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THE CATEGORY OF PARTIAL ACTIONS OF A GROUP: SOME CONSTRUCTIONS

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Abstract: In this paper we introduce the category G-**pAct** of partial actions of a fixed group G. The objects or G-psets are the sets X endowed with a partial action of G on X and the morphisms, or preferably G-pmorphisms, are the maps preserving this action. As a special achievement, we extend several well-known constructions in the category G-**Act**, of global actions of G, to this new context. In particular, we characterize products, coproducts, equalizers and pullbacks for arbitrary G-pmorphisms. We also characterize coequalizers and pushouts for strong G-pmorphisms (category G-**fpAct**). Last, we prove that the category G-**pAct** is complete and the category G-**fpAc**t is cocomplete.

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1. Introduction

Category theory was introduced in 1945 by Eilenberg and MacLane ([4]). However, its relevance was not recognized until the 1960s, when the works of Lawvere

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on logic, foundations of mathematics, the category of sets and closed Cartesian categories became known. During the 1970s it was finally accepted that category theory was a genuine branch of mathematics, with tools, problems and specialized techniques. The success of the theory is due mainly to the language, which generalizes many constructions that occur in many areas of mathematics. Furthermore, it shelters many types of structures that were, at first, considered dissimilar.

More recently, in 1998, Exel ([5]) introduced the notion of partial action of a group in his works on operator theory. He used collections of subsets of the base space and bijections between them (partial bijections of the space). The importance of this theory lies in its applicability to many fields of mathematics, as numerous recent publications have shown (see [3], [7] and its references). In particular, these notions have been successfully used to extend classical results on dynamical systems ([6]), topological and metric spaces ([8],[9]), rings ([7]) and representation theory ([3]), among others fields.

In the current literature there is no categorical context for partial actions nor for characterizations of objects and morphisms. In Section 2 we present first the category of global actions of a fixed group G (G-**Act**). Then, we construct the category of partial actions of G (G-**pAct**). In Section 3 we characterize the products, coproducts and equalizers in the category G-**pAct**. Additionally, we introduce the notion of strong G-pmorphism (category G-**fpAct**) and characterize the coequalizers and pushouts in the category G-**fpAct**. Finally, we prove that the category G-**pAct** is complete and the category G-**fpAct** is cocomplete.

2. The Category G-pAct

Recall that a global action of a group G, with identity element 1, on the set X is a function $\cdot : G \times X \to X$ such that:

1. $1 \cdot x = x$ for every $x \in X$.
2. $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and all $x \in X$.

In this case X is called G-set.

The category of global actions of a fixed group G, denoted by G-**Act**, is one whose objects are the sets X endowed with a global action of G on X. The morphisms between the objects X, Y, called G-morphisms, are the maps $f : X \to Y$ such that for all $x \in X$ and all $g \in G$, $f(g \cdot x) = g \cdot f(x)$. This category is very important because it is a topoi. Moreover, there is significant literature on applications of this theory in other sciences like physics and computer science. In mathematics, many constructions within this category as products, equalizers and pullbacks, among others concepts, are natural extensions of those well-known in the category of sets.

Global actions can be generalized in several ways, including the actions of monoids and the grupoid actions. More recently, Exel in 1998 ([5]) in his studies on operator theory introduced the notion of partial action of a group by considering partial bijections.

Definition 1. A partial action α of the group G on the set X is a collection of subsets S_g, $g \in G$, of X and bijections $\alpha_g : S_{g^{-1}} \to S_g$ such that for all $g, h \in G$ the following statements hold:

1. $S_1 = X$ and α_1 is the identity of X.
2. $S_{(gh)^{-1}} \supseteq \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.
3. $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for all $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

In this case the set X is called G-pset. The partial action α, will be denoted by $\alpha = \{S_g, \alpha_g\}_{g \in G}$. Note that item 3. implies $\alpha_g^{-1} = \alpha_{g^{-1}}$ for all $g \in G$.

