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ON THE FORM OF CORRELATION FUNCTION FOR A CLASS OF NONSTATIONARY FIELD WITH A DISCRETE AND MIXED SPECTRUM

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Abstract: The present paper is devoted to the derivation of an explicit form of linearly representable random fields in the form $h(x_1, x_2) = \exp\{i(x_1A_1 + x_2A_2)\}$ h, where $h \in H$, H is a Hilbert space, operators A_1, A_2 are such that $A_1A_2 = A_2A_1$ and $C^3 = 0$ where $C = A_1^*A_2 - A_2A_1^*$.

The results obtained are the generalization of theorem proved in [3], [5], [7].

It is shown that a rank of nonstationary of field $h(x_1, x_2)$ depends not only on a degree of non-self adjoint of A_1, A_2 but on a degree of nilpotency of commutator $C(C^3 = 0)$.

In the present paper an explicit form of correlation function for discrete spectrum of A_1 and A_2 is derived. A form in the case of spectrum of operator A_1 is constructed in zero and that of the operator A_2 is pure discrete of $A_1$and A_2 is zero and the other is discrete, is obtained.

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1. Preliminary Information

1.1.

Consider a vector field $h(x)$ where $x = (x_1, x_2) \in \mathbb{R}^2$, with values in Hilbert space H.
In this paper we suppose that $h(x) = Z_x h$ where $Z_x = \exp\left[i(x_1A_1 + x_2A_2)\right]$. In this case A_1 and A_2 are such operators in the Hilbert space H for which $A_1A_2 = A_2A_1$, and the operator function Z_x is called a two-parametric commutative semigroup. The main tool of correlation theory for vector fields in a Hilbert space H is a correlation function [3]:

$$K(x, y) = \langle h(x), h(y)\rangle \,, \tag{1}$$

where $x, y \in \mathbb{R}^2$, for twice permutational classes of linear operators $\{A_1, A_2\}$, $(A_1A_2 = A_2A_1, A_1^*A_2 = A_2A_1^*)$.
Generalizing the results given [3,5,7] has introduced partial infinitesimal correlation functions (ICF) by relations (under the assumption that $K(x_1, x_2, y_1, y_2)$ is twice differentiable function):

$$\begin{aligned} W_1(x_1, x_2, y_1, y_2) &= -\frac{\partial K(x_1+\tau_1, x_2, y_1+\tau_1, y_2)}{\partial \tau_1}\,|_{\tau_1=0} \\ W_2(x_1, x_2, y_1, y_2) &= -\frac{\partial K(x_1, x_2+\tau_2, y_1, y_2+\tau_2)}{\partial \tau_2}\,|_{\tau_2=0} \\ W(x_1, x_2, y_1, y_2) &= -\frac{\partial^2 K(x_1+\tau_1, x_2+\tau_2, y_1+\tau_1, y_2+\tau_2)}{\partial \tau_1 \partial \tau_2}\,|_{\tau_1=\tau_2=0}\,, \end{aligned} \tag{2}$$

where W_1, W_2 and W are not independent.
Indeed:

$$\begin{aligned} &\int_o^{-y_1} W(x_1+\tau_1, x_2+\tau_2, y_1+\tau_1, y_2+\tau_2)d\tau_1 \\ = &\left[\frac{\partial}{\partial \tau_2}K(x_1-y_1, x_2+\tau_2, 0, y_2+\tau_2) - \frac{\partial}{\partial \tau_2}K(x_1, x_2+\tau_2, y_1, y_2+\tau_2)\right] \\ = &-W_2(x_1-y_1, x_2+\tau_2, 0, y_2+\tau_2) + W_2(x_1, x_2+\tau_2, y_1, y_2+\tau_2). \end{aligned}$$

Similarly it is easy to get

$$\begin{aligned} &\int_o^{-y_2} W(x_+\tau_1, x_2+\tau_2, y_1+\tau_1, y_2+\tau_2)d\tau_2 \\ &= -W_1(x_1+\tau_1, x_2-y_2, y_+\tau_1, 0) + W_1(x_1+\tau_1, x_2, y_1+\tau_1, y_2). \end{aligned}$$

$$\int_o^{-y_1}\int_o^{-y_2} W(x_1+\tau_1,x_2+\tau_2,y_1+\tau_1,y_2+\tau_2)d\tau_1 d\tau_2$$
$$= K(x_1-y_1,x_2,y_2,0,0) - K(x_1-y_1,x_2,0,y_2)$$
$$- K(x_1,x_2-y_2,y_1,0) - K(x_1,x_2,y_1,y_2). \tag{3}$$

Recall that the field $h(x)$ in H is called dissipative if $(A_1)_I \geq 0$. As in one-dimension case it is easy to establish [3, 7] that

$$\lim_{\tau_1\to\infty} K(x_1+\tau_1,x_2,y_1+\tau_1,y_2) = K^1_\infty(x_1-y_1,x_2,y_2);$$
$$\lim_{\tau_2\to\infty} K(x_1,x_2+\tau_1,y_1,y_2+\tau_1) = K^2_\infty(x_2-y_2,x_1,y_1);$$
$$\lim_{\tau_1,\tau_2\to\infty} K(x+\tau,y+\tau) = K^1_\infty(x-y). \tag{4}$$

If the correlation function depends only on the difference of arguments then a field is called a stationary field [3] (just in this way a stationary was defined by Kolmogorov, A.N.).

