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POSTULATION OF DOUBLE POINTS WITH RESTRICTED SUPPORT: THE CASE OF A SMOOTH QUADRIC 3-FOLD

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Abstract: Let D be an integral hypersurface of the projective variety, $D \subset X$ a hypersurface and L a line bundle on X. Let $E_{x,y} \subset X$ be a general union of x double points of X and y double points of X. We classifies the triple (L, x, y) for which $h^1(\mathcal{I}_{E_{x,y}} \otimes L) \cdot h^0(\mathcal{I}_{E_{x,y}} \otimes L) > 0$ when X is a smooth quadric 3-fold and D is a hyperplane section of X and in some cases with X an integral quadric surface.

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1. Introduction

For any irreducible and projective variety X and any $P \in X_{\mathrm{reg}}$ let $(2P, X)$ denote the closed subscheme of X with $(\mathcal{I}_{P,X})^2$ as its ideal sheaf. The scheme $(2P, X)$ is a zero-dimensional scheme, $(2P, X)_{\mathrm{red}} = \{P\}$ and $\deg((2P, X)) = 1 + \dim(X)$. We say that $(2P, X)$ is a 2-point of X. For any finite set $S \subset X_{\mathrm{reg}}$ set $(2S, X) := \cup_{P \in S}(2P, X)$. We say that $(2P, X)$ is a 2-point of X.

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Theorem 1. *Let $X \subset \mathbb{P}^4$ be a smooth quadric hypersurface and let $D \subset Q_3$ be an integral hyperplane section of Q_3. Fix integers k, x, y such that $k \geq 1$, $x \geq 0$ and $0 \leq 3y \leq (k+1)^2$. Let $Z \subset Q_3$ be a general union of x 2-points of Q_3 and y 2-points of D. Then either $h^0(\mathcal{I}_Z(k)) = 0$ or $h^1(\mathcal{I}_Z(k)) = 0$, except in the following cases:*

(a) $(k, x, y) = (2, 3, 0)$; *we have* $h^0(\mathcal{I}_Z(2)) = 3$ *and* $h^1(\mathcal{I}_Z(2)) = 1$.

(b) $(k, x, y) = (2, 4, 0)$; *we have* $h^0(\mathcal{I}_Z(2)) = 1$ *and* $h^1(\mathcal{I}_Z(2)) = 3$.

(c) $(k, x, y) = (2, 0, 3)$; *we have* $h^0(\mathcal{I}_Z(2)) = 6$ *and* $h^1(\mathcal{I}_Z(2)) = 1$.

(d) $(k, x, y) = (2, 3, 1)$; *we have* $h^0(\mathcal{I}_Z(2)) = 1$ *and* $h^1(\mathcal{I}_Z(2)) = 2$.

(e) $(k, x, y) = (2, 2, 2)$; *we have* $h^0(\mathcal{I}_Z(2)) = 1$ *and* $h^1(\mathcal{I}_Z(2)) = 1$.

(f) $(k, x, y) = (2, 1, 3)$; *we have* $h^0(\mathcal{I}_Z(2)) = 2$ *and* $h^1(\mathcal{I}_Z(2)) = 1$.

Take X, D, k, x, y as in Theorem 1. Let $\phi_k : X \to \mathbb{P}^r$, $r := \binom{k+4}{4} - \binom{k+2}{4}$, be the embedding of X by the complete linear system $|\mathcal{O}_X(k)|$. By [1] the statement of Theorem 1 says that the join of x copies of $\phi_k(X)$ and y copies of $\phi_k(D)$ has the expected dimension if and only if (k, x, y) is not one of the exceptional cases listed in Theorem 1.

The case $y = 0$ of Theorem 1 is true ([3]).

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. We use at several steps this assumption.

2. Surfaces

For any line bundle L on X, any closed subscheme Z of X and any vector space $V \subseteq H^0(X, L)$ set $V(-Z) := V \cap H^0(X, \mathcal{I}_Z \otimes L)$. Let $D \subset X$ be an effective Cartier divisor of X. For every closed subscheme $Z \subset X$ the residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is a closed

For every line bundle L on X we have an exact sequence of coherent sheaves:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes L_{|D} \to 0 \qquad (1)$$

From (1) we get the following inequalities

- $h^0(X, \mathcal{I}_Z \otimes L) \leq h^0(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^0(D, \mathcal{I}_{Z\cap D,D} \otimes L_{|D})$;
- $h^1(X, \mathcal{I}_Z \otimes L) \leq h^1(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^1(D, \mathcal{I}_{Z\cap D,D} \otimes L_{|D})$.

We will call " Castelnuovo's inequality " any of these inequalities and call (1) the Castelnuovo's sequence.

Notation 1. When X and D are obvious from the context, $E_{x,y}$ denote a general union of x 2-points of X and y 2-points of D. Set $E_x := E_{x,0}$.

Remark 1. Fix any line bundle L on X. The semicontinuity theorem for cohomology gives $h^0(\mathcal{I}_{E_{x+y}} \otimes L) \leq h^0(\mathcal{I}_{E_{x,y}} \otimes L)$.

Lemma 1. *Fix a zero-dimensional scheme $Z \subset X$, a line bundle L on X and take a general $P \in X$. Set $n := \dim(X)$, $a := h^0(\mathcal{I}_Z \otimes L)$ and $b := h^0(\mathcal{I}_{Z \cup 2P} \otimes L)$. If $a - b = n+1$, then $h^0(\mathcal{I}_{Z \cup \{2P,T\}} \otimes L) = a - n$ for every divisor T of X with $P \in T_{\mathrm{reg}}$. Now assume $b \geq a - n$. Set $G := Aut(X)$. Assume that the action of G on X has an open orbit, U, and that for every $O \in U$ the stabilizer $G_O \subset G$ of O acts with an open orbit U_O on the set of all codimension one linear subspaces of the tangent space of X at O. Assume $D \cap U \neq \emptyset$ and that for a general $O \in D \cap U$ the tangent space $T_O D$ of D at O is contained in U_O. Then there is $g \in G$ such that $h^0(\mathcal{I}_{g(Z) \cup \{2O,D\}} \otimes L) = b$ for a general $O \in D$.*

Proof. Only the last assertion is not obvious. Assume $b \geq a - n$. Hence there is a degree n subscheme $W \subset 2P$ such that $h^0(\mathcal{I}_{Z \cup W} \otimes L) = b$. Moreover, this is true for an open subset U'_P of these degree n subscheme. Every degree n subschemes of $2P$ corresponds to a unique $(n-1)$-dimensional linear subspace of the tangent space of X at P and the converse holds. Hence we may find $h \in G$ with $h(O) = P$ and $h(T_O D) = W$. Use $g := h^{-1}$. □

Lemma 2. *Let T be an integral projective surface and $D \subset T$ be an integral Cartier divisor. Fix a zero-dimensional scheme $Z \subset X$, a linear subspace V of $H^0(X, L)$ and integers $a \geq 0$ and $b \geq 0$. Let $W \subseteq H^0(D, L_{|D})$ be the image of the restriction map $V(-Z) \to H^0(D, L_{|D})$. Fix $(a,b) \in \mathbb{N}^2$ and let $E \subset D$ be a general union of a 2-points of D and b points of D. Set $\alpha := \dim(V)$ and $\beta := \dim(W)$.*

(i) We have $\dim(V(-(Z \cup E)) = \alpha - \min\{\beta, 2a + b\}$.