For a fixed group G we call objets to G-psets. Now, for X and Y two G-psets, with partial actions $\alpha_X = \{X_g, {}_X\alpha_g\}_{g \in G}$ and $\alpha_Y = \{Y_g, {}_Y\alpha_g\}_{g \in G}$ respectively, we call G-pmorphism to maps $f : X \to Y$ such that:

$$x \in X_{g^{-1}} \text{ implies } f(x) \in Y_{g^{-1}}, \text{ and } f({}_X\alpha_g(x)) = {}_Y\alpha_g(f(x)).$$

The usual composition between G-pmorphisms is a G-pmorphism and the identity map id_X is a G-pmorphism for each object X. Thus, the G-psets together with the G-pmorphisms, the usual identity maps and the usual composition define a category, which will be called category of partial actions of G and it will be denoted by G-**pAct**.

Although the category G-**pAct** is a natural extension of the category G-**Act**, it has not been explicitly introduced in the current literature. Similarly, there are no studies on characterizations of objects and morphisms in this category. Since the category G-**Act** is a subcategory of G-**pAct**, it is natural to extend known results of the category G-**Act** to category G-**pAct**.

3. Results

In this section we extend some well known constructions in the set category and the global actions category. In particular, we characterize the products, coproducts, equalizers and pullbacks for arbitrary G-pmorphisms and the coequalizers and pushouts for strong G-pmorphisms. We assume that the reader is familiarized with the techniques and basic concepts of category theory, which can be found in [1].

3.1. Products

It is clear that the Cartesian product of a family of sets is a set and more generally the Cartesian product of a family of G-sets is also a G-set. In this first part we show that the former results can be naturally generalized to the category G-**pAct**. We begin by extending Affirmation 1 of [2] for an arbitrary family of G-psets.

Proposition 2. *If $\{X_i\}_{i\in I}$ is a collection of G-psets, then $\prod_{i\in I} X_i = \{f : I \to \bigcup_{i\in I} X_i \mid f(i) \in X_i$ for every $i \in I\}$ is a G-pset.*

Proof. Consider the collection of G-psets $\{X_i\}_{i\in I}$, where the partial action on X_i is given by $\alpha_i = \{{}_iX_g, {}_i\alpha_g\}_{g\in G}$ for each $i \in I$. For $\prod_{i\in I} X_i$, take $\prod_{i\in I} {}_iX_g$ and $\prod_{i\in I} {}_i\alpha_g : \prod_{i\in I} {}_iX_{g^{-1}} \to \prod_{i\in I} {}_iX_g$ defined by $\prod_{i\in I} {}_i\alpha_g(x) = ({}_i\alpha_g(x_i))_{i\in I}$ for each $x \in \prod_{i\in I} {}_iX_{g^{-1}}$. It is clear that $\prod_{i\in I} {}_i\alpha_g$ is a well defined function and it is a bijection. Now we verify 2. and 3. of Definition 1, since 1. is evident.

2. If $x = (x_i)_{i\in I} \in (\prod_{i\in I} {}_i\alpha_h)^{-1}\ (\prod_{i\in I} {}_iX_h \cap \prod_{i\in I} {}_iX_{g^{-1}})$ then

$$(\prod_{i\in I} {}_i\alpha_h)(x) = ({}_i\alpha_h(x_i))_{i\in I} \in \prod_{i\in I}({}_iX_h \cap {}_iX_{g^{-1}}).$$

Thus for each $i \in I$ it has $x_i \in {}_i\alpha_h^{-1}({}_iX_h \cap {}_iX_{g^{-1}}) \subseteq {}_iX_{(gh)^{-1}}$. That is, $x = (x_i)_{i\in I} \in \prod_{i\in I} {}_iX_{(gh)^{-1}}$.