Then (3) may be presented in the form

$$K(x,y) = \int_o^\infty\int_o^\infty W(x+\tau,y+\tau)d\tau_1 d\tau_2 + K^1_\infty(x_1-y_1,x_2,y_2) \tag{5}$$
$$+ K^2_\infty(x_2-y_2,x_1,y_1) + K_\infty(x-y),$$

where $K_\infty(x-y)$ is a Hermitian-positive function which may be considered as a stationary field correlation function, $K^1_\infty(x_1-y_1,x_2,y_2)$ (as well as $K^2_\infty(x_1-y_1,x_2,y_2)$) with variable x_1-y_1 is a Hermitian-positive function for each x_2,y_2, and , as a function of x_2,y_2, it is a dissipative curve of one variable in H. Thus, it is determined by the infinitesimal correlation function $W(x,y)$.

1.2.

Recall [1, 2] that the rank of nonstationary function $h(x)$ of twice permutational system of linear operators A_1, A_2 is the greatest rank of all the quadratic form

$$\sum_{\alpha,\beta=1}^{n} W(x_\alpha,x_\beta)\zeta_\alpha\overline{\zeta}_\beta, \quad x_\alpha\in\mathbb{R}^2, \zeta_\alpha\in\mathbb{C}, n<\infty$$

It is not difficult to show that the rank of nonstationarity for the present case coincides with the dimension of the space $H_o = \overline{(A_1)_I H}\bigcap\overline{(A_2)_I H}$ (here as usually $(A_k)_I = \frac{A_k - A_k^*}{2i}$ as in [3]) and in addition

$$W(x,y) = 4\left\langle (A_1)_I (A_2)_I h(x), h(y)\right\rangle. \tag{6}$$

Derivation of formula (6):

From formula (2) it follows that

$$\begin{aligned} W_1(x_1,x_2,y_1,y_2) &= -\frac{\partial K(x+\tau_1,x_2,y_1+\tau_1,y_2)}{\partial \tau_1}\ |_{\tau_1=0} \\ &= -(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1})K(x_1,x_2,y_1,y_2) \\ &= -(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1})\left\langle Z_x h, Z_y h\right\rangle \\ &= -\left\langle iA_1 Z_x h, Z_y h\right\rangle - \left\langle Z_x h, iA_1 Z_y h\right\rangle \\ &= \left\langle \frac{A_1-A_1^*}{i} Z_x h_1 Z_y h\right\rangle \\ &= 2\left\langle (A_1)_I h(x), h(y)\right\rangle. \end{aligned}$$

Similarly

$$W_2(x_1,x_2,y_1,y_2) = 2\left\langle (A_2)_I h(x), h(y)\right\rangle.$$

Therefore,

$$W(x_1,x_2,y_1,y_2) = -(\frac{\partial}{\partial x_2}+\frac{\partial}{\partial y_2})W_1(x_1,x_2,y_1,y_2).$$

then we get that

$$\begin{aligned} W(x_1,x_2,y_1,y_2) &= -(\frac{\partial}{\partial x_2}+\frac{\partial}{\partial y_2})\left\langle 2(A_1)_I h(x), h(y)\right\rangle \\ &= -\left\langle 2(A_1)_I iA_2 h(x), h(y)\right\rangle - \left\langle 2(A_1)_I h(x), iA_2 h(y)\right\rangle \\ &= 2\left\langle \frac{(A_1)_I A_2 - A_2^*(A_1)_I}{i} h(x), h(y)\right\rangle. \end{aligned}$$

Since A_1 and A_2 are twice permutable then,

$$W(x,y) = 2\left\langle (A_1)_I \frac{A_2-A_2^*}{i} h(x), h(y)\right\rangle = 4\left\langle (A_1)_I (A_2)_I h(x), h(y)\right\rangle.$$

In case $\dim H_0 = 1$, i.e. when the rank of nonstationarity of vector field $h(x)$ is equal to one we get

$$W(x,y) = \Phi(x)\overline{\Phi(y)}, \tag{7}$$

where

$$\Phi(x) = \left\langle h(x), h_0\right\rangle.$$

2. Correlation Functions and Spectral Representation of the Twice Permutational Fields of Rank 1

2.1.

Consider the vector field $h(x_1, x_2) = \exp(ix_1A_1 + ix_2A_2)h$, where $h \in H, H_0 = \overline{(A_1)_I H} \bigcap \overline{(A_2)_I H}$, $\dim H_0 = 1$ and the operators A_1, A_2 are twice permutable.

As it was shown in [5, 7], the ICF of vector field $h(x_1, x_2)$ has the form

$$W(x_1, x_2, y_1, y_2) = \Phi(x_1, x_2)\overline{\Phi(y_1, y_2)},$$

where, $\Phi(x_1, x_2) = \langle \exp(ix_1A_1 + ix_2A_2)h, h_0 \rangle, h_0 \in H_0, \|h_0\| = 1, 2ImA_1 2$ $ImA_2h_0 = \lambda_0 h_0$ and λ_0 is real number.

Using the relation (3)

$$\exp(tA) = -\frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)(A - \lambda I)^{-1} d\lambda,$$

where Γ is a closed path that contains all the spectrum of operator A , one can represent the function $\Phi(x_1, x_2)$ in the form

$$\Phi(x_1, x_2) = (\frac{1}{2\pi i})^2 \oint_{\Gamma_1}\oint_{\Gamma_2} \exp(i\lambda_1 x_1 + i\lambda_2 x_2)$$
$$\left\langle (A_1 - \lambda_1 I)^{-1}(A_2 - \lambda_2 I)^{-1}h, h_0 \right\rangle d\lambda_1, d\lambda_2. \quad (8)$$

The closed path Γ_k includes the spectrum of the operator $A_k (k = 1, 2)$. While calculating integrals in (8) one can pass to any system of operators $\overset{\bullet}{A}_1, \overset{\bullet}{A}_2$, acting in Hilbert space $\overset{\bullet}{H}$, which are unitary equivalent to the original operators A_1, A_2:

$$((A_1 - \lambda_1 I)^{-1}(A_2 - \lambda_2 I)^{-1}h, h_o)_H = ((\overset{\bullet}{A}_1 - \lambda_1 I)^{-1}(\overset{\bullet}{A}_2 - \lambda_2 I)^{-1}g, g_o)_{\overset{\bullet}{H}},$$

where $\overset{\bullet}{A}_k U = UA_k (k = 1, 2)$, and U is a unitary operator acting from H into $\overset{\bullet}{H}$, where $\overset{\bullet}{H}$ is a model space, $Uh_0 = g_0$.

