

**COEFFICIENT ESTIMATE OF CERTAIN SUBCLASSES
OF CONVEX p -VALENT FUNCTIONS WITH
A BOUNDED POSITIVE REAL PART**

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Abstract: We estimate the bounds of coefficients and solve Fekete-Szegő problem for p -valent Mocanu-convex and Pascu-type functions in the open unit disk Δ which maps Δ onto the strip domain w with $p\alpha < \Re w < p\beta$.

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1. Introduction

Let \mathcal{A}_p denote the class of all functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic and p -valent in the open unit disk $\Delta = \{z : z \in \mathbb{C} : |z| < 1\}$. Note that $\mathcal{A}_1 := \mathcal{A}$ the class of analytic functions further \mathcal{S} the subclass of \mathcal{A} consisting of all univalent functions f in Δ . A function $f \in \mathcal{A}$ is said to be starlike of order α ($0 \leq \alpha < 1$) in Δ if it satisfies $\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha$. This class is denoted by $\mathcal{S}(\alpha)$ and $\mathcal{S}(0) = \mathcal{S}$. The class $\mathcal{S}(\alpha)$ was introduced by Robertson [4]. It is well-known that $\mathcal{S}(\alpha) \subset \mathcal{S} \subset \mathcal{A}$. Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f \in \mathcal{A}$ which satisfy $\Re \left(\frac{zf'(z)}{f(z)} \right) < \beta$ ($z \in \Delta$) for some real number β with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi et. al [6].

Let $P(z)$ and $Q(z)$ be analytic in Δ . Then the function $P(z)$ is said to subordinate to $Q(z)$ in Δ written by

$$P(z) \prec Q(z) \quad (z \in \Delta), \quad (1)$$

if there exists a function $w(z)$ which is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \Delta$), and such that $P(z) = Q(w(z))$ ($z \in \Delta$). From the definition of the subordinations, it is easy to show that the subordination (1) implies that

$$P(0) = Q(0) \quad \text{and} \quad P(\Delta) \subset Q(\Delta). \quad (2)$$

In particular, if $Q(z)$ is univalent in Δ , then the subordination (1) is equivalent to the condition (2).

Motivated by the classes $\mathcal{S}(\alpha)$ and $\mathcal{M}(\beta)$, we define a new class for certain p -valent functions.

Definition 1. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{S}^p(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < \Re \left(\frac{zf'(z)}{pf(z)} \right) < \beta \quad (z \in \Delta, p \in \mathbb{N}). \quad (3)$$

Definition 2. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{C}_\lambda^p(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < \Re \left\{ \frac{1}{p} \left(1 + \frac{zf'(z)}{f(z)} \right) \right\} < \beta \quad (z \in \Delta, \lambda \geq 0). \quad (4)$$

Definition 3. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{M}_\lambda^p(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < \Re \left\{ \frac{(1-\lambda)}{p} \frac{zf(z)}{f(z)} + \frac{\lambda}{p} \left(1 + \frac{zf(z)}{f(z)} \right) \right\} < \beta \quad (z \in \Delta, \lambda \geq 0, p \in \mathbb{N}). \quad (5)$$

Remark 4. When $p = 1$, $\mathcal{M}_\lambda^p(\alpha, \beta)$ reduces to $\mathcal{M}_\lambda(\alpha, \beta)$, the class of Mocanu-convex functions with bounded positive real part. Further we note that $\mathcal{M}_0^p(\alpha, \beta) = \mathcal{S}^p(\alpha, \beta)$ [1], $\mathcal{M}_0^1(\alpha, \beta) = \mathcal{S}(\alpha, \beta)$ [3], $\mathcal{M}_1^p(\alpha, \beta) = \mathcal{C}^p(\alpha, \beta)$, the class of p -valent convex functions with bounded positive real part and $\mathcal{M}_1^1(\alpha, \beta) = \mathcal{C}(\alpha, \beta)$, the class of convex functions with bounded positive real part.

Definition 5. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{N}_\lambda^p(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < \Re \left\{ \frac{(1-\lambda)\frac{1}{p}zf(z) + \lambda\frac{1}{p^2}z(zf(z))}{(1-\lambda)f(z) + \lambda\frac{1}{p}zf(z)} \right\} < \beta \quad (z \in \Delta, \lambda \geq 0, p \in \mathbb{N}). \quad (6)$$

Remark 6. When $p = 1$, $\mathcal{N}_\lambda^p(\alpha, \beta)$ reduces to $\mathcal{N}_\lambda(\alpha, \beta)$, the class of Pascu-type functions with positive real part. Further we note that $\mathcal{N}_0^p(\alpha, \beta) = \mathcal{S}^p(\alpha, \beta)$ [1], $\mathcal{N}_0^1(\alpha, \beta) = \mathcal{S}(\alpha, \beta)$ [3], $\mathcal{N}_1^p(\alpha, \beta) = \mathcal{C}^p(\alpha, \beta)$ and $\mathcal{N}_1^1(\alpha, \beta) = \mathcal{C}(\alpha, \beta)$.