3. For all $x = (x_i)_{i\in I} \in (\prod_{i\in I} {}_i\alpha_h)^{-1}\ (\prod_{i\in I} {}_iX_h \cap \prod_{i\in I} {}_iX_{g^{-1}})$ we have that $(\prod_{i\in I} {}_i\alpha_g \circ \prod_{i\in I} {}_i\alpha_h)(x) = \prod_{i\in I} {}_i\alpha_g(\prod_{i\in I} {}_i\alpha_h(x)) = \prod_{i\in I} {}_i\alpha_g(({}_i\alpha_h(x_i))_{i\in I}) = w$ with $w = ({}_i\alpha_g(({}_i\alpha_h(x_i))_{i\in I}))_{i\in I}$. Since for each $i \in I$ we obtain

$$x_i \in ({}_i\alpha_h)^{-1}({}_iX_h \cap {}_iX_{g^{-1}}),$$

then

$${}_i\alpha_g({}_i\alpha_h(x_i)) = {}_i\alpha_{gh}(x_i).$$

Hence, $w = ({}_i\alpha_{gh}(x_i))_{i\in I} = \prod_{i\in I} {}_i\alpha_{gh}(x)$. □

Note that each projection $\pi_i : \prod_{i\in I} X_i \longrightarrow X_i$, $i \in I$, defined by $\pi_i(x) = x_i$ for each $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$ is a G-pmorphism.

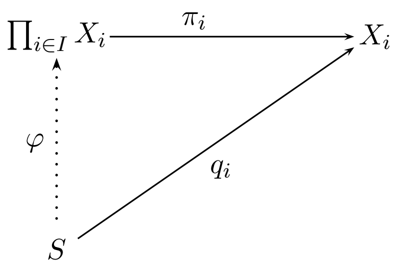
Theorem 3. *If $\{X_i\}_{i\in I}$ is a collection of G-psets, then the source $\mathcal{P} = (\prod_{i\in I} X_i \stackrel{\pi_i}{\to} X_i)_{i\in I}$ is the product of the familly $\{X_i\}_{i\in I}$ in the category G-**pAct**.*

Proof. Consider the source $\mathcal{P} = (\prod_{i\in I} X_i \stackrel{\pi_i}{\to} X_i)_{i\in I}$ and suppose that there exists other source $\mathcal{S} = (S \stackrel{q_i}{\longrightarrow} X_i)_{i\in I}$. Define $S \stackrel{\varphi}{\longrightarrow} \prod_{i\in I} X_i$ by $\varphi(s) = (q_i(s))_{i\in I}$ for each $s \in S$. First, we prove that φ is a G-pmorphism. If $s \in S_{g^{-1}}$ then for each $i \in I$ we have $q_i(s) \in {}_iX_{g^{-1}}$ and thus $\varphi(s) = (q_i(s))_{i\in I} \in \prod_{i\in I} {}_iX_{g^{-1}}$. Moreover,

$$\begin{aligned}\prod_{i\in I} {}_i\alpha_g(\varphi(s)) &= \prod_{i\in I} {}_i\alpha_g(r) \text{ with } r = (q_i(s))_{i\in I} \\ &= w \text{ with } w = ({}_i\alpha_g(r))_{i\in I}.\end{aligned}$$

Then, $w = ({}_i\alpha_g((q_i(s))_{i\in I}))_{i\in I} = ({}_i\alpha_g(q_i(s)))_{i\in I}$ and since for each $i \in I$, q_i is a G-pmorphism then $({}_i\alpha_g(q_i(s))) = q_i({}_S\alpha_g(s))$ and $w = (q_i({}_S\alpha_g(s)))_{i\in I} = \varphi({}_S\alpha_g(s))$.

Now we must prove that the following diagram commutes for each $i \in I$:



For $s \in S$ we have $(\pi_i \circ \varphi)(s) = \pi_i(\varphi(s)) = \pi_i((q_i(s))_{i\in I}) = q_i(s)$. That is, $\pi_i \circ \varphi = q_i$. Moreover, φ is the unique morphism that makes the diagram commutative. In fact, if there exists $\varphi' : S \to \prod_{i\in I} X_i$, $\varphi'(s) = x$ with $x = (x_i)_{i\in I}$ such that $\pi_i \circ \varphi' = q_i$ for each $i \in I$, then $q_i(s) = (\pi_i \circ \varphi')(s) = \pi_i(\varphi'(s)) = \pi_i(x) = x_i$, and thus $\varphi'(s) = (q_i(s))_{i\in I} = \varphi(s)$ for all $s \in S$. Hence, $\varphi = \varphi'$. □

3.2. Coproductos

For the following proposition we assume, without loss of generality, that each pair of different G-psets have empty intersection. The following result extends Affirmation 2 of [2] for an arbitrary family of G-psets.