Then the function $\Phi(x_1, x_2)$ can be presented in the form

$$\Phi(x_1, x_2) = (-\frac{1}{2\pi i})^2 \oint_{\Gamma_1}\oint_{\Gamma_2} \exp(i\lambda_1 x_1 + i\lambda_2 x_2)$$
$$\left\langle (\overset{\bullet}{A}_1 - \lambda_1 I)^{-1}(\overset{\bullet}{A}_2 - \lambda_2 I)^{-1})g, g_0 \right\rangle d\lambda_1 d\lambda_2.$$

2.2.

Consider the case, when the field $h(x_1,x_2)$ belongs to the class $K_{22}^{(1)}$ [7], i.e. the spectrum of each of the operator A_k $(k=1,2)$ is pure discrete. In this case, the model space $\overset{\bullet}{H}$ coincides with

$$\ell^2(\beta_1,\beta_2)=\Big\{f(p,q),p=1,\dots,N_1,q=1,\dots,N_2,(N_k\le\infty),\\ \sum_{p=1}^{N_1}\sum_{q=1}^{N_2}|f(p,q)|^2\left(\beta_p^{(1)}\right)^2\left(\beta_q^{(2)}\right)^2<\infty\Big\},$$

where $\left\{\lambda_p^{(1)}\right\}_{p=1}^{N_1}$ is a sequence of non-real points of spectrum $A_1\,|_{H_2}$, where $H_2=2\overline{\mathrm{Im}A_2H}$,

$\lambda_p^{(1)}=\alpha_p^{(1)}+\frac{i}{2}\left(\beta_p^{(1)}\right)^2$; $\left\{\lambda_q^{(2)}\right\}_{q=1}^{N_2}$ is a sequence of non-real points of spectrum $A_2\,|_{H_1}$, where $H_1=2\overline{\mathrm{Im}A_1H}$, $\lambda_q^{(2)}=\alpha_q^{(2)}+\frac{i}{2}\left(\beta_q^{(2)}\right)^2$.

The operators $\overset{\bullet}{A}_1$ and $\overset{\bullet}{A}_2$can be represented as follows

$$\begin{aligned}\overset{\bullet}{A}_1f(p,q)&=\lambda_p^{(1)}f(p,q)+i\sum_{p=1}^{N_1}f(s,q)\left(\beta_s^{(1)}\right)^2,\\ \overset{\bullet}{A}_2f(p,q)&=\lambda_q^{(2)}f(p,q)+i\sum_{q=1}^{N_2}f(p,r)\left(\beta_r^{(2)}\right)^2.\end{aligned}$$

Since the intersection of non-Hermitian subspaces of operators $\overset{\bullet}{A}_1$and $\overset{\bullet}{A}_2$coincides with the subspace of functions that does not depend on the arguments then $\overset{\bullet}{h}_0(p,q)\equiv 1$, and $\overset{\bullet}{h}=f(p,q)$. It is easy to show that

$$\left(\overset{\bullet}{A}{}_1^{*}-\lambda_1I\right)^{-1}\left(\overset{\bullet}{A}{}_1^{*}-\lambda_2I\right)^{-1}h_0(p,q)\\ =\frac{1}{\lambda_p^{(1)}-\lambda_1}\frac{1}{\lambda_q^{(2)}-\lambda_2}\prod_{s=1}^{p-1}\frac{\lambda_1-\overline{\lambda_{p-s}^{(1)}}}{\lambda_1-\lambda_{p-s}^{(1)}}\prod_{r=1}^{q-1}\frac{\lambda_2-\overline{\lambda_{q-r}^{(2)}}}{\lambda_2-\lambda_{q-r}^{(12)}}.$$

Then there exists a representation

$$\Phi(x_1,x_2) = (-\frac{1}{2\pi i})^2 \oint_{\Gamma_1}\oint_{\Gamma_2} \exp(i\lambda_1 x_1 + i\lambda_2 x_2)$$
$$\times \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} \frac{\left(\beta_p^{(1)}\right)^2 \left(\beta_q^{(2)}\right)^2}{(\lambda_p^{(1)}-\lambda_1)(\lambda_q^{(2)}-\lambda_2)} \prod_{s=1}^{p-1} \frac{\lambda_1 - \overline{\lambda_{p-s}^{(1)}}}{\lambda_1 - \lambda_{p-s}^{(1)}} \prod_{r=1}^{q-1} \frac{\lambda_2 - \overline{\lambda_{q-r}^{(2)}}}{\lambda_2 - \lambda_{q-r}^{(12)}} f(p,q)d\lambda_1 d\lambda_2.$$

Thus,

$$\Phi(x_1,x_2) = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} f(p,q)\Lambda_p^{(1)}(x_1)\Lambda_q^{(2)}(x_2)\left(\beta_p^{(1)}\beta_q^{(2)}\right)^2,$$

where, the functions $\Lambda_p^{(1)}(x_1)$ and $\Lambda_q^{(2)}(x_2)$ are defined as follows

$$\Lambda_p^{(1)}(x_1) = -\frac{1}{2\pi i}\oint_{\Gamma_1} \exp(i\lambda_1 x_1)\frac{1}{(\lambda_p^{(1)}-\lambda_1)} \prod_{s=1}^{p-1} \frac{\lambda_1 - \overline{\lambda_{p-s}^{(1)}}}{\lambda_1 - \lambda_{p-s}^{(1)}} d\lambda_1,$$

$$\Lambda_q^{(2)}(x_2) = -\frac{1}{2\pi i}\oint_{\Gamma_2} \exp(i\lambda_2 x_2)\frac{1}{(\lambda_q^{(2)}-\lambda_2)} \prod_{r=1}^{q-1} \frac{\lambda_2 - \overline{\lambda_{q-r}^{(2)}}}{\lambda_2 - \lambda_{q-r}^{(12)}} d\lambda_2.$$

3. Correlation Functions of Commutative Systems of Operators in Case of the Nilpotentness the Commutator $C = [A_1^*, A_2](C^3 = 0)$ with a Discrete Spectrum

3.1.