Now, we define an analytic function $\mathcal{S}_{\alpha, \beta}(z) : \Delta \rightarrow \mathbb{C}$ by

$$\mathcal{S}_{\alpha, \beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} z}} \right) \quad (7)$$

due to Kuroki and Owa [3] and they proved $\mathcal{S}_{\alpha, \beta}(z)$ maps Δ onto a convex domain w with $\alpha < \Re(w) < \beta$, conformally.

2. Coefficient Estimates for $f \in \mathcal{M}_\lambda^p(\alpha, \beta)$

Applying the function $\mathcal{S}_{\alpha, \beta}(z)$ defined by (7), we give a necessary and sufficient condition for $f(z) \in \mathcal{A}_p$ to belong to the class $\mathcal{M}_\lambda^p(\alpha, \beta)$.

Lemma 7. *Let $f(z) \in \mathcal{A}_p$ and $0 \leq \alpha < 1 < \beta$. Then $f(z) \in \mathcal{M}_\lambda^p(\alpha, \beta)$ if and only if*

$$\frac{(1-\lambda)zf(z)}{pf(z)} + \frac{\lambda}{p} \left(1 + \frac{zf(z)}{f(z)}\right) \prec 1 + \frac{\beta-\alpha}{\pi} i \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z} \right) \quad (8)$$

in Δ .

By taking $\lambda = 0$ and $\lambda = 1$ we state the following lemmas respectively:

Lemma 8. *[1] Let $f \in \mathcal{A}_p$. Then $f(z) \in \mathcal{S}^p(\alpha, \beta)$ if and only if*

$$\frac{zf(z)}{pf(z)} \prec 1 + \frac{\beta-\alpha}{\pi} i \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z} \right) \quad (z \in \Delta) \quad (9)$$

where $\alpha < 1$ and $\beta > 1$.

Lemma 9. *Let $f \in \mathcal{A}_p$. Then $f(z) \in \mathcal{C}^p(\alpha, \beta)$ if and only if*

$$\frac{1}{p} \left(1 + \frac{zf(z)}{f(z)}\right) \prec 1 + \frac{\beta-\alpha}{\pi} i \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z} \right) \quad (z \in \Delta) \quad (10)$$

where $\alpha < 1$ and $\beta > 1$.

We note that

$$\mathcal{S}_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi} i \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z} \right) = 1 + \sum_{n=1} B_n z^n,$$

where

$$B_n = \frac{2(\beta-\alpha)}{n\pi} \sin \frac{n\pi(1-\alpha)}{\beta-\alpha} \quad (n = 1, 2, 3, \dots). \quad (11)$$

Using the subordination (8), we find sharp bounds on the second and third coefficients for $f(z) \in \mathcal{M}_\lambda^p(\alpha, \beta)$, by applying the following lemma due to Rogosinski [5].

Lemma 10. *Let $P(z) = \sum_{n=1} A_n z^n$ and $Q(z) = \sum_{n=1} B_n z^n$ be analytic in Δ . If $P(z) \prec Q(z)$ ($z \in \Delta$), then*

$$\sum_{k=1}^m |A_k|^2 \leq \sum_{k=1}^m |B_k|^2 \quad (m = 1, 2, \dots).$$

Theorem 11. *If the function $f(z) = z^p + \sum_{n=1} a_{p+n}z^{p+n} \in \mathcal{M}_\lambda^p(\alpha, \beta)$, then*

$$|a_{p+1}| \leq \frac{2p^2(\beta - \alpha)}{\pi(p + \lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_{p+2}| \leq \frac{2p^2(\beta - \alpha)}{\pi(p + \lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \times \left(\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p(p^2 + 2p\lambda + \lambda)(\beta - \alpha)}{\pi(p + \lambda)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right).$$