Proposition 4. *If $\{X_i\}_{i\in I}$ is a collection of G-psets, then $\coprod_{i\in I} X_i$ is a G-pset.*

Proof. Let $\{X_i\}_{i\in I}$ be a collection of G-psets, where each X_i, $i \in I$, has the partial action $\alpha_i = \{{}_iX_g, {}_i\alpha_g\}_{g\in G}$. Take $\coprod_{i\in I} X_i$ and for each $g \in G$, define $\coprod_{i\in I} {}_iX_g$. Since for each $x \in \coprod_{i\in I} {}_iX_{g^{-1}}$ there exists a unique $i \in I$ such that $x \in {}_iX_{g^{-1}}$, then we define $\coprod_{i\in I} {}_i\alpha_g : \coprod_{i\in I} {}_iX_{g^{-1}} \to \coprod_{i\in I} {}_iX_g$ as $\coprod_{i\in I} {}_i\alpha_g(x) = {}_i\alpha_g(x)$ for each $x \in \coprod_{i\in I} {}_iX_{g^{-1}}$. It is clear that $\coprod_{i\in I} {}_i\alpha_g$, $g \in G$, is a well defined function and it is a bijection. We must prove that $\{\coprod_{i\in I} {}_iX_g, \coprod_{i\in I} {}_i\alpha_g\}_{g\in G}$ defines a partial action on $\coprod_{i\in I} X_i$. We verify 2. and 3. of Definition 1 since 1. is evident.

2. If $x \in (\coprod_{i\in I} {}_i\alpha_h)^{-1} (\coprod_{i\in I} {}_iX_h \cap \coprod_{i\in I} {}_iX_{g^{-1}})$, then there exists a unique $i \in I$ such that ${}_i\alpha_h(x) \in ({}_iX_h \cap {}_iX_{g^{-1}})$. So, $x \in {}_i\alpha_h{}^{-1}({}_iX_h \cap {}_iX_{g^{-1}}) \subseteq {}_iX_{(gh)^{-1}}$ and thus $x \in \coprod_{i\in I} {}_iX_{(gh)^{-1}}$.

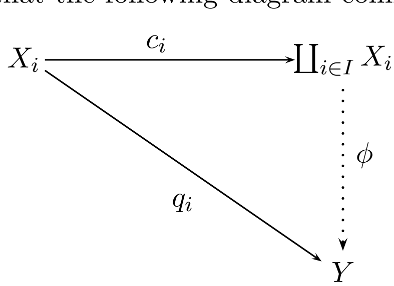
3. If $x \in (\coprod_{i\in I} {}_i\alpha_h)^{-1} (\coprod_{i\in I} {}_iX_h \cap \coprod_{i\in I} {}_iX_{g^{-1}})$ then there exists a unique $i \in I$ such that $x \in ({}_i\alpha_h)^{-1}({}_iX_h \cap {}_iX_{g^{-1}})$. Thus $(\coprod_{i\in I} {}_i\alpha_g \circ \coprod_{i\in I} {}_i\alpha_h)(x) = {}_i\alpha_g({}_i\alpha_h(x)) = {}_i\alpha_{gh}(x) = \coprod_{i\in I} {}_i\alpha_{gh}(x)$ □

Note that each $c_i : X_i \to \coprod_{i\in I} X_i$, $i \in I$, defined by $c_i(x) = x$ for each $x \in X_i$ is a G-pmorphism.

