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ON NEGATIVE BINOMIAL APPROXIMATION FOR GEOMETRIC RANDOM SUMS

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Abstract: We give a non-uniform bound for the distance between the distribution function of random sums of independent geometric random variables and an appropriate negative binomial distribution function. Two examples have been given to illustrate the result obtained.

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1. Introduction

Let S_n be a sum $\sum_{i=1}^{n} X_i$ of independently distributed geometric random variables, where $P(X_i = k) = (1-p_i)^k p_i$ for $k = 0, 1, \ldots$. It is well-known that if all $q_i = (1-p_i)$ are small, the distribution of S_n can be approximated by the Poisson distribution with mean $E(S_n) = \sum_{i=1}^{n} \frac{q_i}{p_i}$ [2]. In addition, the distribution of S_n can also be approximated by the negative binomial distribution with parameters n and $p = 1-q = \frac{1}{n}\sum_{i=1}^{n} p_i$ [5]. For $x \in \mathbb{N} \cup \{0\}$, let $\mathbb{S}_n(x) = P(S_n \leq x)$ and $\mathbb{NB}_{n,p}(x) = P(NB_{n,p} \leq x)$, where $NB_{n,p}$ is the negative binomial random variable with parameters n and p. Suppose N is a non-negative integer valued random variable and independent of the X_i's,

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then S_N is the random sums of independent geometric random variables. Let $\lambda_N = \sum_{i=1}^{N} \frac{q_i}{p_i}$, $\widehat{n} = E(N)$ and $\widehat{q} = 1 - \widehat{p} = \frac{\lambda}{\widehat{n}+\lambda}$, where $\lambda = E(\lambda_N)$. In this paper, we are interested to give a non-uniform bound for the distance between $\mathbb{S}_N(x)$ and $\mathbb{NB}_{\widehat{n},\widehat{p}}(x)$, $|\mathbb{S}_N(x) - \mathbb{NB}_{\widehat{n},\widehat{p}}(x)|$, for $x \in \mathbb{N}$, which is in Section 2. In Section 3, two examples have been given to illustrate the desired result. The conclusion of this study is presented in the last section.

2. Result

The following theorem presents a non-uniform bound for the distance between $\mathbb{S}_N(x)$ and $\mathbb{NB}_{\widehat{n},\widehat{p}}(x)$.

Theorem 2.1. *For $x \in \mathbb{N}$ and $\lambda = E(\lambda_N)$, then*

$$\left|\mathbb{S}_N(x) - \mathbb{NB}_{\widehat{n},\widehat{p}}(x)\right| \le \min\left\{E\left(\frac{1-e^{-\lambda_N}}{\lambda_N}\sum_{i=1}^{N}\left(\frac{q_i}{p_i}\right)^2\right), \frac{1}{x}E\left(\sum_{i=1}^{N}\left(\frac{q_i}{p_i}\right)^2\right)\right\} + \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x}\right\}\left(Var(\lambda_N) + \frac{\lambda^2}{\widehat{n}}\right), \tag{1}$$

where $\mathbb{S}_N(0) = E(\prod_{i=1}^{N} p_i)$.

Proof. Let $\mathbb{P}_\lambda(x)$ be the Poisson distribution function with mean λ at x. It follows the fact that

$$\begin{aligned}\left|\mathbb{S}_N(x) - \mathbb{NB}_{\widehat{n},\widehat{p}}(x)\right| &\le |\mathbb{S}_N(x) - \mathbb{P}_{\lambda_N}(x)| + |\mathbb{P}_{\lambda_N}(x) - \mathbb{P}_\lambda(x)| \\ &\quad + \left|\mathbb{P}_\lambda(x) - \mathbb{NB}_{\widehat{n},\widehat{p}}(x)\right| \\ &\le \sum_{n=0}^{\infty} P(N=n)\,|\mathbb{S}_n(x) - \mathbb{P}_{\lambda_n}(x)| + |\mathbb{P}_{\lambda_N}(x) - \mathbb{P}_\lambda(x)| \\ &\quad + \left|\mathbb{P}_\lambda(x) - \mathbb{NB}_{\widehat{n},\widehat{p}}(x)\right|. \end{aligned} \tag{2}$$

