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A GROUP WHERE C-PERMUTABILITY DOES NOT IMPLY C-NORMALITY

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Abstract: The $c-$permutability of subgroups in a finite group was recently discovered and a number of related results have been published. As there is a close relationship between permutability and normality, it is not surprising that a property called $c-$normality was defined and linked with $c-$permutability. By definition, all $c-$normal subgroups are $c-$permutable. In this paper, we show that the converse is not true by creating a group of order 32 containing a $c-$permutable subgroup that is not $c-$normal.

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1. Introduction

We first note that all groups and subgroups considered in this paper are finite.

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A subgroup H of a group G permutes with a subgroup K of G if HK is also a subgroup of G. In addition, if H permutes with every subgroup of G then H is said to be a permutable subgroup of G. By [3], every permutable subgroup H of G is subnormal. The study of permutable subgroups has resulted to many interesting properties, especially in the finite cases.

$c-$normality and $c-$permutability are two properties that was discovered and defined by Wang [5] and Al-sharo [2], respectively. Wang uses $c-$normality, which is a "weaker" version of normality, to generalize some known theorems. In particular, he proved that a group G is solvable if and only if every maximal subgroup of G is $c-$normal in G. Al-sharo instead proved that a group G is solvable if and only if every maximal subgroup of G is $c-$permutable in G. Studies have also been done on transitivity of these properties. In [2], Al-sharo introduced the $CPT-$group, a group in which $c-$permutability is a transitive relation. Transitive relation here means that for all subgroups H and K of G, if H is $c-$permutable in K and K is $c-$permutable in G, then it implies H is $c-$permutable in G. Alsharo [1] have instead studied on $CT-$groups, i.e groups in which $c-$normality is a transitive relation. They stated and proved a number of new characterizations of finite solvable $CT-$groups.

From [2] and [5], we deduce that if G is a solvable group then every maximal subgroup of a group G is $c-$normal and $c-$permutable. From the definitions of $c-$normal and $c-$permutable subgroups, it is known that every $c-$normal subgroup of a group G is $c-$permutable. In this paper, we show that the converse is not true. The Groups, Algorithms and Programming (GAP) software was used for the computations.

2. Preliminaries

We assume that the reader has prior knowledge of normality and subnormality of subgroups. Here, we give some preliminaries on $c-$normality and $c-$permutability of subgroups in a finite group G.

Definition 2.1. (see [5]) Let H be a subgroup of a finite group G. The core of H in G , denoted as H_G, is defined to be the largest normal subgroup of G contained in H, (or equivalently to $H_G = \cap\{H^g : g \in G\}$).

Definition 2.2. (see [5]) Let G be a group. We will call a subgroup H of G $c-$normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$. H $c-$norm G denotes H is normal in G.

Wang [5] deduced that every normal subgroup of a finite group G is $c-$normal. However, the following example shows that the converse is not true.

Example 2.3. Let $G = S_3$. Consider $H =< (12) >$, a Sylow 2-subgroup of G. Now, H is $c-$normal in G since $HA_3 = S_3$ and $H \cap A_3 = 1 \leq HS_3 = 1$. However, since H is not subnormal in G, then H is not a normal subgroup of G.

Definition 2.4. (see [2]) Let H be a subgroup of a finite group G. The ${}_pCore$ of H in G is defined to be the largest permutable subgroup of G contained in H, and denoted by ${}_pCore_G(H)$, or H_{PG}.

Definition 2.5. (see [2]) Let G be a group. We call a subgroup H $c-$permutable if there exists a permutable subgroup P of G such that $HP = G$ and $H \cap P \leq H_{PG}$. We use H $c-$perm G to denote H is $c-$permutable in G.

Permutable subgroups are $c-$permutable but the converse is not true. The following example shows this.

Example 2.6 The symmetric group S_3, take H the Sylow $2-$subgroup of S_3, we can see that H is not permutable, but it is $c-$permutable with all subgroups of S_3.

It has been proven that a normal subgroup is permutable. Therefore, since a normal subgroup is always $c-$normal, then a $c-$normal subgroup must be permutable and hence is $c-$permutable. What follows is a computational proof, using GAP, showing that a $c-$permutable subgroup need not be $c-$normal.

3. Main Result

To prove that $c-$permutability does not imply $c-$normality, we create a group G of order 32 and list its subgroups. We show that all subgroups of G are $c-$permutable but there exists among them a subgroup which fails to be $c-$normal.

Theorem 3.7. *Let G be a group with all its subgroups $c-$permutable. Then, there exists a subgroup H of G where H is not $c-$normal.*

Proof. Let $G =< a, b, c, a^8 = b^2 = c^2 = 1, b^a = bc, c^a = ca^4, (a^2)^b = ba^4 >$

The order of G is $32 = 2^5$, and G is isomorphic to the semidirect product: $(Z_8 \rtimes Z_2) \rtimes Z_2$

Now since $p-$groups are nilpotent and every subgroup of a nilpotent group is subnormal then every subgroup of G is subnormal.

We use Groups, Algorithms Programming (GAP) for the computation, we see that G has 42 subgroups. They are:

$<>, < a^4 >, < a^7bab >, < a^6baba >, < b >, < a^{-1}ba >, < a^{-2}ba^2 >,$
$< a^{-3}ba^3 >, < a^6b >, < a^5ba >, < b^{-1}a^6b^2 >, < b^{-1}a^5bab >< a^{-2} >,$
$< c, aca^{-1} >, < a^2c^{-1} >, < b, a^{-4} >, < a^{-1}ba, a^{-4} >, < a^2b^{-1}, a^{-2}b^{-1} >,$
$< ab^{-1}a, a^{-3}b^{-1}a >, < c, a^{-2}b^{-1} >, < b^{-1}cb, b^{-1}a^{-2} >,$
$< b^{-1}a^{-1}cab, b^{-1}a^{-3}b^{-1}ab >, < a^{-1}ca, a^{-3}b^{-1}a >, < b, c >, < a^{-3}ba^3, a^{-3}ca^3 >$
$, < a^{-2}ba^2, a^{-2}ca^2 >, < a^{-1}ba, a^{-1}ca >, < b, c, aba^{-1} >, < a^{-2}, c >,$
$< c, a^2b^{-1}aca^{-1} >, < a^{-2}, b >, < a^{-2}, a^{-1}ba >, < b, a^2c^{-1} >,$
$< a^{-1}ba, ac^{-1}a >, < a >, < b^{-1}ab >, < ba^{-1} >, < a^{-1}b >, < a^{-2}, b, c >,$
$< a, c >, < a^{-2}, ba^{-1} >, < G >$.

All these subgroups are $c-$permutable in G. But if we take $H =< a^7bab >= \{1, a^7bab\} \simeq Z_2$ subgroup of order 2, then H is not normal in G. Since $aH \neq Ha$,and H is contained in every subgroup of order 6 , then H cannot be $c-$normal in G.

Hence, there is a $c-$permutable subgroup H of G that is not $c-$normal.

4. Conclusion

We have shown that there exists a group G of order 32 for which $c-$ permutability does not imply $c-$normality. Hence, this proves that c-permutability does not imply $c-$normality.

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