Similar to, the class of twice permutable system of linear operator of the vector field

$$h(x) = Z_x h, x = (x_1,x_2) \in \mathbb{R}^2, \quad Z_x = \exp[i(x_1A_1 + x_2A_2)], h \in H,$$

where the system of operators $\{A_1, A_2\}$ satisfies

$$[A_1,A_2] = 0, C = [A_1^*, A_2], (C^3 = 0), C^2 \neq 0. \tag{9}$$

We introduce the correlation functions

$$W_1(x_1,x_2,y_1,y_2) = -\frac{\partial}{\partial\tau_1}K(x_1+\tau_1,x_2,y_1+\tau_1,y_2)\mid_{\tau_1=0},$$

$$W_2(x_1,x_2,y_1,y_2) = -\frac{\partial}{\partial\tau_2}K(x_1,x_2+\tau_2,y_1,y_2+\tau_2)\mid_{\tau_2=0,}$$
$$W(x_1,x_2,y_1,y_2) = -\frac{\partial^2}{\partial\tau_1\partial\tau_2}K(x_1+\tau_1,x_2+\tau_2,y_1+\tau_1,y_2 +\tau_2)\mid_{\tau_1=\tau_2=0.} \tag{10}$$

It is not difficult to see that for the case of the vector field $h(x)$ one can obtain [4]

$$\begin{aligned} W_1(x_1,x_2,y_1,y_2) &= 2\left\langle (A_1)_I h(x), h(y)\right\rangle, \\ W_2(x_1,x_2,y_1,y_2) &= 2\left\langle (A_2)_I h(x), h(y)\right\rangle, \\ W(x_1,x_2,y_1,y_2) &= \left\langle Dh(x), h(y)\right\rangle. \end{aligned} \tag{11}$$

Here the operator D is self-adjoint and is of the form

$$D = 2i(A_2^*(A_1)_I - (A_1)_I(A_2) = 2i(A_1^*(A_2)_I - (A_2)_I A_1). \tag{12}$$

Let us show that D may be represented as (12). From formula (10) using differentiation rules we can easily get

$$\begin{aligned} W_1(x_1,x_2,y_1,y_2) &= -(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1})K(x,y) \\ &= -\left\langle \frac{\partial}{\partial x_1}h(x), h(y)\right\rangle - \left\langle h(x), \frac{\partial}{\partial y_1}h(y)\right\rangle \\ &= \left\langle -iA_1h(x), h(y)\right\rangle + \left\langle h(x), iA_1h(y)\right\rangle \\ &= 2\left\langle (\frac{A_1-A_1^*}{i})h(x), h(y)\right\rangle. \end{aligned}$$

Then we can find

$$\begin{aligned} W(x,y) &= -\frac{\partial}{\partial x_2}2\left\langle (A_1)_I h(x), h(y)\right\rangle - \frac{\partial}{\partial y_2}\left\langle 2(A_1)_I h(x), h(y)\right\rangle \\ &= -\left\langle 2i(A_1)_I A_2 h(x), h(y)\right\rangle - \left\langle 2(A_1)_I h(x), iA_2h(y)\right\rangle. \end{aligned}$$

This completes the proof (12). Elementary evaluations show that the operator D in (12) can be reduced to

$$D = C + 4(A_2)_I(A_1)_I = C^* + 4(A_1)_I(A_2)_I. \tag{13}$$

In order to obtain a concrete form of operator D we confine ourselves, from now on, to systems of linear operators that satisfy the following theorem which is proved in [4]. A system of operators A_1, A_2 is called a simple system [3] if there is no subspace in H, which reduces the operators A_1 and A_2, a contraction on which is self-adjoint at least for one of the operators A_k.

Theorem 3.1. [4] *Let us assume that a simple commuting system of linear operators* A_1, A_2 *is such that:*

1. $C^3 = 0$, $\dim CH = 2$
2. $\dim H_0 = 1,\ H_0 = 1, H_0 = \overline{(A_1)_I H} \bigcap \overline{(A_2)_I H}$
3. $\overline{(A_1)_I C^k H} \subset C^k H, (A_2)_I C^{*p} H \subset C^{*p} H, (k, p = 1, 2).$

Then, the space H *is decomposed into orthogonal sum* $H = H_1 \oplus H_2 \oplus H_3$, *where* H_k *reduce* A_1 *and subspaces* H_3 *and* $H_2 \oplus H_3$ *are invariant under* A_2 *and the contractions of system* $\{A_1, A_2\}$ *on* H_k *are twice permutable.*

In what follows, we assume that a system of linear operators $\{A_1, A_2\}$ satisfies the assumptions of Theorem 3.1, to conclude the result stated in lemma 3.1, below. Let $C^2 H = \{\lambda h_3\}, CH \ominus C^2 H = \{\mu h_2\}$ and $C^{*2} H = \{\lambda g_3\}, C^* H \ominus C^{*2} H = \{\mu g_2\}$. It is obvious that $h_3 \perp g_3, g_2$ and $g_3 \perp h_3, h_2$. Obviously, this follows from the condition $C^3 = C^{*3} = 0$. One can easily see that $H_3 \bigcap H_0 = \{\lambda h_3\}, H_2 \bigcap H_0 = \{2h_2\}$. Let us denote by h_1 as a vector such that $\{\lambda \tilde{h}_1\} = H_1 \bigcap H_0$ and introduce the following vectors:

$$\begin{aligned} h_1 &= \tilde{h}_1 = \left\langle \tilde{h}_1, g_3 \right\rangle g_3, \\ g_1 &= \tilde{g}_1 = \langle \tilde{g}_1, h_3 \rangle h_3, \end{aligned}$$

where the vector g_1 is such that $g_1 + h_3 + g_2 + g_3 = h_0$, and h_0 is a basis vector of space H_0.