(ii) If $\dim(V(-Z)) = \dim(V) - \deg(Z)$ and $h^0(X, \mathrm{Res}_D(Z) \otimes L(-D)) = 0$, then $\dim(V(-(Z \cup E))) = \max\{0, \alpha - 2a - b\}$.

Remark 2. Let X be an integral projective variety, D an integral Cartier divisor of X , Z a closed subscheme of X, L a line bundle on X and $V \subseteq H^0(X, L)$ a linear subspace. Set $V(-D) := \{f \in V : f|D \equiv 0\}$. Fix an integer $y > 0$ and take a general $E \subset D$ with $\sharp(E) = y$. We have $V(-E) =$

$\max\{\dim(V(-D)), \dim(V) - y\}$. If $V = H^0(X, \mathcal{I}_Z \otimes L)$, then $V(-D) = H^0(X, \mathrm{Res}_D(Z) \otimes L(-D))$.

Proof. Since $\dim(Z \cap D) < \dim(D)$ and E is general in D, we have $E \cap Z = \emptyset$ and hence the notation $Z \cup E$ is not ambiguos. Since $\mathrm{char}(K) = 0$ and D is integral, we have $\dim(W(-E)) = \max\{0, \alpha - \beta - 2a - b\}$ ([8]). Part (ii) follows from part (i), because $\beta = \alpha$ in this case by the residual exact sequence (1). □

Corollary 1. *Let T be an integral projective surface and $D \subset T$ be an integral Cartier divisor. Fix $L \in Pic(X)$ and a zero-dimensional scheme $Z \subset X$ such that $h^1(X, \mathcal{I}_Z \otimes L) = 0$ and $h^0(\mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) = 0$. Set $\alpha := h^0(X, \mathcal{I}_Z \otimes L) - h^0(\mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D))$. Fix integer $a \ge 0$, $b \ge 0$ and take a general $S \cup S' \subset D$ such that $\sharp(S) = a$, $\sharp(S') = b$ and $S \cap S' = \emptyset$.*

Then $h^0(X, \mathcal{I}_{Z \cup \{2S,D\} \cup S'} \otimes L) = \max\{0, \alpha - 2a - b\}$ and $h^1(X, \mathcal{I}_{Z \cup \{2S,D\} \cup S'} \otimes L) = \max\{0, 2a + b - \alpha\} + h^1(\mathcal{I}_Z \otimes L)$.

By our definition of the scheme E_x we may always assume $E_x \cap D = \emptyset$. Hence as a particular case of Corollary 1 we get the following result (part (d) being obvious).

Proposition 1. *Let X be an integral projective surface, L a line bundle on X and $D \subset X$ an effective Cartier divisor of X. Fix $(x, y) \in \mathbb{N}^2$ such that $2y \le \eta$. Set $\alpha := h^0(\mathcal{I}_{E_x} \otimes L) - h^0(\mathcal{I}_{E_x} \otimes L(-D))$ and $\eta := h^0(L) - h^0(L(-D))$.*

- *(a) If $h^1(\mathcal{I}_{E_x} \otimes L(-D)) = 0$ and $2y \le \eta$, then $h^1(\mathcal{I}_{E_{x,y}} \otimes L) = 0$.*
- *(b) If $h^0(\mathcal{I}_{E_x} \otimes L(-D)) = 0$ and $2y \le \eta$, then either $h^0(\mathcal{I}_{E_{x,y}} \otimes L) = 0$ or $h^1(\mathcal{I}_{E_{x,y}} \otimes L) = 0$.*
- *(c) If $h^0(\mathcal{I}_{E_x} \otimes L) = 0$, then $h^0(\mathcal{I}_{E_{x,y}} \otimes L) = 0$.*
- *(d) We have $h^1(\mathcal{I}_{E_{x,y}} \otimes L) = h^1(\mathcal{I}_{E_x} \otimes L) + \max\{2y - \alpha, 0\}$ and $h^0(\mathcal{I}_{E_{x,y}} \otimes L) = h^0(\mathcal{I}_{E_x} \otimes L) - \min\{\alpha, 2y\}$.*

The integer η in the statement of Proposition 1 is the image of the restriction map $H^0(L) \to H^0(D, L_{|D})$.

Proposition 2. *Let $X \subset \mathbb{P}^3$ be an integral quadric surface and let $D \subset X$ be a smooth hyperplane section. Fix $(k, x, y) \in \mathbb{N}^3$ such that $k \ge 1$ and $2y \le 2k + 1$, i.e. $y \le k$. Let $Z \subset X$ be a general union of x 2-points of X and y 2-points of D. Then either $h^0(\mathcal{I}_Z(k)) = 0$ or $h^1(\mathcal{I}_Z(k)) = 0$, unless $(k, x, y) = (2, 3, 0)$. If $(k, x, y) = (2, 3, 0)$, then $h^0(\mathcal{I}_Z(2)) = h^1(\mathcal{I}_Z(2)) = 1$.*

Proof. Set $\epsilon := 3x+2y-(k+1)^2$. Increasing or decreasing x (for a fixed y) we may assume $-2 \le \epsilon \le 2$ (unless we landle in an exceptional case). If $k=1$, then it is sufficient to notice that $h^1(\mathcal{I}_{E_{0,0}}(1)) = h^1(\mathcal{I}_{E_{0,1}}(1)) = h^1(\mathcal{I}_{E_{1,0}}(1)) = 0$ and $h^0(\mathcal{I}_{E_{2,0}}(1)) = h^0(\mathcal{I}_{E_{1,1}}(1)) = 0$.

(a) First assume $k=2$. We have $0 \le y \le 2$.

