SINGULAR CURVES OVER A FINITE FIELD AND WITH MANY POINTS

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Abstract: Recently Fukasawa, Homma and Kim introduced and studied certain projective singular curves over $\mathbb{F}_q$ with many extremal properties. Here we extend their definition to more general non-rational curves.

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1. Introduction

Fix a prime $p$ and a $p$-power $q$. Recently S. Fukasawa, M. Homma and S. J. Kim introduced a family of singular rational curves defined over $\mathbb{F}_q$, with many singular points over $\mathbb{F}_q$ and, conjecturally, some extremal properties. In this paper we discuss a similar type of curves, discuss their extremal properties and, in some cases, show that they are, more or less, the curves introduced in [5]. The zeta-function $Z_Y(t)$ of a singular curve $Y$ is explicitly given in terms of the Frobenius on a "topological" invariant $H^1_c(Y, \mathbb{Q}_\ell)$ ([2], [1], p. 2). Hence $Z_Y(t)$ does not detect the finer invariants of the singular points of $Y$ (it does not distinguish between unibranch points defined over the same extension of $\mathbb{F}_q$; in particular it does not distinguish between a smooth point and a cusp). Using gluing of points of the normalization with the same residue field we may define a "minimal" singular curve with prescribed normalization and prescribed zeta-function.

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Let $Y$ be a geometrically integral projective curve defined over $\mathbb{F}_q$. Let $u : C \to Y$ denote the normalization. Since any finite field is perfect, $C$ and $u$ are defined over $\mathbb{F}_q$. Hence for every integer $n \geq 1$ we have $u(C(\mathbb{F}_q^n)) \subseteq Y(\mathbb{F}_q^n)$ and for each $P \in Y(\mathbb{F}_q^n)$ the scheme $u^{-1}(P)$ is defined over $\mathbb{F}_q^n$. Hence the finite set $u^{-1}(P)_{\text{red}}$ is defined over $\mathbb{F}_q^n$ (but of course if $\#(u^{-1}(P)_{\text{red}}) > 1$ the points of $u^{-1}(P)_{\text{red}}$ may only be defined over a larger extension of $\mathbb{F}_q$). We are interested in properties of the set $Y(\mathbb{F}_q)$ knowing $C$. A. Weil’s study of the zeta-function of smooth projective curves was extended to the case of singular curves ([2]). We will use the very useful and self-contained treatment given by Y. Aubry and M. Perret ([1]). There are infinitely many curves $Y'$ defined over $\mathbb{F}_q$, with $C$ as their normalization and with the same zeta-function (see Examples 1, 2 and Lemma 2). However, given $Y$, there is one natural such curve if we prescribe also the sets $u^{-1}(P)_{\text{red}}$ as subsets of $C(\mathbb{F}_q)$. Let $w_s : Y_s \to Y$ be the seminormalization of $Y$ ([2], [10]). We recall that $Y_s$ is an integral projective curve with $C$ as its normalization and that $u = w_s \circ u_s$, where $u_s : C \to Y_s$ is the normalization map. Over an algebraically closed field the one-dimensional seminormal singularities with embedding dimension $n \geq 1$ are exactly the singularities formally isomorphic to the local ring at the origin of the union of the coordinate axis in $\mathbb{A}^n$. Even over a finite field the curve we introduce in this note is defined in the same way, i.e. the curves $C_{[q,n]}$, $n \geq 2$, defined below are obtained in the same way, i.e. the gluing process introduced by C. Traverso ([8]) gives always a seminormal curve and if the base field is algebraically closed, then all seminormal curve singularities are obtained in this way (over an algebraically closed base field a more general construction is given in [7], p. 70). We call axial singularities the curve singularities obtained in this way. Hence by definition we say that $(Y, P)$ is an axial singularity with embedding dimension $n$ if and only if over $\mathbb{F}_q$, it is formally isomorphic at $P$ to the germ at 0 of the union of of the $n$ axis of $\mathbb{A}^n$. An axial singularity of embedding dimension $n > 2$ is not Gorenstein. An axial singularity of embedding dimension $2$ is an ordinary double point except that over a non-algebraically closed base field, say $\mathbb{F}_q$, we need to distinguish if the two branches of $Y$ at $P$ (or the two lines of its tangent cone) are defined over $\mathbb{F}_q$ or not (in the latter case each of them is defined over $\mathbb{F}_{q^2}$). Similarly, for an axial singularity $(Y, P)$ of embedding dimension $t \geq 2$ defined over $\mathbb{F}_q$, the $t$ lines of the tangent cone $C(P, Y)$ are defined over $\mathbb{F}_{q^t}$ and their union is defined over $\mathbb{F}_q$. In the examples we are interested in, none of these lines will be defined over a field $\mathbb{F}_{q^e}$ with $e < t$. If $P \in Y$ is a singular point, then we may associate a non-negative integer $p_a(Y, P)$ (usually called the arithmetic genus of the singularity or the drop of genus the singular point $P$) such that
$p_a(Y) = p_a(C) + \sum_{P \in \text{Sing}(Y)} p_a(Y, P)$. When $Y$ is an axial singularity with embedding dimension $n$, then $p_a(Y, P) = n - 1$.