Proof. Let

$$\begin{aligned} P(z) &= \frac{(1 - \lambda)zf(z)}{p} + \frac{\lambda}{p} \left(1 + \frac{zf(z)}{f(z)} \right) \\ &= 1 + \frac{(1 + p\lambda)}{p} a_{p+1}z + \left[\frac{2(p + 2\lambda)}{p^2} a_{p+2} - \frac{(p^2 + 2p\lambda + \lambda)}{p^3} a_{p+1}^2 \right] z^2 + \dots \end{aligned}$$

and

$$Q(z) = \mathcal{S}_{\alpha, \beta}(z) = 1 + \sum_{n=1} B_n z^n, \tag{12}$$

where B_n is as in (11). Applying Lemma 10 we can get the results as asserted. □

When $\lambda = 0$ and $\lambda = 1$ we state the following corollaries respectively:

Corollary 12. *[1] If the function $f(z) = z^p + \sum_{n=1} a_{p+n}z^{p+n} \in \mathcal{S}^p(\alpha, \beta)$, then*

$$|a_{p+1}| \leq \frac{2p(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_{p+2}| \leq \frac{p(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left(\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right).$$

Corollary 13. *If the function $f(z) = z^p + \sum_{n=1} a_{p+n}z^{p+n} \in \mathcal{C}^p(\alpha, \beta)$, then*

$$|a_{p+1}| \leq \frac{2p^2(\beta - \alpha)}{\pi(p + 1)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_{p+2}| \leq \frac{p^2(\beta - \alpha)}{\pi(p + 2)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left[\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right].$$

When $p = 1$, from Theorem 11, we state the following corollary:

Corollary 14. *If the function $f(z) = z + \sum_{n=2} a_n z^n \in \mathcal{M}_\lambda(\alpha, \beta)$, then*

$$|a_2| \leq \frac{2(\beta - \alpha)}{\pi(1 + \lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_3| \leq \frac{(\beta - \alpha)}{\pi(1 + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left(\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2(1 + 3\lambda)(\beta - \alpha)}{\pi(1 + \lambda)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right).$$

Making use of the following lemma we shall solve the Fekete-Szegő problem for $f(z) \in \mathcal{M}_\lambda^p(\alpha, \beta)$.

Lemma 15. [2] *Let $h(z) = 1 + h_1 z + h_2 z^2 + \dots$ be a function with positive real part in Δ . Then for any complex number ν ,*

$$|h_2 - \nu h_1^2| \leq 2 \max\{1, |1 - 2\nu|\}.$$

Theorem 16. *Let $0 \leq \alpha < 1 < \beta$ and let the function f given by $f(z) = z^p + \sum_{n=1} a_{p+n} z^{p+n}$ be in the class $\mathcal{M}_\lambda^p(\alpha, \beta)$. Then for any complex number μ ,*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^2(\beta - \alpha)}{\pi(p + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p[(p^2 + 2p\lambda + \lambda) - 2\mu p(p + 2\lambda)](\beta - \alpha)}{\pi(p + \lambda)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.$$

Proof. Let $P(z) = \frac{(1 - \lambda)zf(z)}{p} + \frac{\lambda}{p} \left(1 + \frac{zf(z)}{f(z)} \right)$. Then, since $f \in \mathcal{M}_\lambda^p(\alpha, \beta)$, we have $P(z) \prec Q(z)$, where $Q(z)$ is given by (12).

Let

$$h(z) = \frac{1 + Q^{-1}(P(z))}{1 - Q^{-1}(P(z))} = 1 + h_1 z + h_2 z^2 + \dots \quad (z \in \Delta).$$

Then h is analytic and has positive real part in the open disk Δ . We also have

$$P(z) = Q \left(\frac{h(z) - 1}{h(z) + 1} \right) \quad (z \in \Delta). \tag{13}$$

We find from the equation (13) that

$$\begin{aligned}
 a_{p+1} &= \frac{p^2 B_1 h_1}{(p + \lambda)} \\
 a_{p+2} &= \frac{p^2}{2(p + 2\lambda)} \left[\frac{B_2 h_1^2}{4} - \frac{B_1 h_1^2}{4} + \frac{B_1 h_2}{2} + \frac{p(p^2 + 2p\lambda + \lambda)}{(p + \lambda)^2} B_1^2 h_1^2 \right]
 \end{aligned}$$

which imply that

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p^2 B_1}{4(p + 2\lambda)} (h_2 - \nu h_1^2),$$

where

$$\nu = \frac{1}{2} \left(1 - \frac{B_2}{B_1} - \frac{p(p^2 + 2p\lambda + \lambda) B_1}{(p + \lambda)^2} + \frac{2\mu p^2 (p + 2\lambda) B_1}{(p + \lambda)^2} \right).$$