(a1) Assume $y=0$. First assume $x=3$. Let $B \subset X$ be a general subset with cardinality 3. Since B is contained in a unique $C \in |\mathcal{O}_X(1)|$, obviously $h^0(\mathcal{I}_{E_3}(2)) > 0$. The uniqueness of C and [7] easily implies $h^0(\mathcal{I}_{E_3}(2)) = 1$. Hence $h^1(\mathcal{I}_{E_3}(2)) = 1$. We also get $h^1(\mathcal{I}_{E_x}(2)) = 0$ for all $x \le 2$ and $h^0(\mathcal{I}_{E_x}(3)) = 0$ for all $x \ge 4$.

(a2) Assume $y > 0$. Since $h^0(\mathcal{I}_{E_3}(2)) = 1$, we have $h^0(\mathcal{I}_{E_{3,y}}(2)) = 0$ for all $y > 0$. Let $B \subset X$ be a general union of two 2-points. Let V be the image of the restriction map $H^0(\mathcal{I}_B(2)) \to H^0(D, \mathcal{O}_D(2))$. Since $h^0(\mathcal{I}_B(2)) = 0$ and $h^0(\mathcal{I}_B(3)) = 3$ (step (a1)), we have $\dim(V) = 3$. Lemma 2 (i.e. [8]) gives $h^0(\mathcal{I}_{E_{2,1}}(2)) = 1$ (i.e. $h^1(\mathcal{I}_{E_{2,1}}(2)) = 0$ and $h^0(\mathcal{I}_{E_{2,2}}(2)) = 0$. Fix a general 2-point A of X and a general $A \subset D$ with $\sharp(E) = y$. Since $y \le 2$, we have $h^1(D, \mathcal{I}_{\{2E,D\}}(2)) = 0$. Since $h^1(\mathcal{I}_A(1)) = 0$, the Castelnuovo's inequality gives $h^1(\mathcal{I}_{E_{1,y}}(2)) = 0$. Taking $\emptyset$ instead of A we get $h^1(\mathcal{I}_{E_{0,y}}(2)) = 0$.

(b) Now assume $k \ge 3$. We also assume to have proved the proposition for the line bundle $\mathcal{O}_X(k-1)$ and (if $k \ne 3$) for the line bundle $\mathcal{O}_X(k-2)$. Set $f := k-y$. Let $A \subset X$ be a general union of $x-f-1$ 2-points. Fix a general $S \cup S' \subset D$ such that $\sharp(S) = y$, $\sharp(S') = f$ and $S \cap S' = \emptyset$. Fix a general $O \in D$. Let Y be a general union of $A \cup 2S' \sup\{2S, D\}$ and one 2-point of X. By the semicontinuity theorem for cohomology ([9], III.12.8), to prove the proposition for the triple (k,x,y) it is sufficient to prove that either $h^0(\mathcal{I}_Y(k)) = 0$ or $h^1(\mathcal{I}_Y(k)) = 0$. Since $h^i(D, \mathcal{I}_{\{2(S\cup S')\cup\{O\}}(k)) = 0$, $i = 0,1$, by the differential Horace lemma for 2-points ([4], [6], Lemma 5, [5]), it is sufficient to prove that either $h^0(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) = 0$ or $h^1(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) = 0$. Notice that there is no exceptional case $(k,x,y) = (2,x,y)$ with $y > 0$. Hence by the inductive assumption either $h^0(\mathcal{I}_{A\cup\{2O,D\}}(k-1)) = 0$ or $h^1(\mathcal{I}_{A\cup\{2O,D\}}(k-1)) = 0$. If $h^0(\mathcal{I}_{A\cup\{2O,D\}}(k-1)) = 0$, then $h^0(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) = 0$. Hence we may assume $h^1(\mathcal{I}_{A\cup\{2O,D\}}(k-1)) = 0$, i.e. $h^0(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) = k^2 - y - 3(x-f-1) - 2$ and $h^0(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) > 0$. If $f = 0$, then we are done. Hence we may assume $f > 0$. Notice that $\deg(A \cup \{2O, D\}) = k^2 + \epsilon$; hence if $\epsilon \ge 0$, then our assumption imply $f > \epsilon$. By Remark 2 to check that either $h^0(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) = 0$ or $h^1(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_A(k-2)) \le \max\{0, h^0(\mathcal{I}_{A\cup S'\cup\{2O,D\}}(k-1)) - f\}$.

(b1) Assume $k=3$. We have $14 \le 3x+2y \le 18$ and $y+f=3$. Hence $x-f-1 \ge 2$. Hence $h^0(\mathcal{I}_A(1)) = 0$, concluding the case $k=3$.

(b2) Assume $k = 4$. We have $3x + 2y = 25 + \epsilon$ and $y + f = 4$. Hence $3x-2f = 17+\epsilon$. We have $h^0(\mathcal{I}_{A\cup\{2O,D\}}(3)) = 16-3(x-f-1)-2 = 17-3x+3f$. First assume $x - f \le 3$. We get $x \ge 11 + \epsilon$ and hence $3x \ge 33 + 3\epsilon > 25 + \epsilon$, a contradiction. If $x - f - 1 \ge 4$, then $h^0(\mathcal{I}_A(2)) = 0$ and hence we are done. Now assume $x - f - 1 = 3$, i.e. $x = 9 + \epsilon$. Hence $2y = -2\epsilon - 2$. Hence $\epsilon = -2$, $y = 1$ and $x = 7$. We first check that $h^1(\mathcal{I}_{E_7}(4)) = 0$. Take a general union $A' \subset X$ of 5 2-points. Let $S_1 \subset D$ be a general subset with $\sharp(S_1) = 2$. Step (b1) gives $h^1(\mathcal{I}_{A'}(3)) = 0$. Since $h^0(\mathcal{I}_{A'}(2)) = 0$ (step (a)), Remark ++ gives $h^1(\mathcal{I}_{A'\cup S_1}(3)) = 0$. Since $h^1(D, \mathcal{I}_{\{2S_1,D\}}(3)) = 0$ and $\mathrm{Res}_D(A' \cup 2S_1) = A' \cup S_1$, Castelnuovo's inequality gives $h^1(\mathcal{I}_{A'\cup 2S_1}(3))00$. Fix a general union $F \subset X$ of 7 2-points. We just proved that $h^1(\mathcal{I}_F(3)) = 0$, i.e. $h^0(\mathcal{I}_F(3)) = 4$. Let $V \subseteq H^0(D, \mathcal{O}_D(4))$ be the image of the restriction map $H^0(\mathcal{I}_F(4)) \to H^0(D, \mathcal{O}_D(4))$. Since $h^0(\mathcal{I}_F(3)) = 0$ (step (b1)), we have $\dim(V) = 4$. Hence $\dim(V(-2P)) = 2$ for a general $P \in D$ ([8]). Hence $h^1(\mathcal{I}_{E_{7,1}}(4)) = 0$, concluding the case $k = 4$.