Then it is easy to see that

$$DH = H_D = span\{h_3, h_2, h_1, g_1, g_2, g_3, \}.$$

Thus, the operator D, corresponding to defect of non-stationary, maps H into a six-dimensional space.
Let us find an explicit form of self-adjoint operator D defined in H_D. Really, it is easy to see that

$$Dh_3 = Ch_3 + 4(A_2)_I (A_1)_I h_3 = 4(A_2)_I \alpha_3 h_3,$$

where $(A_1)_I h_3 = \alpha_3 h_3$. Therefore,

$$\langle Dh_3, g_2 \rangle = 0 \quad \text{and} \quad \langle Dh_3, g_3 \rangle = 0.$$

Similarly, we can obtain

$$Dh_2 = Ch_2 + 4(A_2)_I (A_1)_I h_2 = \mu h_3 + 4(A_2)_I \alpha_2 h_2,$$

where $(A_1)_I h_2 = \alpha_2 h_2$.

Thus

$$\langle Dh_2, g_3 \rangle = 0.$$

By repeating the same arguments one can obtain

$$\langle Dh_3, h_1 \rangle = 0, \langle Dg_3, h_2 \rangle = 0, \langle Dg_2, h_3 \rangle = 0.$$

Hence, we have proved the following lemma.

Lemma 3.1. *The matrix of the operator D in the basis $\{h_1, h_2, h_3, g_1, g_2, g_3\}$ of the space H_D can be written by the form*

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & 0 & 0 \\ d_{12} & d_{22} & d_{23} & d_{24} & d_{25} & 0 \\ d_{13} & d_{23} & d_{33} & d_{34} & d_{35} & d_{36} \\ d_{14} & d_{24} & d_{34} & d_{44} & d_{45} & d_{64} \\ 0 & d_{25} & d_{35} & d_{45} & d_{55} & d_{56} \\ 0 & 0 & d_{36} & d_{46} & d_{56} & d_{66} \end{pmatrix} \tag{14}$$

where $d_{\alpha,\beta} \in \mathbb{R}$ are real numbers.

Thus, D is a generalization of Jacobian matrix, namely D is a semi-diagonal matrix.
Consequently,

$$Dh = \sum_{\alpha,\beta=1}^{6} \langle h, \ell_\alpha \rangle \, d_{\alpha,\beta} \ell_\beta, \tag{15}$$

where $\ell_1 = h_3, \ell_2 = h_2, \ell_3 = h_1, \ell_4 = g_1, \ell_5 = g_2, \ell_6 = g_3$.
Here, as above, we denote $\|\ell_\alpha\| = 1, (\alpha = 1, \cdots, 6)$, and $d_{\alpha,\beta} = \langle D\ell_\alpha, \ell_\beta \rangle$.

3.2.

Let us consider the infinitesimal correlation function $W(x, y)$(11):

$$W(x, y) = \langle Dh(x), h(y) \rangle .$$

Then, by virtue of (15) one can obtain

$$W(x, y) = \sum_{\alpha\beta=1}^{6} \langle h(x), \ell_\alpha \rangle \, d_{\alpha,\beta} \left\langle \overline{h(y), \ell_\beta} \right\rangle .$$

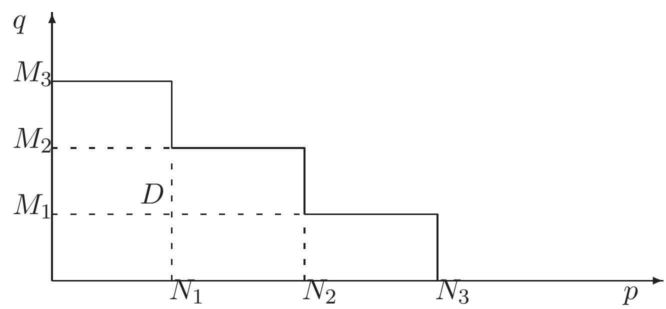
Denote by

$$\Phi_\alpha(x) = \langle h, \exp\left[i(x_1 A_1^* + x_2 A_2^*)\right]\ell_\alpha\rangle \; (\alpha = 1, 2, 3, \ldots, 6),$$

then

$$W(x, y) = \sum_{\alpha,\beta=1}^{6} \Phi_\alpha(x) \cdot d_{\alpha,\beta} \overline{\Phi_\beta(y)}. \tag{16}$$

Let us find the form of functions $\Phi_\alpha(x)$. First, note that the functions $\Phi_\alpha(x)$ are invariant under unitary equivalence and hence we can use the presentation model which is derived in [7]. As is obvious from the models the vector-functions $\exp\left[-i(x_1 A_1^* + x_2 A_2^*)\right]\ell_\alpha$ generate subspaces L_α invariant under the operators A_1^* and A_2^* where the restrictions of the operators A_1^* and A_2^* to L_α are twice permutable. Denote the images of the vectors $\{\ell_\alpha\}$ under unitary equivalence (which is realized by the model restriction) by $\{h_\alpha\}$, and the image of h by $f(p, q)$, where the functions $f(p, q)$ are defined in the following domain