Let $C$ be a smooth and geometrically connected projective curve defined over $\mathbb{F}_q$. Let $F_q : C(\overline{\mathbb{F}}_q) \to C(\overline{\mathbb{F}}_q)$ be the action of the Frobenius of order $q$. For each $P \in C(\overline{\mathbb{F}}_q)$ let $\text{ord}(P, q)$ be the cardinality of the orbit of $P$ by the action of $F_q$. For every integer $t \geq 1$ we have we have $F_q^t = (F_q)^t$ and $C(\overline{\mathbb{F}}_q^t) = \{P \in C(\overline{\mathbb{F}}_q) : (F_q)^t(P) = P\}$. Hence $\text{ord}(P, q)$ is the minimal integer $t$ such that $P \in C(\overline{\mathbb{F}}_q^t)$ and $P \in C(\overline{\mathbb{F}}_q^t)$ if and only if $\text{ord}(P, q)|s$.

We fix $q, C$ and an integer $n \geq 2$. For all integers $i \geq 1$ set $N_i := \sharp(C(\overline{\mathbb{F}}_q^i))$. Let $N'_i$ be the number of all $P \in C(\overline{\mathbb{F}}_q)$ with $\text{ord}(P, q) = i$. Since $N_t = \sum_{s|t} N'_s$, Möbius inversion formula gives $N'_t = \sum_{s|t} \mu(s)N_{t/s}$ for all $t$.

We construct a singular curve $C[q, n]$ with $C$ as its normalization and $\sharp(C[q, n](\overline{\mathbb{F}}_q))$ very large in the following way. Fix an integer $t$ such that $2 \leq t \leq n$. For each $P \in C(\overline{\mathbb{F}}_q^t)$ with $\text{ord}(P, q) = t$ the orbit of the Frobenius $F_q$ has order $t$, say $\{P, \ldots, (F_q)^{t-1}(P)\}$. Let $C[q, n]$ be the only curve obtained by gluing each of these orbits (for all possible $t \leq n$) (see Remark 4). By construction $C[q, n]$ is a seminormal curve defined over $\mathbb{F}_q$, each singular points of $C[q, n]$ is defined over $\mathbb{F}_q$ and $\sharp(C[q, n](\overline{\mathbb{F}}_q)) = N_1 + \sum_{i=2}^n N'_i/i$. The integers $N'_i$, $i \geq 1$, are uniquely determined by the integers $N_i$, $i \geq 1$, because $N_t = \sum_{s|t} N'_s$ and hence $\sum_{s|t} \mu(s)N_{t/s}$. Fix $P \in C[q, n]$ with embedding dimension $t \geq 2$. The Frobenius $F_q$ acts on the local ring $\mathcal{O}_{C[q, n], P}$ and hence on the $t$ branches of $C[q, n]$ at $P$ (i.e. the $t$ smooth branches through $0$ of the tangent cone of $C[q, n]$ at $P$). Since $P$ is an axial singularity, the action of Frobenius is the restriction to $u^{-1}(P)$ of the action of the Frobenius $F_q : C(\overline{\mathbb{F}}_q) \to C(\overline{\mathbb{F}}_q)$. Hence this action is cyclic, i.e. it has a unique orbit. Thus if $O = u(P)$ with $\text{ord}(P) = t$, then $p_a(C[q, n]) = t - 1$ and none of the $t$ branches of $C[q, n]$ at $u(O)$ is defined over a proper subfield of $\mathbb{F}_q^t$. See Propositions 1, 2, Question 1 and Remark 2 for the relations between $\mathbb{P}^1_{[q, n]}$ and the curves $B$ and $B_n$ studied in [5].