Following [4], we have

$$|\mathbb{S}_n(x) - \mathbb{P}_{\lambda_n}(x)| \le \min\left\{\frac{1-e^{-\lambda_n}}{\lambda_n}, \frac{1}{x}\right\}\sum_{i=1}^{n}\left(\frac{q_i}{p_i}\right)^2. \tag{3}$$

Applying Theorem 1.C in [1] together with Lemma 2.1 in [3] and using Corollary 4.2 in [3], we also obtain

$$|\mathbb{P}_{\lambda_N}(x) - \mathbb{P}_\lambda(x)| \le \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x}\right\} Var(\lambda_N) \tag{4}$$

and

$$\left|\mathbb{P}_\lambda(x) - \mathbb{NB}_{\widehat{n},\widehat{p}}(x)\right| \le \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x}\right\}\frac{\widehat{n}\widehat{q}^2}{\widehat{p}^2} = \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x}\right\}\frac{\lambda^2}{\widehat{n}}, \tag{5}$$

respectively. Hence, the inequality (1) is obtained by taking the bounds in (3), (4) and (5) into (2). □

If X_i's are identically distributed, then the following corollary is an immediately consequence of the Theorem 2.1.

Corollary 2.1. *For $x \in \mathbb{N}$, if $p_1 = p_2 = \cdots = p$, then we have the following:*

$$\begin{aligned}\left|\mathbb{S}_N(x) - \mathbb{NB}_{\widehat{n},p}(x)\right| &\le \min\left\{E\left(1-e^{-\frac{Nq}{p}}\right), \frac{\widehat{n}q}{xp}\right\}\frac{q}{p} \\ &+ \min\left\{\frac{1-e^{-\frac{\widehat{n}q}{p}}}{\frac{\widehat{n}q}{p}}, \frac{1}{x}\right\}(Var(N)+\widehat{n})\left(\frac{q}{p}\right)^2, \end{aligned}\tag{6}$$

where $\mathbb{S}_N(0) = E(p^N)$.

3. Examples

We give two examples to illustrate the result in the case of X_i's are identically distributed.

Example 3.1. For n ($n \in \mathbb{N}$) is fixed, let N be a positive integer-valued random variable with probability function

$$P(N=k) = \begin{cases} \frac{1}{2} & , k = n, \\ \frac{1}{2} & , k = 2n, \\ 0 & , \text{otherwise.} \end{cases}$$

Therefore $\widehat{n} = E(N) = \frac{3n}{2}$ and $Var(N) = \frac{n^2}{4}$. Let $p_1 = p_2 = \cdots = p$, then for $x \in \mathbb{N}$, we have

$$\begin{aligned}\left|\mathbb{S}_N(x) - \mathbb{NB}_{\frac{3n}{2},p}(x)\right| &\le \min\left\{1 - \frac{e^{-\frac{nq}{p}} + e^{-\frac{2nq}{p}}}{2}, \frac{3nq}{2xp}\right\}\frac{q}{p} \\ &+ \min\left\{\frac{1-e^{-\frac{\widehat{n}q}{p}}}{\frac{\widehat{n}q}{p}}, \frac{1}{x}\right\}\left(\frac{n^2}{4}+\frac{3n}{2}\right)\left(\frac{q}{p}\right)^2.\end{aligned}$$

Example 3.2. Let N be a positive integer-valued random variable with probability function

$$P(N=n)=\frac{1}{2^n},\ n=1,2,...,$$

then we have $\widehat{n}=E(N)=2$ and $Var(N)=2$. If $p_1=p_2=\cdots=p$, then for $x\in\mathbb{N}$, we obtain

$$\left|\mathbb{S}_N(x)-\mathbb{NB}_{\widehat{n},p}(x)\right|\le\min\left\{1,\frac{2q}{xp}\right\}\frac{q}{p}+\min\left\{\frac{1-e^{-\frac{2q}{p}}}{\frac{2q}{p}},\frac{1}{x}\right\}\left(\frac{2q}{p}\right)^2.$$

4. Conclusion

In this study, a non-uniform bound for the distance between the distribution function of random sums of independent geometric random variables and an appropriate negative binomial distribution function with parameters $\widehat{n}$ and $\widehat{p}$ cloud be obtained. The bound points out that the distribution function of random sums of independent geometric random variables can be approximated by the negative binomial distribution function with parameters $\widehat{n}$ and $\widehat{p}$ when $\widehat{q}=1-\widehat{p}$ is small.

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