Applying Lemma 15, we obtain

$$\begin{aligned}
 |a_{p+2} - \mu a_{p+1}^2| &= \frac{p^2}{4(p + 2\lambda)} |B_1| |h_2 - \nu h_1^2| \\
 &\leq \frac{p^2}{2(p + 2\lambda)} B_1 \max\{1, |1 - 2\nu|\}.
 \end{aligned} \tag{14}$$

Substituting $B_1 = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$ and $B_2 = \frac{(\beta - \alpha)}{\pi} \sin \frac{2\pi(1 - \alpha)}{\beta - \alpha}$ in (14), we can obtain the results as asserted. \square

By taking $\lambda = 0$ and $\lambda = 1$ we state the following corollaries respectively:

Corollary 17. [1] Let $0 \leq \alpha < 1 < \beta$ and let the function f given by $f(z) = z^p + \sum_{n=1} a_{p+n} z^{p+n}$ be in the class $S^p(\alpha, \beta)$. Then for any complex number μ ,

$$\begin{aligned}
 |a_{p+2} - \mu a_{p+1}^2| &\leq \frac{p(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \\
 &\times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p(1 - 2\mu)(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.
 \end{aligned}$$

Corollary 18. Let $0 \leq \alpha < 1 < \beta$ and let the function f given by $f(z) = z^p + \sum_{n=1} a_{p+n} z^{p+n}$ be in the class $C^p(\alpha, \beta)$. Then for any complex number μ ,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^2(\beta - \alpha)}{\pi(p + 2)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

$$\times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p[(p + 1)^2 - 2\mu p(p + 2)](\beta - \alpha)}{\pi(p + 1)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.$$

Putting $p = 1$ in Theorem 16, we get the following corollary.

Corollary 19. *Let $0 \leq \alpha < 1 < \beta$ and let the function f given by $f(z) = z + \sum_{n=2} a_n z^n$ be in the class $\mathcal{M}_\lambda(\alpha, \beta)$. Then for any complex number μ ,*

$$|a_3 - \mu a_2^2| \leq \frac{(\beta - \alpha)}{\pi(1 + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

$$\times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2[(1 + 3\lambda) - 2\mu(1 + 2\lambda)](\beta - \alpha)}{\pi(1 + \lambda)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.$$

3. Coefficient Estimates for $f \in \mathcal{N}_\lambda^p(\alpha, \beta)$

First, by applying the function $\mathcal{S}_{\alpha, \beta}(z)$ defined by (7), we give a necessary and sufficient condition for $f(z) \in \mathcal{A}_p$ to belong to the class $\mathcal{N}_\lambda^p(\alpha, \beta)$.

Lemma 20. *Let $f(z) \in \mathcal{A}_p$ and $0 \leq \alpha < 1 < \beta$. Then $f(z) \in \mathcal{N}_\lambda^p(\alpha, \beta)$ if and only if*

$$\frac{(1 - \lambda)\frac{1}{p}zf(z) + \lambda\frac{1}{p^2}z(zf(z))}{(1 - \lambda)f(z) + \lambda\frac{1}{p}zf(z)} \prec 1 + \frac{\beta - \alpha}{\pi}i \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z} \right) \quad (15)$$

in Δ .

Remark 21. For $\lambda = 0$ and $\lambda = 1$, we get Lemma 8 and Lemma 9 respectively.

Using the subordination (15), we find sharp bounds on the second and third coefficients for $f(z) \in \mathcal{N}_\lambda^p(\alpha, \beta)$, by applying Lemma 10.

Theorem 22. *If the function $f(z) = z^p + \sum_{n=1} a_{p+n}z^{p+n} \in \mathcal{N}_\lambda^p(\alpha, \beta)$, then*

$$|a_{p+1}| \leq \frac{2p^2(\beta - \alpha)}{\pi(p + \lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_{p+2}| \leq \frac{p^2(\beta - \alpha)}{\pi(p + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left(\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right).$$

Proof. Let

$$\begin{aligned} P(z) &= \frac{(1 - \lambda)\frac{1}{p}zf(z) + \lambda\frac{1}{p^2}z(zf(z))}{(1 - \lambda)f(z) + \lambda\frac{1}{p}zf(z)} \\ &= 1 + \frac{(p + \lambda)}{p}a_{p+1}z + \left[\frac{2(p + 2\lambda)}{p^2}a_{p+2} - \frac{(p + \lambda)^2}{p^3}a_{p+1}^2 \right] z^2 + \dots \end{aligned} \tag{16}$$

and

$$Q(z) = \mathcal{S}_{\alpha, \beta}(z) = 1 + \sum_{n=1} B_n z^n, \tag{17}$$

where B_n is as in (11). Applying Lemma 10 we can get the results as asserted. \square

Remark 23. For $\lambda = 0$ and $\lambda = 1$, we get Corollary 12 and Corollary 13 respectively.