2. The Curves $C[q, n]$ and their Maximality Properties

Let $u : C \to Y$ denote the normalization map. We often write $u^{-1}(P)$ instead of $u^{-1}(P)_{\text{red}}$. Set $\Delta_Y := \sharp(u^{-1}(\text{Sing}(Y)(\overline{\mathbb{F}}_q))) - \sharp(\text{Sing}(Y)(\overline{\mathbb{F}}_q))$. The zeta-function $Z_Y(t)$ of $Y$ is the product of the zeta-function $Z_C(t)$ of $C$ and a degree $\Delta_Y$ polynomials whose inverse roots are roots of unity ([2], [1], Theorem 2.1 and Corollary 2.4). Let $\omega_i, 1 \leq i \leq 2q$, be the inverse roots of numerator of $Z_C(t)$ and $\beta_j, 1 \leq i \leq \Delta_Y$ the inverse roots of the polynomial $Z_Y(t)/Z_C(t)$. For every integer $n \geq 1$ we have $\sharp(Y(\mathbb{F}_q^n)) = q^n + 1 - \sum_{i=1}^{2q} \omega_i^n - \sum_{j=1}^{\Delta_Y} \beta_j^n$. 
We have $\sharp(C(\mathbb{F}_{q^n})) = q^n + 1 - \sum_{i=1}^{2g} \omega_i^n$. Recall that $|\beta_j| = 1$ for all $j$. Assume for the moment that $n$ is odd. In this case among all curves with fixed normalization $C$ and with fixed $\Delta_Y$ the integer $\sharp(Y(\mathbb{F}_q))$ is maximal (resp. minimal) for a curve with $\beta_j = -1$ for all $j$ (resp. $\beta_j = 1$) for all $j$, if any such curve exists. In $n$ is even then the minimum is achieved if there is $Y$ with $\beta_j \in \{-1, 1\}$ for all $j$.

**Lemma 1.** Let $Y$ be a geometrically integral projective curve and $u : C \to Y$ its normalization. The degree $\Delta_Y$ polynomial $Z_Y(t)/Z_C(t)$ has all its inverse roots equal to $-1$ if and only if for each $P \in \text{Sing}(Y)$ either $\sharp(u^{-1}(P)) = 1$ or $P \in Y(\mathbb{F}_q)$ and $u^{-1}(P)$ is formed by two points of $C(\mathbb{F}_q^2)$ (in the latter case these two points are exchanged by the Frobenius and they are in $C(\mathbb{F}_q^2)\setminus C(\mathbb{F}_q)$).