Then, using the results of Section 2, we can get an explicit expression for the functions $\Phi_\alpha(x_1, x_2)$

$$\begin{aligned}
\Phi_1(x_1, x_2) &= \sum_{p=1}^{N_1} \sum_{q=M_2}^{M_3} f(p, q) \Lambda_p^{(1)}(x_1) \Lambda_q^{(2)}(x_2) \left(\beta_p^{(1)} \beta_q^{(2)}\right)^2, \\
\Phi_2(x_1, x_2) &= \sum_{p=1}^{N_2} \sum_{q=M_1}^{M_2} f(p, q) \Lambda_p^{(1)}(x_1) \Lambda_q^{(2)}(x_2) \left(\beta_p^{(1)} \beta_q^{(2)}\right)^2, \\
\Phi_3(x_1, x_2) &= \sum_{p=1}^{N_3} \sum_{q=1}^{M_1} f(p, q) \Lambda_p^{(1)}(x_1) \Lambda_q^{(2)}(x_2) \left(\beta_p^{(1)} \beta_q^{(2)}\right)^2, \qquad (17) \\
\Phi_4(x_1, x_2) &= \sum_{p=1}^{N_1} \sum_{q=1}^{M_3} f(p, q) \Lambda_p^{(1)}(x_1) \Lambda_q^{(2)}(x_2) \left(\beta_p^{(1)} \beta_q^{(2)}\right)^2,
\end{aligned}$$

$$\Phi_5(x_1,x_2) = \sum_{p=N_1}^{N_2}\sum_{q=1}^{M_2} f(p,q)\Lambda_p^{(1)}(x_1)\Lambda_q^{(2)}(x_2)\left(\beta_p^{(1)}\beta_q^{(2)}\right)^2,$$

$$\Phi_6(x_1,x_2) = \sum_{p=N_2}^{N_3}\sum_{q=1}^{M_1} f(p,q)\Lambda_p^{(1)}(x_1)\Lambda_q^{(2)}(x_2)\left(\beta_p^{(1)}\beta_q^{(2)}\right)^2,$$

where $1 < N_1 < N_2 < N_3$ and $1 < M_1 < M_2 < M_3$ and the functions $\Lambda_p^{(1)}(x_1)$, $\Lambda_q^{(2)}(x_2)$ are given by

$$\Lambda_p^{(1)}(x_1) = -\frac{1}{2\pi i}\oint_{\Gamma_1} \exp(i\lambda_1 x_1)\frac{1}{(\lambda_p^{(1)} - \lambda_1)}\prod_{s=1}^{p-1}\frac{\lambda_1 - \overline{\lambda_{p-s}^{(1)}}}{\lambda_1 - \lambda_{p-s}^{(1)}}d\lambda_1,$$

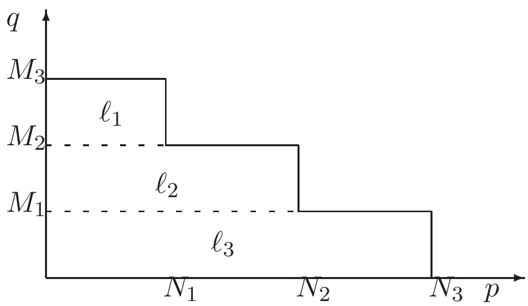
$$\Lambda_q^{(2)}(x_2) = -\frac{1}{2\pi i}\oint_{\Gamma_2} \exp(i\lambda_2 x_2)\frac{1}{(\lambda_q^{(2)} - \lambda_2)}\prod_{r=1}^{q-1}\frac{\lambda_2 - \overline{\lambda_{q-r}^{(2)}}}{\lambda_2 - \lambda_{q-r}^{(12)}}d\lambda_2.$$

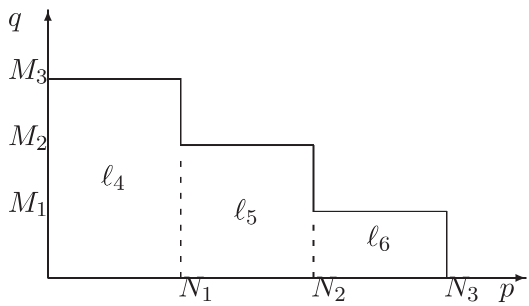
and at last $\beta_p^{(1)} = \sqrt{2\mathrm{Im}\lambda_p^{(1)}}$, $\beta_q^{(2)} = \sqrt{2\mathrm{Im}\lambda_q^{(2)}}$.

Thus, we can formulate the following theorem.

Theorem 3.2. *Assume that a system of linear operators $\{A_1, A_2\}$ satisfies to the assumptions of theorem 1 where the spectrum of each operator A_k ($k = 1, 2$). Then the infinitesimal correlation function $W(x, y)$ (11) is represented in the form (16) where $d_{\alpha,\beta} \in \mathbb{R}$ and the functions $\Phi_\alpha(x)$ are defined in (17).*

To evaluate $d_{\alpha,\beta}$ we represent ℓ_α graphically into the figures





where ℓ_α's are constants in the indicated areas.

So that

$$\ell_\alpha = \frac{\chi_{D_\alpha}}{\sqrt{S_{D_\alpha}}},$$

where χ_{D_α} is the characteristic function of the domain D_α, which is shown in the pictures for ℓ_α and S_{D_α} is the area.
For example,

$$\begin{aligned}
\ell_1 &= \frac{\chi_{[1,N_1]\times[M_2,M_3]}}{\sqrt{\sum\limits_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \sum\limits_{q=M_2}^{M_3}\left(\beta_q^{(2)}\right)^2}}, \\
\ell_2 &= \frac{\chi_{[1,N_2]\times[M_1,M_2]}}{\sqrt{\sum\limits_{p=1}^{N_2}\left(\beta_p^{(1)}\right)^2 \sum\limits_{q=1}^{M_2}\left(\beta_q^{(2)}\right)^2}},
\end{aligned}$$

etc.