(b3) Now assume $k \ge 5$. The inductive assumption gives that either $h^0(\mathcal{I}_A(k-2)) = 0$ or $h^1(\mathcal{I}_A(k-2)) = 0$. In the latter case we have $h^0(\mathcal{I}_A(k-2)) = (k-1)^2 - 3(x-f-1)$. Hence it is sufficient to check that $f + 2 \le k^2 - (k-1)^2 = 2k - 1$, i.e. $f \le 2k - 3$. This inequality is true, because $y + f = k$. □

Proposition 3. *Fix non-negative integers u, v, a, b, x, y such that $u > 0$, $v > 0$ and $(a, b) \notin \{(0, c), (c, 0)\}$ for some $c \ne 1$. Assume $2y \le \eta$, where $\eta := (u+1)(v+1)-(u-a+1)(v-b+1)$ if $u \ge a$ and $v \ge b$ and $\eta := (u+1)(v+1)$ otherwise. Fix an integral $D \in |\mathcal{O}_X(a, b)|$. Then either $h^0(\mathcal{I}_{E_{x,y}}(u, v)) = 0$ or $h^1(\mathcal{I}_{E_{x,y}}(u, v)) = 0$, except in the following cases:*

(a) $y = 0$, $u \equiv v \equiv 0 \pmod 2$, either $u = 2$ or $v = 2$, and $x = (u+1)(v+1)/3$; in this case $h^0(\mathcal{I}_{E_{x,0}}(u, v)) = h^1(\mathcal{I}_{E_{x,0}}(u, v)) = 1$.

(b) $u = a$, $b \le v$, $2x \le v-b$ and $\eta - x < 2y \le \eta$; in this case $h^0(\mathcal{I}_{E_{x,y}}(u, v)) = v - b + 1 - 2x$ and $h^1(\mathcal{I}_{E_{x,y}}(u, v)) = x + 2y - v + b - (u + 1)(v + 1)$.

(c) $v = b$, $a \le u$, $2x \le u-a$ and $\eta-x < 2y \le \eta$; in this case $h^0(\mathcal{I}_{E_{x,y}}(u, v)) = u - a + 1 - 2x$ and $h^1(\mathcal{I}_{E_{x,y}}(u, v)) = x + 2y + u - a - (u + 1)(v + 1)$.

(d) $u-a \ge 2$, $v-b \ge 2$, $u-a \equiv v-b \equiv 0 \pmod 2$, $x = (u-a+1)(v-b+1)/3$, $2y = \eta$; in this case we have $h^0(\mathcal{I}_{E_{x,y}}(u, v) = h^1(\mathcal{I}_{E_{x,y}}(u, v)) = 1$.

Proof. The assumptions on a and b are equivalent to assuming the existence of an integral element of $|\mathcal{O}_X(a, b)|$.The integer η is the dimension of the image of the restriction map $H^0(\mathcal{O}_X(u, v)) \to H0(\mathcal{O}_X(u, v))$. Hence by [8] we may assume $x > 0$. If $h^0(\mathcal{I}_{E_{x,0}}(u,v)) = 0$, then $h^0(\mathcal{I}_{E_{x,y}}(u,v)) = 0$ for all $y \geq 0$. Hence from now on we assume $h^0(\mathcal{I}_{E_x}(u,v)) > 0$.

(a) Assume $y = 0$. By [2], Proposition 4.1, or [10] or [7] the only exceptional cases are when u and v are even, either $u = 2$ or $v = 2$ and $x = (u+1)(v+1)/3$. In this case we have $h^1(\mathcal{I}_{E_x}(u,v)) = h^0(\mathcal{I}_{E_x}(u,v)) = 1$ (e.g. if $u = 2$ and $F \subset X$ is a general subset of X with $\sharp(B) = x$ the only element of $|\mathcal{I}_{2B}(2,v)|$ is the curve $2C$, where C is the unique element of $|\mathcal{I}_B(1,v/2)|$. Since $C \neq D$ (for a general E_x) we have $h^0(\mathcal{I}_{E_{x,y}}(u,v)) = 0$ if $y > 0$ and $h^0(\mathcal{I}_{E_x}(u,v)) = 1$. Hence from now on we assume $y > 0$.

(b) Assume $h^0(\mathcal{I}_{E_x}(u-a,v-b)) = 0$. If $h^1(\mathcal{I}_{E_x}(u,v)) = 0$, then apply Proposition 1. If $h^1(\mathcal{I}_{E_x}(u,v)) > 0$ and (as we are assuming) $h^0(\mathcal{I}_{E_x}(u,v)) > 0$, then step (a) gives $h^0(\mathcal{I}_{E_{x,y}}(u,v)) = 0$ for all $y > 0$.

(c) By step (b) we may assume $h^0(\mathcal{I}_{E_x}(u-a,v-b)) > 0$, $x > 0$, and $y > 0$. Since $h^0(\mathcal{I}_{E_x}(u-a,v-b)) > 0$, we have $h^0(\mathcal{I}_{E_{x,y}}(u,v)) > 0$. If $h^1(\mathcal{I}_{E_x}(u-a,v-b)) = 0$, then $h^1(\mathcal{I}_{E_{x,y}}(u,v)) = 0$. Hence we may assume $h^1(\mathcal{I}_{E_x}(u-a,v-b)) > 0$. Since $h^0(\mathcal{I}_{E_x}(u-a,v-b)) > 0$ and $x > 0$, we have $u \geq a$, $v \geq b$ and $(u,v) \neq (a,b)$. Since $h^1(\mathcal{I}_{E_x}(u-a,v-b)) > 0$, then either $u \leq a+2$ or $v \leq b+2$. Fix a general union $B \subset X$ of x 2-points of X. In particular we have $B \cap D = \emptyset$ and $h^i(\mathcal{I}_B(z,w)) = h^i(\mathcal{I}_{E_x}(z,w))$ for all z, w. Let $S \subset D$ be a general subset with $\sharp(S) = y$. Let α be the dimension of the image V of the restriction map $H^0(\mathcal{I}_B(u,v)) \to H^0(D, \mathcal{O}_D(u,v))$.