**Proof.** The explicit form of the polynomial $Z_Y(t)/Z_C(t)$ is given in [1], Theorem 2.1. The polynomial $Z_Y(t)/Z_C(t)$ is a product of polynomials, each of them associated to a different singular point of $Y$. Hence it is sufficient to consider separately the contribution of each singular point of $Y$. Fix $P \in \text{Sing}(Y)$ and call $Z_P(t)$ the associated polynomial. Let $d_P$ be the minimal integer $t \geq 1$ such that $P \in Y(\mathbb{F}_{q^t})$. We have $(1 - t^{d_P})Z_P(t) = \prod_{Q \in u^{-1}(P)}(1 - t^{\text{ord}(Q,n)})$. Since $Y$ is defined over $\mathbb{F}_q$, the orbit of $P$ by the Frobenius of $Y$ has order $d_P$. For any point $P' \neq P$ in this orbit, say $F_q^{x}(P)$ for some $x \in \{1, \ldots, d_P - 1\}$ we have $u^{-1}(P') = F_q^{x}(u^{-1}(P))$ and $d_{P'} = d_P$. Since the normalization map is defined over $\mathbb{F}_q$, we have $d_P/\text{ord}(Q,q)$ for each $Q \in u^{-1}(P)$.

First assume $\sharp(u^{-1}(P)) = 1$. The only point, $Q$, of $u^{-1}(P)$ is defined over $\mathbb{F}_{q^{d_P}}$. Since $d_{u(Q)}|\text{ord}(Q,q)$, we get $\text{ord}(Q,q) = d_P$. We easily get $\sharp(u^{-1}(P)) = 1$ if and only if the constant 1 is the factor of $Z_Y(t)/Z_C(t)$ associated to the orbit of $P$. Hence from now on we assume $\alpha := \sharp(u^{-1}(P)) \geq 2$.

If $\text{ord}(Q,q) > d_P$ for some $Q \in u^{-1}(P)$ and either $d_P \geq 2$ or $\text{ord}(Q,q) \geq 3$, then we get that $(1 - t^{d_P})Z_P(t)$ has a root of order $> \max\{d_P, 2\}$ and hence $Z_P(t)$ has a root $\neq -1$.

Now assume $d_P \geq 2$ and $\text{ord}(Q,q) = d_P$ for all $Q \in u^{-1}(Q)$. We get $Z_P(t) = (1 - t^{d_P})^\alpha$. Since we assumed $\alpha \geq 2$, even in this case $Z_P(t)$ has a root $\neq -1$.

Now assume $d_P = 1$. It remains to analyze the case $\text{ord}(Q,q) \in \{1, 2\}$ for any $Q \in u^{-1}(P)$. If $\text{ord}(Q,q) = 1$ for at least one $Q \in u^{-1}(P)$, then $Z_P(1) = 0$. If $\text{ord}(Q,q) = 2$ for all $Q \in u^{-1}(P)$, then $Z_P = (1 + t)(1 - t^2)^{\alpha-1}$ has $\alpha - 1$ roots equal to 1.

**Remark 1.** Fix $q$, $g$, $C$ with genus $g$ and an integer $n \geq 2$. Set $N_i := \sharp(C(\mathbb{F}_q))$. We have $\sharp(C_{[q,n]}(\mathbb{F}_q)) = N_1 + \sum_{i=2}^{n} N_i^i / i$. Now assume that $g > 0$ and
that \( q \) is a square. If \( C \) is a minimal curve for \( \mathbb{F}_q \), then it is a minimal curve for each \( \mathbb{F}_{q^i} \), \( i \geq 2 \) (use that \( C \) is minimal if and only if \( Z_C(t) = \frac{(t-q)^{2g}}{(1-t)(1-t^q)} \) ([9])). Hence for fixed \( q \) and \( n \) with \( n \) an odd prime the integer \( \sharp(C_{q,n}(\mathbb{F}_q)) \) is minimal varying \( C \) among all smooth curves of genus \( g \) if and only if \( C \) is a minimal curve. If \( n = 2 \) and \( q \) is a square, then \( Z_Y(t) \) is the same for all minimal curves over the same genus. Since \( N_2' = N_2 - N_1 \), we get \( N_1 + N_2'/2 = N_2 + N_1/2 \) and hence when \( n = 2 \) and \( q \) is a square for a fixed genus \( g \) the minimal among all \( \sharp(C_{q,2}(\mathbb{F}_q)) \) with fixed genus \( g \) is obtained if and only if \( C \) is a minimal curve.