Let us define $D\ell_1$:

$$D\ell_1 = (C + 4(A_2)_I(A_1)_I)\ell_1 = 4(A_2)_I(A_1)_I\ell_1 = 2(A_2)_I \sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \ell_1.$$

After $(A_2)_I$ we get

$$\begin{aligned}
D\ell_1 = \sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \sum_{q=M_2}^{M_3}\left(\beta_q^{(2)}\right)^2 \chi_{[1,N_1]\times[1,M_3]} \\
= \sqrt{\sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2}\sqrt{\sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \sum_{q=1}^{M_3}\left(\beta_q^{(2)}\right)^2}\,\ell_4.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d_{1,1} &= \langle D\ell_1, \ell_1\rangle = \sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \sum_{q=M_2}^{M_3}\left(\beta_q^{(2)}\right)^2 \left\langle \chi_{[1,N_1]\times[1,M_3]}, \ell_1 \right\rangle \\
&= \sqrt{\sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \sum_{q=1}^{M_3}\left(\beta_q^{(2)}\right)^2};
\end{aligned}$$

$$d_{1,2} = \langle D\ell_1, \ell_2\rangle = \sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \sum_{q=M_2}^{M_3}\left(\beta_q^{(2)}\right)^2 \left\langle \chi_{[1,N_1]\times[1,M_3]}, \ell_2\right\rangle$$
$$= \sum_{p=1}^{N_1}\left(\beta_p^{(1)}\right)^2 \sqrt{\frac{\sum\limits_{q=M_2}^{M_3}\left(\beta_q^{(2)}\right)^2}{\sum\limits_{p=1}^{N_2}\left(\beta_p^{(1)}\right)^2}}.$$

The other coefficients are similar.

4. Correlation Functions of Commutative Systems of Operators in Case of the Nilpotentness the Commutator $C = [A_1^*, A_2](C^3 = 0)$ with a Mixed Spectrum

4.1.

Let the field $h(x_1, x_2)$ belongs to the class $K_{12}^{(1)}$ [7], i.e. the spectrum of operator A_1 is concentrated in zero and that of operator A_2 is pure discrete. A model space $\overset{\bullet}{H}$ for this case is a totality:

$$\overset{\bullet}{H} = \left\{ f_k(y), k = 1, \dots, N; N \le \infty, y \in [0, \ell] : \sum_{k=1}^{N}\left[\int_0^{\ell} |f_k(y)|^2\, dy\right]\beta_k^2 < \infty \right\},$$

$\left\{\lambda_k = \alpha_k + \frac{i}{2}\beta_k^2\right\}_{k=1}^{N}$ is a sequence of non-real points of spectrum of operator $A_2\,|_H$. The model operators $\overset{\bullet}{A}_1$and $\overset{\bullet}{A}_2$ are defined by formulas:

$$\left(\overset{\bullet}{A}_1 f\right)_k (y) = -i\int_y^{\ell} f_k(t)dt,$$
$$\left(\overset{\bullet}{A}_2 f\right)_k (y) = \lambda_k f_k(y) + i\sum_{s=1}^{k-1} f_s(y)\beta_s^2.$$

It is easy to see that in this case $\overset{\bullet}{h}_0(y) = const$, then

$$f_k(y) = \left[\left(\overset{\bullet}{A}_1 - \mu_1 I\right)^{-1}\left(\overset{\bullet}{A}_2 - \mu_2 I\right)^{-1} g_0\right]_k (y)$$
$$= -\frac{1}{\mu_1(\lambda_k - \mu_2)} \exp\left(\frac{iy}{\mu_1}\right) \prod_{s=1}^{k-1} \frac{\mu_2 - \overline{\lambda}_{k-s}}{\mu_2 - \lambda_{k-s}}.$$

So,

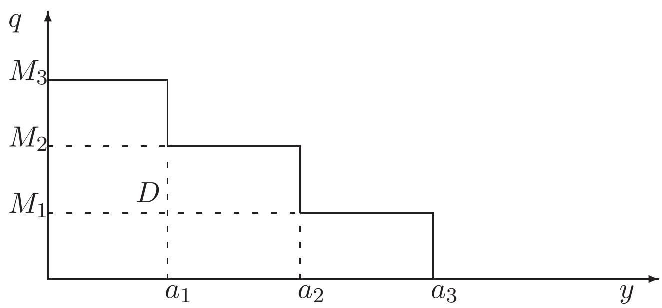
$$\Phi(x_1, x_2) = -\sum_{k=1}^{N} \beta_k^2 \Lambda_k(x_2) \int_0^{\ell} f(x,y) J_0(2\sqrt{x_1 y}) dy,$$

where functions Λ_k is defined in Sections 3.2 and
$J_0(z) = \sum\limits_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{(k!)^2}$ is the Bessel function of zero order.

4.2.

One may deduce the results analogous to that of Section 3, for Theorem 3.1 and Lemma 3.1 where $W(x,y)$ is in the form, given in (16).