(c1) Assume $u = a$. We have $h^0(\mathcal{I}_{E_x}(0,v-b)) > 0$ if and only if $2x \leq v-b$. Assume $2x \leq v-b$. Since $u > 0$, we get $3x < (u+1)(v+1)$. Hence $h^1(\mathcal{I}_{E_x}(u,v)) = 0$ (step (a)). We have $h^0(\mathcal{I}_B(0,v-b)) = v-b+1-2x$. We have $\eta = (u+1)(b+1) - (v-b+1)$ and $\alpha = \eta - x$. By [8] we have $\dim(V(-2S)) = \max\{0, \alpha - 2y)$. By part (b) of Proposition 1 we get that $h^1(\mathcal{I}_{E_{x,y}}(u,v)) > 0$ if and only if $2y > \eta - x$. Since $h^1(\mathcal{I}_{E_x}(u,v)) = 0$, we also get that if $2y > \eta - x$, then $h^0(\mathcal{I}_{E_{x,y}}(u,v)) = v-b+1-2x$ and $h^1(\mathcal{I}_{E_{x,y}}(u,v)) = x + 2y + v - b + 1 - (u+1)(v+1)$. In the same way we handle the case $v = b$.

(c2) Assume $u = a+1$. By step (c1) we may assume $v > b$. Assume for the moment that $h^1(\mathcal{I}_{E_x}(u,v)) = 0$. By part (b) of Proposition 1 we have $h^0(\mathcal{I}_{E_{x,y}}(u,v)) \cdot h^1(\mathcal{I}_{E_{x,y}}(u,v)) = 0$, unless $h^0(\mathcal{I}_{E_x}(1,v-b+1)) > 0$, i.e. unless $3x \leq 2v-2b+3$. Assume $3x \leq 2v-2b+2$. Since $h^1(\mathcal{I}_{E_x}(1,v-b+1)) = 0$, we get $h^1(\mathcal{I}_{E_{x,y}}(u,v)) = 0$. Now assume that $h^1(\mathcal{I}_{E_x}(u,v)) > 0$. Since $h^0(\mathcal{I}_{E_x}(u,v)) > 0$ and $y > 0$, step (a) gives $h^0(\mathcal{I}_{E_{x,y}}(u,v)) = 0$.

In the same way we handle the case $v = b+1$.

(c3) Assume $u = a+2$. By steps (c1) and (c2) we may assume $v \geq b+2$. Since $u \geq 3$, $v \geq 3$ and $h^0(\mathcal{I}_{E_x}(u,v)) > 0$, we have $h^1(\mathcal{I}_{E_x}(u,v)) > 0$. By part (b) of Proposition 1 we have $h^0(\mathcal{I}_{E_{x,y}}(u,v)) \cdot h^1(\mathcal{I}_{E_{x,y}}(u,v)) = 0$, unless $h^0(\mathcal{I}_{E_x}(2, v-b)) > 0$, i.e. unless $x \le v-b+1$. If $x \le v-b$, then $h^1(\mathcal{I}_{E_x}(2, v-b)) = 0$ and hence $h^1(\mathcal{I}_{E_{x,y}}(u,v)) = 0$ (Proposition 1). Now assume $x = v-b+1$. If $2y = \eta$, then $h^i(D, \mathcal{I}_{\{2S,D\}}(u,v)) = 0$, $i = 0,1$, and hence $h^0(\mathcal{I}_{E_{x,y}}(u,v)) = h^1(\mathcal{I}_{E_{x,y}}(u,v)) = 1$. Now assume $2y < \eta$. In this case $\alpha = \eta - 1$. By [7] for general S the scheme $\{2S, D\}$ imposes $2y$ independent conditions to V. Since $h^1(\mathcal{I}_{E_x}(u,v)) = 0$, we get $h^1(\mathcal{I}_{E_{x,y}}(u,v)) = 0$. In the same way we handle the case $v = b+2$. □

Proposition 4. *Fix integers $k > 0$, $a > 0$, $x \ge 0$ and y such that $0 \le 2y \le \eta$, where $\eta = (k+1)^2 - (k-a+1)^2$ if $a \le k$ and $\eta := (k+1)^2$ if $a > k$. Let $X \subset \mathbb{P}^3$ be an integral quadric cone. Fix an integral curve $D \in |\mathcal{O}_X(k)|$. Then either $h^0(\mathcal{I}_{E_{x,y}}(k)) = 0$ or $h^1(\mathcal{I}_{E_{x,y}}(k)) = 0$, except if either $(k,x,y) = (2,3,0)$ or η is even $k = a+2$, $x = 3$ and $y = \eta/2$. In these two exceptional cases we have $h^0(\mathcal{I}_{E_{x,y}}(k)) = h^0(\mathcal{I}_{E_{x,y}}(k)) = 1$.*

Proof. Since $h^0(\mathcal{O}_X(k-a)) = 0$ if $k < a$ and $h^0(\mathcal{O}_X(k-a)) = (k-a+1)$ if $k \ge a$, the integer η is the dimension of the image of the restriction map $H^0(\mathcal{O}_X(k)) \to H^0(D, \mathcal{O}_D(k))$. By Proposition 2 we may assume $a \ge 2$ and $y > 0$. By Propositions 1 and 2 it is sufficient to study the case $k = a+2$. Assume $k = a+2$. We have $h^0(\mathcal{I}_{E_x}(2)) = 0$ if and only if $x \ge 4$. Hence we are done if $x \ge 4$ (part (b) of Proposition 1). If $x \le 2$, then $h^1(\mathcal{I}_{E_x}(2)) = 0$ and hence $h^1(\mathcal{I}_{E_{x,y}}(k)) = 0$ for all $y \le \lfloor \eta/2 \rfloor$ (part (a) of Proposition 1). Now assume $x = 3$. First assume $2y = \eta$. Since $h^0(\mathcal{I}_{E_3}(2)) > 0$ and $h^0(\mathcal{I}_{E_3}(2)) > 0$, we have $h^0(\mathcal{I}_{E_{x,\eta/2}}(k)) > 0$. Notice that $h^0(\mathcal{I}_{E_{x,\eta/2}}(k)) = h^1(\mathcal{I}_{E_{x,\eta/2}}(k))$. Since $h^1(D, \mathcal{I}_{\{2S,D\}}(k)) = 0$ and $h^1(\mathcal{I}_B(2)) = h^1(\mathcal{I}_{E_3}(2)) = 1$, (1) gives $h^1(\mathcal{I}_{E_{3,\eta/2}}(a+2)) \le 1$. Hence $h^0(\mathcal{I}_{E_{x,\eta/2}}(k)) = h^1(\mathcal{I}_{E_{x,\eta/2}}(k)) = 1$.