**Proposition 1.** The curve \( \mathbb{P}^1_{[q,2]} \) is isomorphic over \( \mathbb{F}_q \) to the plane curve \( B \subset \mathbb{P}^2 \) defined in [4] and [5].

**Proof.** Let \( u : \mathbb{P}^1 \to \mathbb{P}^1_{[q,2]} \) denote the normalization map. The normalization map \( \Phi : \mathbb{P}^1 \to B \) is unramified, because the composition of it with the inclusion \( B \hookrightarrow \mathbb{P}^2 \) is unramified (part (i) of [5], Theorem 2.2). By [5], part (iii) of Theorem 2.2, \( B \) is a degree \( q + 1 \) plane curve with \( (q^2 - q)/2 \) singular points and \( \Phi(P) = \Phi(Q) \) with \( P \neq Q \) if and only if \( u(P) = u(Q) \). Hence the universal property of the seminormalization gives the existence of a morphism \( \psi : \mathbb{P}^1_{[q,2]} \to B \) such that \( \psi \) is a bijection. Since \( p_a(\mathbb{P}^1_{[q,2]}) = (q^2 - q)/2 \), we have \( p_a(\mathbb{P}^1_{[q,2]}) = p_a(B) \). Hence \( \psi \) is an isomorphism. \( \square \)

**Proposition 2.** Fix an integer \( n \geq 3 \). Then \( \mathbb{P}^1_{[q,n]} \) is the seminormalization of the curve \( B_n \subset \mathbb{P}^n \), defined in [5], §6, and there is a birational morphism \( \psi_{q,n} : \mathbb{P}^1_{[q,n]} \to B_n \) defined over \( \mathbb{F}_q \) such that \( \psi_{n,q} : \mathbb{P}^1_{[q,n]}(K) \to B_n(K) \) is bijective for every field \( K \supseteq \mathbb{F}_q \).

**Proof.** Let \( u : \mathbb{P}^1 \to \mathbb{P}^1_{[q,n]} \) and \( \Phi_n : \mathbb{P}^1 \to B_n \) denote the normalization maps. By [5], part (ii) of Theorem 6.4, each point \( P \in \text{Sing}(B_n) \) corresponds to an integer \( t \in \{2, \ldots, n\} \) and an orbit of the Frobenius on \( \mathbb{P}^1(\mathbb{F}_q^t) \setminus \mathbb{P}^1(\mathbb{F}_{q^{t-1}}) \). Hence the definition of \( \mathbb{P}^1_{[q,n]} \) and the universal property of the seminormalization gives a birational morphism \( \psi_{q,n} : \mathbb{P}^1_{[q,n]} \to B_n \) defined over \( \mathbb{F}_q \) such that \( \psi_{n,q} : \mathbb{P}^1_{[q,n]}(K) \to B_n(K) \) is bijective for every field \( K \supseteq \mathbb{F}_q \). \( \square \)

**Question 1.** We guess that \( \psi_{q,n} \) is an isomorphism.

**Remark 2.** Fix a prime power \( q \) and the integer \( n \geq 3 \). Let \( \Phi_n : \mathbb{P}^1 \to B_n \) denote the normalization map. By [5], part (i) of Theorem 6.4, \( \Phi_n \) is unramified (this is a necessary condition for being \( \psi_{q,n} \) an isomorphism). The following conditions are equivalent:

(i) the morphism \( \psi_{q,n} \) is an isomorphism;
(ii) \( p_a(B_n) = p_a(\mathbb{P}^1_{[q,n]}) \);

(iii) for each \( P \in \text{Sing}(B_n) \), say with \( P = \Phi_n(Q) \) and \( \text{ord}(Q,q) = t \), the singularity \((B_n,P)\) has arithmetic genus \( t - 1 \);

(iv) for each \( P \in \text{Sing}(B_n) \), say with \( P = \Phi_n(Q) \) and \( \text{ord}(Q,q) = t \) the tangent cone \( C(P,B_n) \subset \mathbb{P}^n \) is formed by \( t \) lines through \( P \) spanning a \( t \)-dimensional linear subspace.