The functions $f_k(y)$ defined in the following domain



Then it's easy to write out the explicit form of the functions $\Phi_\alpha(x_1, x_2)$:

$$\Phi_1(x_1, x_2) = \sum_{q=M_2}^{M_3} \beta_q^2 \Lambda_q(x_2) \int_0^{a_1} f(\xi, q) J_0(2\sqrt{x_1 \xi}) d\xi,$$

$$\Phi_2(x_1, x_2) = \sum_{q=M_1}^{M_2} \beta_q^2 \Lambda_q(x_2) \int_0^{a_2} f(\xi, q) J_0(2\sqrt{x_1 \xi}) d\xi,$$

$$
\begin{aligned}
\Phi_3(x_1,x_2) &= \sum_{q=1}^{M_1} \beta_q^2 \Lambda_q(x_2) \int_0^{a_3} f(\xi,q) J_0(2\sqrt{x_1\xi}) d\xi, \\
\Phi_4(x_1,x_2) &= \sum_{q=1}^{M_3} \beta_q^2 \Lambda_q(x_2) \int_0^{a_1} f(\xi,q) J_0(2\sqrt{x_1\xi}) d\xi, \\
\Phi_5(x_1,x_2) &= \sum_{q=1}^{M_2} \beta_q^2 \Lambda_q(x_2) \int_{a_1}^{a_2} f(\xi,q) J_0(2\sqrt{x_1\xi}) d\xi, \\
\Phi_6(x_1,x_2) &= \sum_{q=1}^{M_1} \beta_q^2 \Lambda_q(x_2) \int_{a_2}^{a_3} f(\xi,q) J_0(2\sqrt{x_1\xi}) d\xi,
\end{aligned}
\tag{18}
$$

where the numbers $1 < M_1 < M_2 < M_3$, $0 < a_1 < a_2 < a_3$,

$$
\Lambda_q^{(2)}(x_2) = -\frac{1}{2\pi i} \oint_{\Gamma_2} \exp(i\lambda_2 x_2) \frac{1}{(\lambda_q^{(2)} - \lambda_2)} \prod_{r=1}^{q-1} \frac{\lambda_2 - \overline{\lambda_{q-r}^{(2)}}}{\lambda_2 - \lambda_{q-r}^{(12)}} d\lambda_2,
$$

$\beta_q^{(2)} = \sqrt{2\mathrm{Im}\lambda_q^{(2)}}$ and

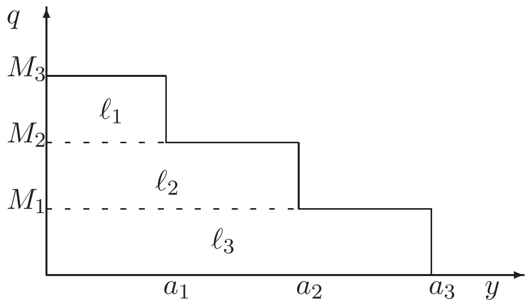
$$
J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{(k!)^2}
$$

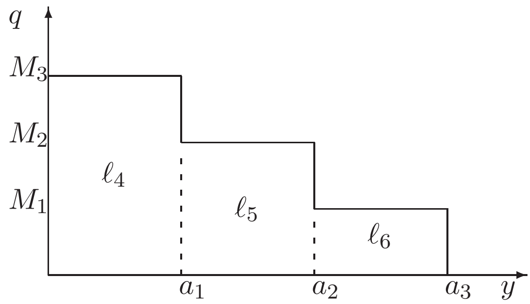
is the Bessel function of zero order.

Thus, we can formulate the following theorem.

Theorem 4.1. *Assume that a system of linear operators $\{A_1, A_2\}$ satisfies to the assumptions of Theorem 3.1 where the spectrum of operator A_1 is concentrated in zero and that of operator A_2 is non-real. Then the infinitesimal correlation function $W(x,y)$(11) is represented in the form* (16) *where $d_{\alpha,\beta} \in \mathbb{R}$ and the functions $\Phi_\alpha(x)$ are defined in* (18).

To evaluate $d_{\alpha,\beta}$ we represent ℓ_α graphically into two figures:





where, the ℓ_α's are constants in the indicated areas, i.e.

$$\ell_\alpha = \frac{\chi_{D_\alpha}}{\sqrt{S_{D_\alpha}}},$$

where, $\chi_{D\alpha}$ is the characteristic function of the domain D_α, which is show in the figures for ℓ_α and S_{D_α} is the area.

For example:

$$\ell_5 = \frac{\chi_{[a_1,a_2]\times[1,M_2]}}{\sqrt{(a_2-a_1)\sum\limits_{q=1}^{M_1}\left(\beta_q^{(2)}\right)^2}};$$

$$\ell_6 = \frac{\chi_{[a_2,a_3]\times[1,M_1]}}{\sqrt{(a_3-a_2)\sum\limits_{q=1}^{M_1}\left(\beta_q^{(2)}\right)^2}}.$$

Now, we calculate $D\ell_6$:

$$\begin{aligned} D\ell_6 &= (C^* + 4(A_1)_I(A_2)_I)\,\ell_6 = 2(A_1)_I \sum_{q=1}^{M_1}\left(\beta_q^{(2)}\right)^2 \ell_6 \\ &= \sum_{q=1}^{M_1}\left(\beta_q^{(2)}\right)^2 (a_3-a_2)\chi_{[0,a_3]\times[1,M_1]}. \end{aligned}$$

Then,

$$d_{6,6} = \langle D\ell_6, \ell_6\rangle = \sum_{q=1}^{M_1}\left(\beta_q^{(2)}\right)^2 (a_3-a_2)\left\langle \chi_{[0,a_3]\times[1,M_1]}, \ell_6\right\rangle$$

$$= \quad (a_3 - a_2) \sum_{q=1}^{M_1} \left(\beta_q^{(2)}\right)^2 .$$

Similarly,

$$d_{6,5} = \langle D\ell_6, \ell_5 \rangle = \sum_{q=1}^{M_1} \left(\beta_q^{(2)}\right)^2 \sqrt{\frac{(a_3 - a_2)}{\sum_{q=1}^{M_2} \left(\beta_q^{(2)}\right)^2}}.$$

Other coefficients are calculated similarly.

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