Now assume $2y \le \eta - 2$. Let B_1 be a general union of two 2-points. Take a general $S \cup S_1 \subset D$ such that $\sharp(S) = y$, $\sharp(S_1) = 1$ and $S \cap S_1 = \emptyset$. To prove that $h^1(\mathcal{I}_{E_{3,y}}(k)) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{B_1 \cup 2S_1 \cup \{2S,D\}}(k)) = 0$. Since $(B_1 \cup 2S_1 \cup \{2S,D\}) \cap D = (2S_1 \cup S, D\}$, $\sharp(S_1 \cup S) \le \eta/2$ and $S_1 \cup S$ is general in D, we have $h^1(D, \mathcal{I}_{\{2(S \cup S_1),D\}}(k)) = 0$. Since $f > 0$, we have $x' \le 2$. Hence $h^1(\mathcal{I}_{B_1}(2)) = 0$. We have $h^0(\mathcal{I}_{B_1}(k-2a) = 0$, because $k - 2a \le 0$ and $B_1 \ne \emptyset$. Apply Remark 2.

Now assume $\eta = 2y+1$. Fix a general $P \in D$. To copy the proof of the case $2t \le \eta - 2$ it is sufficient to check that $h^1(\mathcal{I}_{B_1 \cup \{2P,D\}}(2)) = 0$. Hence it is sufficient to prove that $h^1(\mathcal{I}_{E_{2,1}}(2)) = 0$. This is true by Proposition 1, because $h^0(\mathcal{I}_{E_2}(k-a)) = 0$, □

3. Proof of Theorem 1

Proof of Theorem 1. For all integers $k > 0$ we have $h^0(\mathcal{O}_X(k)) = \binom{k+4}{4} - \binom{k+2}{4} = (k+2)(k+1)(2k+3)/6$. Set $\epsilon := 4x + 3y - (k+2)(k+1)(2k+3)/6$. If $k = 1$, then it is sufficient to notice that $h^1(\mathcal{I}_{E_{0,0}}(1)) = h^1(\mathcal{I}_{E_{0,1}}(1)) = h^1(\mathcal{I}_{E_{1,0}}(1)) = 0$ and $h^0(\mathcal{I}_{E_{1,1}}(1)) = h^0(\mathcal{I}_{E_{2,0}}(1)) = 0$. All cases with $y = 0$ are known ([3]). Hence we may assume $y > 0$. First assume $x = 0$. We have $h^0(\mathcal{I}_{E_{0,y}}(k)) > 0$ for all $k > 1$. Since $3y \le (k+1)^2$, Proposition 2 gives that $h^1(\mathcal{I}_{E_{0,y}}(k)) = 0$ if and only if $k = 2$ and $y = 3$. Since $h^i(\mathcal{O}_X(1)) = 0$, $i = 1, 2$, we have $h^1(\mathcal{I}_{E_{0,3}}(2)) = 1$ and $h^0(\mathcal{I}_{E_{0,3}}(2)) = 6$. From now on we assume $x > 0$.

(a) Assume $k = 2$. First assume $x = 3$. Since $h^1(\mathcal{I}_{E_x}(3)) > 0$ for all $x \ge 3$ ([3]), we have $h^1(\mathcal{I}_Z(2)) > 0$ if $x \ge 3$. Since $h^0(\mathcal{I}_{E_4}(2)) > 0$, Remark 1 gives $h^0(\mathcal{I}_{E_{3,1}}(2)) > 0$. Since $h^0(\mathcal{I}_{E_3}(2)) = 3$ and $h^0(\mathcal{I}_{E_4}(2)) = 1$, Lemma 1 gives $h^0(\mathcal{I}_{E_{3,1}}(2)) = 1$, i.e. $h^1(\mathcal{I}_{E_{3,1}}(2)) = 2$. Since $h^0(\mathcal{I}_{E_{3,1}}(2)) = 1$ and $h^0(\mathcal{I}_{E_3}(1)) = 0$, we have $h^0(\mathcal{I}_{E_{3,y}}(2)) = 0$ for all $y \ge 2$. Since $h^0(\mathcal{I}_{E_4}(2)) = 1$, then $h^0(\mathcal{I}_Z(2)) = 0$ if $x \ge 4$ and $y > 0$.

Now assume $x = 2$. Since $h^0(\mathcal{I}_{E_2}(1)) = 0$ and $h^1(\mathcal{I}_{E_2}(2)) = 0$, the image V of the restriction map $H^0(\mathcal{I}_{E_2}(2)) \to H^0(D, \mathcal{O}_D(2))$ has dimension 8. Since $h^0(\mathcal{I}_{E_3}(2)) = 3$, Lemma 1 gives $h^0(\mathcal{I}_{E_{2,1}}(2)) = 3$, i.e. $h^1(\mathcal{I}_{E_{2,1}}(2)) = 0$. Since $h^0(\mathcal{I}_{E_4}(2)) = 1$, we have $h^0(\mathcal{I}_{E_{2,2}}(2)) \ge 1$ by the semicontinuity theorem. Since $h^0(\mathcal{O}_X(2)) = 14$, we have $h^0(\mathcal{I}_{E_{2,2}}(2)) = h^1(\mathcal{I}_{E_{2,2}}(1))$. Assume $h^0(\mathcal{I}_{E_{2,2}}(2)) \ge 2$. Since $h^0(\mathcal{I}_{E_2}(1)) = 0$ and $h^0(\mathcal{I}_{E_2}(2)) = 6$, the image V of the restriction map $h^0(\mathcal{I}_{E_2}(2)) \to H^0(D, \mathcal{O}_D(2))$ has dimension 6. Take general $O, P \in D$. Since $h^0(\mathcal{I}_{E_{2,1}}(2)) = 3$, we have $\dim(V(-2O)) = 3$. To check that $h^0(\mathcal{I}_{E_{2,2}}(2)) \le 1$ and hence to conclude the case $(x, y) = (2, 2)$ it is sufficient to prove that $\dim(V(-2O - 2P)) \le 1$. Assume that $\dim(V(-2O - 2P)) \ge 2$. Since P is general in D, it means that the rational map $D \dashrightarrow \mathbb{P}^2$ induced by the linear system $V(-2O)$ has generically differential zero. Since $\dim(V(-2O)) \ge 2$ and $\mathrm{char}(\mathbb{K}) = 0$, this is absurd.

Now assume $x = 1$. Since$h^1(\mathcal{I}_{E_1}(1)) = 0$, we get $h^1(\mathcal{I}_{E_{1,y}}(2)) = 0$ if $y \le 2$ and $h^1(\mathcal{I}_{E_{1,3}}(2)) \le 1$. Now assume $y = 3$. Take a general set $S \subset D$ with $\sharp(S) = 3$. Since $h^1(D, \mathcal{I}_{\{2S,D\}}(2)) = 1$, $h^0(\mathcal{I}_{E_1}(2)) = 1$ and $h^i(\mathcal{I}_{E_1}(1)) = 0$, $i = 1, 2$, the Castelnuovo's exact sequence (1) gives $h^1(\mathcal{I}_{E_{1,3}}(2)) = 1$ and $h^0(\mathcal{I}_{E_{1,3}}(2)) = 2$.