Theorem 5. *Let $\{X_i\}_{i\in I}$ be a collection of G-psets. The sink $\mathcal{C} = (X_i \xrightarrow{c_i} \coprod_{i\in I} X_i)_{i\in I}$ is the coproduct of the family $\{X_i\}_{i\in I}$ in the category G-**pAct**.*

Proof. Consider the sink $\mathcal{C} = (X_i \xrightarrow{c_i} \coprod_{i\in I} X_i)_{i\in I}$ and suppose that there exists another sink $\mathcal{S} = (X_i \xrightarrow{q_i} Y)_{i\in I}$. We define $\phi : \coprod_{i\in I} X_i \longrightarrow Y$ as $\phi(x) = q_i(x)$ for each $x \in X$. First, we must see that ϕ is a G-pmorphism. If $x \in \coprod_{i\in I} {}_iX_{g^{-1}}$ there exists a unique $i \in I$ such that $x \in {}_iX_{g^{-1}}$, then $\phi(x) = q_i(x) \in Y_{g^{-1}}$. Moreover, $\phi(\coprod_{i\in I} {}_i\alpha_g(x)) = \phi({}_i\alpha_g(x)) = q_i({}_i\alpha_g(x)) = {}_Y\alpha_g(q_i(x)) = {}_Y\alpha_g(\phi(x))$.

Now we must prove that the following diagram commutes for each $i \in I$:



If $x \in X_i$ then $(\phi \circ c_i)(x) = \phi(c_i(x)) = \phi(x) = q_i(x)$. That is, $\phi \circ c_i = q_i$ for each $i \in I$. Finally, ϕ is the unique G-pmorphism that makes the diagram commutative. In fact, if there exists $\phi' : \coprod_{i \in I} X_i \to Y$ such that $\phi' \circ c_i = q_i$ for all $i \in I$, then for each $x \in \coprod_{i \in I} X_i$ we have $\phi(x) = (\phi \circ c_i)(x) = (\phi' \circ c_i)(x) = \phi'(x)$. Thus, $\phi = \phi'$. □

3.3. Equalizers

The G-invariant sets, those that remain fixed under the action of G, are very important in the theory of global actions. The analogous notion in partial actions is the following.

Definition 6. Let $\alpha = \{X_g, {}_X\alpha_g\}_{g \in G}$ be a partial action of G on X and $A \subseteq X$. A is called G-pinvariant if ${}_X\alpha_g(A \cap X_{g^{-1}}) = A \cap X_g$ for all $g \in G$.

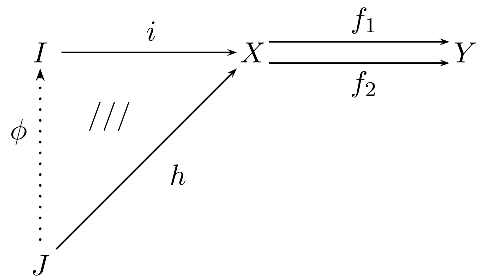
To determine if the subset A is G-pinvariant, it is enough to prove ${}_X\alpha_g(A \cap X_{g^{-1}}) \subseteq A \cap X_g$ for all $g \in G$. Moreover, note that the collection $\alpha_A = \{A \cap X_g, {}_A\alpha_g\}_{g \in G}$ where ${}_A\alpha_g$ is defined by restriction of ${}_X\alpha_g$ to $A \cap X_{g^{-1}}$, defines a partial action of G on A. In other words, every G-pinvariant set is a G-pset.

Proposition 7. *Let X, Y be two G-psets. If f_1 and $f_2 : X \to Y$ are G-pmorphisms, then set $I = \{x \in X \mid f_1(x) = f_2(x)\}$ is a G-pset.*

Proof. It is enough to see that I is G-pinvariant, that is, ${}_X\alpha_g(I \cap X_{g^{-1}}) \subseteq I \cap X_g$ for all $g \in G$. If $g \in G$ and $x \in I \cap X_{g^{-1}}$, then ${}_X\alpha_g(x) \in X_g$. Moreover, $f_1({}_X\alpha_g(x)) = {}_Y\alpha_g(f_1(x)) = {}_Y\alpha_g(f_2(x)) = f_2({}_X\alpha_g(x))$ which implies that ${}_X\alpha_g(x) \in I$. So, ${}_X\alpha_g(x) \in I \cap X_g$ and thus I is a G-pinvariant set. In consequence, I is a G-pset. □

Theorem 8. *Let X, Y be two G-psets and $f_1, f_2 : X \to Y$ two G-pmorphisms. The G-pset $I = \{x \in X \mid f_1(x) = f_2(x)\}$ together with the inclusion G-pmorphism $i : I \to X$, $i(a) = a$ for all $a \in I$, is an equalizer of f_1 and f_2 in the category G-**pAct**.*

Proof. It is clear that i satisfies $f_1 \circ i = f_2 \circ i$. Now, if $h : J \to X$ is another G-pmorphism such that $f_1 \circ h = f_2 \circ h$ then we must see that there exists a unique G-pmorphism $\phi : J \to I$, such that $h = i \circ \phi$.



