \lim_{p \to \infty} (m+n)^p f\left(\frac{x}{(m+n)^p}\right)$ exists and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq (((m+n)^p - (m+n))t)(((m+n)^p - (m+n))t + \theta\|x\|^p)^{-1}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.2 if we take $\Phi_{x,y}(t) = t(t + \theta(\|x\|^p + \|y\|^p))^{-1}$ for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = (m+n)^{-p}$, then we get the desired result.

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References