(b) Assume $k > 2$ and that Theorem 1 is true for the line bundle $\mathcal{O}_X(k-1)$ and, if $k \ge 4$, for the line bundle $\mathcal{O}_X(k-2)$. Set $f := (k+1)^2 - 3y$ and $g := (k+1)^2 - 3y - 3f$. We have $0 \le g \le 2$. Let $A \subset X$ be a general union of $x - f - 1$ 2-points. Fix a general $S \cup S' \cup S'' \subset D$ such that $\sharp(S) = y$, $\sharp(S') = f$, $\sharp(S'') = g$, and $S \cap S' = S \cap S'' = S' \cap S'' = \emptyset$. Let Y be a general union of $A \cup 2S' \cup \{2S, D\}$ and g 2-points of X. By the semicontinuity theorem for cohomology ([9], III.12.8), to prove the proposition for the triple (k,x,y) it is sufficient to prove that either $h^0(\mathcal{I}_Y(k))=0$ or $h^1(\mathcal{I}_Y(k))=0$. Since $h^i(D,\mathcal{I}_{\{2(S\cup S'),D\}\cup S''}(k))=0$, $i=0,1$, by the differential Horace lemma for 2-points ([4], [6], Lemma 5, [5]), it is sufficient to prove that either $h^0(\mathcal{I}_{A\cup S'\cup\{2S'',D\}}(k-1))=0$ or $h^1(\mathcal{I}_{A\cup S'\cup\{2S'',D\}}(k-1))=0$. If $h^0(\mathcal{I}_{A\cup\{2S'',D\}}(k-1))=0$, then $h^0(\mathcal{I}_{A\cup S\cup\{2S'',D\}}(k-1))=0$ and hence we are done. Hence we may assume $h^0(\mathcal{I}_{A\cup\{2S'',D\}}(k-1))>0$. If $k\geq 4$, then the inductive assumption gives $h^1(\mathcal{I}_{A\cup\{2S'',D\}}(k-1))=0$, i.e. $h^0(\mathcal{I}_{A\cup\{2S'',D\}}(k-1))=-\epsilon+f$. By Lemma 2 it is sufficient to prove that $h^0(\mathcal{I}_A(k-2))\leq\max\{0,-\epsilon\}$.

(b1) Assume $k=3$. We have $4x+3y=30+\epsilon$ and $3y+3f+g=16$. Hence $g=1$ and $y+f=5$. Since $h^0(\mathcal{I}_{E_7}(3))=2$ ([3]) and $h^0(\mathcal{I}_{E_8}(3))=0$ ([3]), Lemma 1 gives $h^0(\mathcal{I}_{E_{7,1}}(3))=0$. Since $h^1(\mathcal{I}_{E_7}(3))=0$, we also get $h^1(\mathcal{I}_{E_{6,1}}(3))=0$. Hence we may assume $y\geq 2$.

(b1.1) First assume $y=2$ and hence $f=3$. It is sufficient to do the case $x=6$. We have $x-f-1=2$. Hence $h^0(\mathcal{I}_A(1))=0$. Since $h^1(\mathcal{I}_{E_{2,1}}(2))=0$ (step (a)), we have $h^0(\mathcal{I}_{A\cup\{2S'',D\}}(2))=3$. Since $h^0(\mathcal{I}_A(1))=0$, Lemma 2 gives $h^i(\mathcal{I}_{A\cup S'\cup\{2S'',D\}}(2))=0$, $i=0,1$.

(b1.2) Now assume $y=3$ and hence $f=2$. It is sufficient to do the cases $x=5$ and $x=6$. First assume $x=5$. Since $h^1(\mathcal{I}_{E_{2,1}}(2))=0$ (step (a)), we have $h^1(\mathcal{I}_{A\cup\{2S'',D\}}(2))=0$. Since $x-f-g=2$, we have $h^0(\mathcal{I}_A(1))=0$. Hence Lemma 2 gives $h^1(\mathcal{I}_{A\cup S'\cup\{2S'',D\}}(2))=0$. Hence $h^1(\mathcal{I}_Y(3))=0$ and so $h^1(\mathcal{I}_{E_{5,3}}(3))=0$.

Since $h^0(\mathcal{I}_{E_{5,3}}(3))=1$, we have $h^0(\mathcal{I}_{E_{6,3}}(3))=0$.

(b1.3) Now assume $y=4$ and hence $f=1$. It is sufficient to do the cases $x=4$ and $x=5$. First assume $x=4$. By step (a) we have $h^1(\mathcal{I}_{E_{2,1}}(2))=0$ and hence $h^1(\mathcal{I}_{A\cup\{2S'',D\}}(2))=0$. Since $h^0(\mathcal{I}_A(1))=0$, Lemma 2 gives $h^1(\mathcal{I}_{A\cup S'\cup\{2S'',D\}}(2))=0$. Hence $h^1(\mathcal{I}_{E_{4,4}}(3))=0$. Now assume $x=5$. Since $h^0(\mathcal{I}_{E_{3,1}}(2))=1$ (step (a)), we have $h^0(\mathcal{I}_{A\cup\{2S'',D\}}(2))=1$. Since $x-f-g\geq 2$, we have $h^0(\mathcal{I}_A(1))=0$. Hence Lemma 2 gives $h^0(\mathcal{I}_{A\cup S'\cup\{2S'',D\}}(2))=0$, i.e. $h^0(\mathcal{I}_{E_{5,4}}(3))=0$.