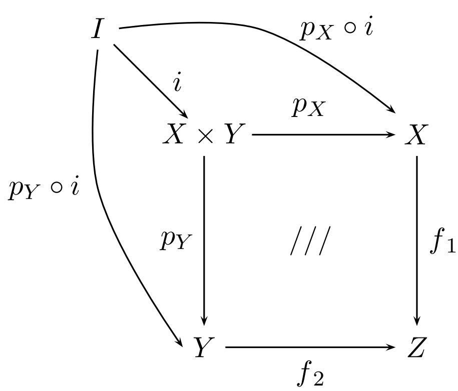
If $j \in J$ then $f_1(h(j)) = f_2(h(j))$, which implies that $h(j) \in I$. So, the map $\phi : J \to I$ defined by $\phi(j) = h(j)$ for all $j \in J$ is a G-pmorphism and $i \circ \phi = h$. Suppose that there exists a G-pmorphism $\phi' : J \to I$ such that $i \circ \phi' = h$. Then, for $j \in J$ it has that $\phi'(j) = i(\phi'(j)) = i(\phi(j)) = \phi(j)$. That is, $\phi = \phi'$ and hence ϕ is unique. □

As a direct consequence of the results above we obtain the following corollary.

Corollary 9. *1. The category G-**pAct** has pullbacks.*

*2. The category G-**pAct** is complete.*

Proof. 1. Suppose that X, Y, Z are G-psets and $f_1 : X \to Z$, $f_2 : Y \to Z$ are G-pmorphisms. Then, a pullback for (Z, f_1, f_2) is given by $(I, p_X \circ i, p_Y \circ i)$, where (I, i) is the equalizer of $f_1 \circ p_X : X \times Y \to Z$ and $f_2 \circ p_Y : X \times Y \to Z$ with p_X, p_Y the usual projections. That construction can be appreciate in the following diagram.



2. It is a direct consequence of the fact that the category G-**pAct** has products and pullbacks (see Theorem 12.3 of [1]). □

4. Coequalizers

We note that if $f : X \to Y$ is a G-pmorphism, then the condition $f(x) \in Y_g$ for some $g \in G$ and some $x \in X$ does not imply that $x \in X_g$. This situation is true in global actions since $X_g = X$ and $Y_g = Y$ for all $g \in G$. However, this condition is crucial to consider partial actions since it is used in the construction of coequalizers and pushouts.

Definition 10. Let X, Y be two G-psets. A G-pmorphism $f : X \to Y$ is called strong if $f(x) \in Y_{g^{-1}}$ implies that $x \in X_{g^{-1}}$.

Definition 11. An equivalence relation $\mathcal{E}$ on the G-pset X is called a G-pcongruence if the following conditions hold:

1. $x \in X_g$ implies $[x] \subseteq X_g$.
2. $x, y \in X_{g^{-1}}$ and $x \,\mathcal{E}\, y$ imply ${}_X\alpha_g(x) \,\mathcal{E}\, {}_X\alpha_g(y)$.

Proposition 12. *Let Y be a G-pset and $\mathcal{E}_Y$ a G-pcongruence on Y. Then:*

1. *$Y/\mathcal{E}_Y = \{[y] \mid y \in Y\}$ is a G-pset.*
2. *The canonical map $\tau_{\mathcal{E}_Y} : Y \to Y/\mathcal{E}_Y$ defined by $\tau_{\mathcal{E}_Y}(y) = [y]$ for each $y \in Y$, is an strong G-pmorphism.*