When $p = 1$, from Theorem 22, we state the following corollary:

Corollary 24. *If the function $f(z) = z + \sum_{n=2} a_n z^n \in \mathcal{N}_\lambda(\alpha, \beta)$, then*

$$|a_2| \leq \frac{2(\beta - \alpha)}{\pi(1 + \lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_3| \leq \frac{(\beta - \alpha)}{\pi(1 + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left(\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2(\beta - \lambda)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right).$$

Again using Lemma 15 we shall solve the Fekete-Szegő problem for $f(z) \in \mathcal{N}_\lambda^p(\alpha, \beta)$.

Theorem 25. *Let $0 \leq \alpha < 1 < \beta$ and let the function f given by $f(z) = z^p + \sum_{n=1} a_{p+n} z^{p+n}$ be in the class $\mathcal{N}_\lambda^p(\alpha, \beta)$. Then for any complex number μ ,*

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &\leq \frac{p^2(\beta - \alpha)}{\pi(p + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \\ &\times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2p[(p + \lambda)^2 - 2\mu p(p + 2\lambda)](\beta - \alpha)}{\pi(p + \lambda)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}. \end{aligned}$$

Proof. Let

$$P(z) = \frac{(1 - \lambda)\frac{1}{p}zf(z) + \lambda\frac{1}{p^2}z(zf(z))}{(1 - \lambda)f(z) + \lambda\frac{1}{p}zf(z)}.$$

Then, since $f \in \mathcal{N}_\lambda^p(\alpha, \beta)$, we have $P(z) \prec Q(z)$, where $Q(z)$ is given by (17).

Let

$$h(z) = \frac{1 + Q^{-1}(P(z))}{1 - Q^{-1}(P(z))} = 1 + h_1z + h_2z^2 + \dots \quad (z \in \Delta).$$

Then h is analytic and has positive real part in the open disk Δ . We also have

$$P(z) = Q\left(\frac{h(z) - 1}{h(z) + 1}\right) \quad (z \in \Delta). \tag{18}$$

We find from the equation (18) that

$$\begin{aligned} a_{p+1} &= \frac{p^2 B_1 h_1}{2(p + \lambda)} \\ a_{p+2} &= \frac{p^2}{2(p + 2\lambda)} \left[\frac{B_2 h_1^2}{4} - \frac{B_1 h_1^2}{4} + \frac{B_1 h_2}{2} + \frac{p B_1^2 h_1^2}{4} \right] \end{aligned}$$

which imply that

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p^2 B_1}{4(p + 2\lambda)} (h_2 - \nu h_1^2),$$

where

$$\nu = \frac{1}{2} \left(1 - \frac{B_2}{B_1} - \frac{p[(p + \lambda)^2 - 2\mu p(p + 2\lambda)]B_1}{(p + \lambda)^2} \right).$$

Applying Lemma 15, we obtain

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &= \frac{p^2}{4(p + 2\lambda)} |B_1| |h_2 - \nu h_1^2| \\ &\leq \frac{p^2}{2(p + 2\lambda)} B_1 \max\{1, |1 - 2\nu|\}. \end{aligned} \tag{19}$$

Substituting $B_1 = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$ and $B_2 = \frac{(\beta - \alpha)}{\pi} \sin \frac{2\pi(1 - \alpha)}{\beta - \alpha}$ in (19), we can obtain the results as asserted. \square

Remark 26. By taking $\lambda = 0$ and $\lambda = 1$ we get Corollary 17 and Corollary 18 respectively.

Putting $p = 1$ in Theorem 25, we get the following corollary.

Corollary 27. *Let $0 \leq \alpha < 1 < \beta$ and let the function f given by $f(z) = z + \sum_{n=2} a_n z^n$ be in the class $\mathcal{N}_\lambda(\alpha, \beta)$. Then for any complex number μ ,*

$$|a_3 - \mu a_2^2| \leq \frac{(\beta - \alpha)}{\pi(1 + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \\ \times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2[(1 + \lambda)^2 - 2\mu(1 + 2\lambda)](\beta - \alpha)}{\pi(1 + \lambda)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.$$

Remark 28. By taking $\lambda = 0$ and $\lambda = 1$ one can deduce the results for functions $f \in \mathcal{S}_\lambda^p(\alpha, \beta)$ [1] and $f \in \mathcal{C}^p(\alpha, \beta)$ respectively. In particular $\lambda = 0, p = 1$ leads to the results obtained in [3].

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