(b1.4) Now assume $y=5$. First assume $x=3$. Since $h^0(\mathcal{I}_{E_3}(2))=3$ ([3]) and $h^0(D,\mathcal{I}_{\{2S,D\}}(3))>0$, we have $h^0(\mathcal{I}_{E_{3,5}}(3))\geq 3$. We have $h^0(\mathcal{I}_{E_{3,5}}(3))=3$ if and only if $h^1(\mathcal{I}_{E_{3,5}}(3))=0$. Assume $h^0(\mathcal{I}_{E_{3,5}}(3))\geq 4$. Let $Z'\subset X$ a general union of 3 2-points. Let $V\subset H^0(D,\mathcal{O}_D(3))$ the image of the restriction map $H^0(\mathcal{I}_{Z'}(3))\to H^0(D,\mathcal{O}_D(3))$. Since $h^0(\mathcal{I}_{Z'}(2))=3$ and $h^1(\mathcal{I}_{Z'}(3))=0$ ([3]), we have $\dim(V)=15$. Since we assumed that $h^0(\mathcal{I}_{E_{3,5}}(3))\geq 4$, we have $V(-2B)\neq 0$ for a general $B\subset D$ with $\sharp(B)=5$, i.e. V is 4-defective in the sense of [7]. We have $D\subset\mathbb{P}^3$ and for each curve $C\subset D$ we call $\deg(C)$ the degree of C as a space curve. Let $H\subset D$ be the zero-locus of a general element of $V(-2B)$. Let Σ be the union of the positive-dimensional components of $\mathrm{Sing}(H)$ containing at least one of the points of B. By [7], $\Sigma \neq \emptyset$, $B \subset \Sigma$ and for all $P, P' \in \Sigma$ the distribution (degrees, number of components and genera) of the components containing P are the same as for the ones containing P'. We have $H = 2\Sigma + E$ with either $E = \emptyset$ or E an effective curve. Since $\deg(H) = 6$, $\Sigma \supset B$ and B is the union of 5 general points, Σ must be a rational normal curve contained in $D \subset \mathbb{P}^3$ (it cannot be a curve of degree ≤ 2, a plane curve of degree 3, the union of 3 lines or the union of a conic and a line). This is impossible if D is a smooth quadric, because in this case we have $H \in |\mathcal{O}_D(3,3)|$ and $H \neq 2\Sigma$, because 3 is odd. Now assume that D is a quadric cone with vertex O and let $m : F_2 \to D$ be its minimal desingularization. The smooth surface F_2 is a Hirzebruch surface with $h := m^{-1}(O)$ as its only irreducible curve with negative self-intersection. We have $\mathrm{Pic}(F_2) \cong \mathbb{Z}^2$ and we take h and a fiber F of the ruling of F_2 as a generator of F_2. Let Σ' and H' denote the strict transform in F_2 of Σ and H, respectively. Since m is induced by the complete linear system $|h+2F|$, we have $\Sigma' \in |h+zF|$ for some $z \leq 3$. Since $h^0(F_2, \mathcal{O}_{F_2}(h+zF)) = 2z$ for all $z > 0$, and Σ contains 5 general points of D, we have $z = 3$. We get $\Sigma' \in |\mathcal{O}_{F_2}(h+3F)|$ and that (as we knew) Σ is a rational normal curve. Since $h^0(F_2, \mathcal{O}_{F_2}(h+3F)) = 0$, we also get that V is the vector space associated to the projective space of all $2C$ with C either a rational normal curve of D (it contains the vertex of D) or the union of a line and a smooth conic not through the vertex of D. In particular V does not depend from the choice of the 3 points of $(Z')_{\mathrm{red}}$. The projective space $|\mathcal{I}_{Z'}(3)|$ contains the union of an element T of $|\mathcal{I}_{Z'}(2)|$ and a hyperplane section of X. Since $h^0(\mathcal{I}_{Z'}(3)) = 3$, the linear system $|\mathcal{I}_{Z'}(2)|$ is the set of all intersection with Q of the quadric hypersurfaces of $\mathbb{P}^4$ whose cone contains the plane spanned by $(Z')_{\mathrm{red}}$. We may assume $(Z')_{\mathrm{red}} \cap D' = \emptyset$. Hence a general $T' \in |\mathcal{I}_{Z'}(2)|$, the divisor $T' \cap D$ of D has only two singular points. Hence for a general $T' \in |\mathcal{I}_{Z'}(2)|$ and a general hyperplane section D' of X the divisor $(T' \cup D') \cap D$ of D is not the double of a rational normal curve, a contradiction. Now assume $E \neq \emptyset$. Since $\deg(H) = 6$, then Σ is either a line or a conic. In both cases the family of all possible Σ' does not contain 5 general points of D (at most 3), a contradiction.

Now assume $x = 4$. We know that $h^0(\mathcal{I}_{E_{4,5}}(3)) \geq 1$ and we need to prove that $h^0(\mathcal{I}_{E_{4,5}}(3)) = 1$, i.e. $h^1(\mathcal{I}_{E_{4,5}}(3)) = 2$. Assume $h^0(\mathcal{I}_{E_{4,5}}(3)) \geq 2$. Let $A' \subset X$ be a general union of 4 2-points and let $V' \subset H^0(D, \mathcal{O}_D(3))$ be the image of the restriction map $H^0(\mathcal{I}_{A'}(3)) \to H^0(D, \mathcal{O}_D(3))$. Since $h^0(\mathcal{I}_{A'}(2)) = 1$ and $h^1(\mathcal{I}_{A'}(3)) = 0$, we have $\dim(V') = 15$. Since we assumed that $h^0(\mathcal{I}_{E_{4,5}}(3)) \geq 2$, we have $V(-2B) \neq 0$ for a general $B \subset D$ such that $\sharp(B) = 5$. Let H be the zero-locus of a general In characteristic zero a general 2-point of an integral variety of positive dimension gives at least two independent conditions to any non-constant linear system ([7]). Hence $h^0(\mathcal{I}_{E_{z,5}}(3)) = 0$ for all $z \geq 5$, even if D is a quadric cone.

(b2) Assume $k = 4$. We have $52 \leq 4x + 3y \leq 58$ and $3y + 3f + 5 = 25$. Hence $g = 1$ and $y+f = 8$. Since $4x+3y = 4x+24-3f \geq 52$, we have $4(x-f) \geq 28$ and hence $x - f \geq 7$. Hence $x - f - g \geq 6$. Therefore $h^0(\mathcal{I}_A(2)) = 0$. It is sufficient to check that either $h^0(\mathcal{I}_{A\cup\{2O,D\}}(3)) = 0$ or $h^1(\mathcal{I}_{A\cup\{2O,D\}}(3)) = 0$. This is true, because no case with $y = 1$ is an exceptional case when $k = 3$ (step. (b2)).

(b3) Assume $k \geq 5$. Since $k - 2 \geq 3$, either $h^0(\mathcal{I}_A(k-2)) = 0$ or $h^1(\mathcal{I}_A(k-2)) = 0$ ([3]). In the latter case we have $h^0(\mathcal{I}_A(k-2)) = h^0(\mathcal{I}_A(k-1)) - k^2 = h^0(\mathcal{I}_{A\cup\{2S'',D\}}(k-1)) - k^2 + 3g$. Hence it is sufficient to check that $f + 3g \leq k^2$. This is true, because $3f + g \leq (k+1)^2$, $g \leq 2$, and $k \geq 5$. □

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