Proof. 1. Suppose that the partial action on Y is given by, $\alpha_Y = \{Y_g, {}_Y\alpha_g\}_{g \in G}$ and consider the G-pcongruence $\mathcal{E}_Y$ on Y. We define $\overline{Y}_g = \{[y] \mid y \in Y_g\}$ and $\overline{\alpha}_g : \overline{Y}_{g^{-1}} \to \overline{Y}_g$ by $\overline{\alpha}_g([y]) = [{}_Y\alpha_g(y)]$. We must see that $\overline{\alpha}_g$, $g \in G$, is a well defined map. If $[x], [y] \in \overline{Y}_{g^{-1}}$ with $[x] = [y]$, then $x, y \in Y_{g^{-1}}$ and $x \,\mathcal{E}_Y\, y$. Since $\mathcal{E}_Y$ is a G-pcongruence it has ${}_Y\alpha_g(x) \,\mathcal{E}_Y\, {}_Y\alpha_g(y)$. Thus, $[{}_Y\alpha_g(x)] = [{}_Y\alpha_g(y)]$. Now, it is clear that each $\overline{\alpha}_g$, $g \in G$, is a bijection.

Finally, for the remains it is enough to apply the properties of the partial action on Y.

2. By definition we have that $y \in Y_{g^{-1}}$ implies $[y] \subseteq \overline{Y}_{g^{-1}}$. Moreover, $\tau_{\mathcal{E}_Y}({}_Y\alpha_g(y)) = [{}_Y\alpha_g(y)] = \overline{\alpha}_g([y]) = \overline{\alpha}_g(\tau_{\mathcal{E}_Y}(y))$. That is, $\tau_{\mathcal{E}_Y}$ is a G-pmorphism. Finally, take $y \in Y$ such that $\tau_{\mathcal{E}_Y}(y) = [y] \in \overline{Y}_{g^{-1}}$. Then, there exists $y' \in Y_g$ such that $[y] = \overline{\alpha}_{g^{-1}}([y']) = [{}_Y\alpha_{g^{-1}}(y')] \subseteq Y_{g^{-1}}$. In consequence, $\tau_{\mathcal{E}_Y}$ is an strong G-pmorphism. □

Note that the G-pmorphisms $f_1, f_2 : X \to Y$ define the relation $R_Y = \{(f_1(x), f_2(x)) \mid x \in X\}$ on Y. The minimal equivalence relation $\mathcal{E}_Y$ which contains R_Y is given by $\mathcal{E}_Y = R_Y \cup R_Y^{-1} \cup \Delta_Y \cup \Omega_Y$, where $\Delta_Y = \{(y, y) \mid y \in Y\}$ and $\Omega_Y = \{(z, w) \mid$ there exist $y_i \in Y$ with $i = 1, 2, ..., n$ such that $z = y_1,\ y_n = w$ and $(y_1, y_2), (y_2, y_3), ..., (y_{n-1}, y_n) \in R_Y \cup R_Y^{-1} \cup \Delta_Y\}$.

Proposition 13. *If $f_1, f_2 : X \to Y$ are strong G-pmorphisms, then the equivalence relation $\mathcal{E}_Y$ defined above is a G-pcongruence.*

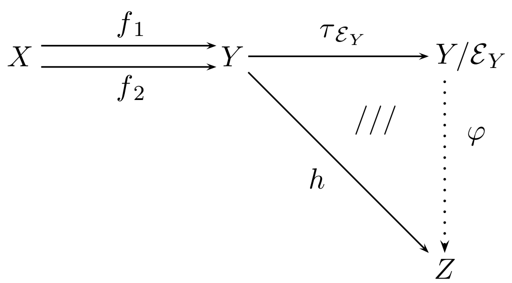
Theorem 14. *Let $f_1, f_2 : X \longrightarrow Y$ be two strong G-pmorphisms and $\mathcal{E}_Y$ the G-pcongruence induced by f_1 and f_2. The G-pset $Y/\mathcal{E}_Y$ together with the canonical map $\tau_{\mathcal{E}_Y}$ is a coequalizer of f_1 and f_2 in the category G-**pAct**.*

Proof. It is clear that $\tau_{\mathcal{E}_Y} \circ f_1 = \tau_{\mathcal{E}_Y} \circ f_2$. If $h : Y \longrightarrow Z$ is an strong G-pmorphism with $h \circ f_1 = h \circ f_2$, then we define $\varphi : Y/\mathcal{E}_Y \longrightarrow Z$ as $\varphi([y]) = h(y)$ for each $[y] \in Y/\mathcal{E}_Y$.

We must see that φ is a well defined function. If $[y_1] = [y_2]$ then $y_1\ \mathcal{E}_Y\ y_2$ and so there exists $x \in X$ such that $f_1(x) = y_1$ and $f_2(x) = y_2$. Thus $(h \circ f_1)(x) = h(y_1) = (h \circ f_2)(x) = h(y_2)$, that is, $\varphi([y_1]) = \varphi([y_2])$.

φ is an strong G-pmorphism. In fact, if $[y] \in \overline{Y}_{g^{-1}}$, then $y \in Y_{g^{-1}}$ which implies that $h(y) = \varphi([y]) \in Z_{g^{-1}}$. In addition, $\varphi(\overline{\alpha}_g([y])) = \varphi([{}_Y\alpha_g(y)]) = h({}_Y\alpha_g(y)) = {}_Z\alpha_g(h(y)) = {}_Z\alpha_g(\varphi([y]))$. Finally, if $\varphi([y]) = h(y) \in Z_{g^{-1}}$ for some $y \in Y$, then $y \in Y_{g^{-1}}$ and hence $[y] \in \overline{Y}_{g^{-1}}$.

If $y \in Y$, $(\varphi \circ \tau_{\mathcal{E}_Y})(y) = \varphi(\tau_{\mathcal{E}_Y}(y)) = \varphi([y]) = h(y)$, that is, $\varphi \circ \tau_{\mathcal{E}_Y} = h$. So, we have the following commutative diagram.



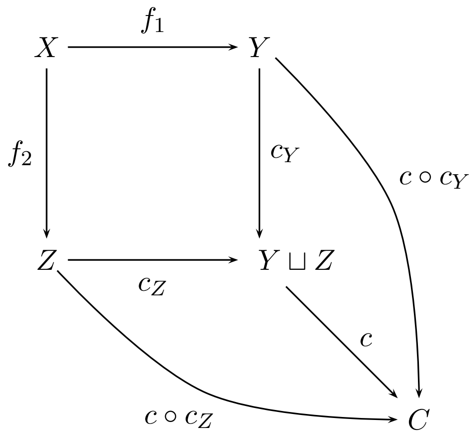
Suppose that $\varphi' : Y/\mathcal{E}_Y \longrightarrow Z$ is another strong G-pmorphism such that $\varphi' \circ \tau_{\mathcal{E}_Y} = h$. Then, for $y \in Y$ we have that $\varphi'([y]) = \varphi'(\tau_{\mathcal{E}_Y}(y)) = \varphi(\tau_{\mathcal{E}_Y}(y)) = \varphi([y])$, which implies that $\varphi' = \varphi$. In consequence, the strong G-pmorphism φ is unique. □

As a direct consequence of the results above we obtain the following corollary. Note that we can consider the category G-**fpAct**, in which the objects are the G-psets and the morphisms are the strong G-pmorphisms. Moreover, it is clear that the category G-**fpAct** has coproducts since the canonical injections are strong G-pmorphisms.

Corollary 15. *1. The category G-**fpAct** has pushouts.*

*2. The category G-**fpAct** is cocomplete.*

Proof. 1. Suppose that X, Y, Z are G-psets and $f_1 : X \to Y$ and $f_2 : X \to Z$ are strong G-pmorphisms. Then, a pushout for (X, f_1, f_2) is given by $(C, c \circ c_Y, c \circ c_Z)$, where (C, c) is the coequalizer of the morphisms $c_Y \circ f_1 : X \to Y \sqcup Z$ and $c_Z \circ f_2 : X \to Y \sqcup Z$ with c_Y, c_Z the usual inclusions. This construction can be appreciate in the following diagram.



2. The result follows by duality since the category G-**fpAct** has coproducts and pushouts (see Theorem 12.3 of [1]). □

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