Part (iv) is just the definition of seminormal singularity given in [2]. Since \( \Phi_n \) is unramified, \( B_n \) has at \( P \) \( t \) smooth branches.

**Proposition 3.** Let \( C \) be a smooth and geometrically irreducible projective curve defined over \( \mathbb{F}_q \). Set \( 2\delta := \#(C(\mathbb{F}_{q^2})) - \#(C(\mathbb{F}_q)) \). Let \( Y \) be a projective curve defined over \( \mathbb{F}_q \) with \( C \) as its normalization. We have \( \#(Y(\mathbb{F}_q)) \geq \#(C(\mathbb{F}_q)) + \delta \) and \( p_a(Y) \leq g + \delta \) if and only if \( Y \) is isomorphic to \( C_{[q,2]} \) over \( \mathbb{F}_q \).

**Proof.** The “if” part is true, because \( p_a(C_{[q,2]}) = g + \delta \) and \( \#(C_{[q,2]}(\mathbb{F}_q)) = g + \delta \). Assume \( \#(Y(\mathbb{F}_q)) \geq \#(C(\mathbb{F}_q)) + \delta \) and \( p_a(Y) \leq g + \delta \). Let \( u : C \to Y \) be the normalization map. The morphism \( u \) is defined over \( \mathbb{F}_q \), i.e. over a field on which \( Y \) is defined, because any finite field is perfect. We have \( \#(\text{Sing}(Y)) \leq p_a(Y) - \delta \) and equality holds only if each singular point of \( Y \) is formally isomorphic over \( \mathbb{F}_q \) either to a node or an ordinary cusp. Set \( \Delta_Y := \#(u^{-1}(\text{Sing}(Y))(\mathbb{F}_q)) - \#(\text{Sing}(Y)(\mathbb{F}_q)) \). The polynomial \( Z_Y(t)/Z_C(t) \) has degree \( \Delta_Y \) and \( \#(Y(\mathbb{F}_q)) \leq \#(C(\mathbb{F}_q)) + \delta \) and equality holds if and only if each inverse root of \( Z_Y(t)/Z_C(t) \) is equal to \(-1\). Since \( \Delta_Y \leq p_a(Y) - g \), we get \( \Delta_Y = \delta \) and \( p_a(Y) = g + \delta \). Since \( p_a(Y) = g + \Delta_Y \) and \( \#(\text{Sing}(Y)(\mathbb{F}_q)) \geq \delta \), we get \( \#(\text{Sing}(Y)(\mathbb{F}_q)) = \#(\text{Sing}(Y)/(\mathbb{F}_q)) = \#(\text{Sing}(Y)(\mathbb{F}_q)) = \delta \) and that for each \( P \in \text{Sing}(Y)(\mathbb{F}_q) \) the set \( u^{-1}(P) \) is formed by two points of \( C(\mathbb{F}_{q^2}) \setminus C(\mathbb{F}_q) \) exchanged by the Frobenius (Lemma 1). Since \( p_a(Y) = g + \#(\text{Sing}(Y)(\mathbb{F}_q)) \) and \( \#(u^{-1}(P)) \geq 2 \) for each \( P \in \text{Sing}(Y)(\mathbb{F}_q) \), we also get that \( Y \) is nodal. Hence \( Y \) is seminormal. The structure of the fibers of \( u^{-1}(P) \), \( P \in \text{Sing}(Y)(\mathbb{F}_q) \), gives \( Y = C_{[q,2]} \). \( \square \)

**Proposition 4.** Let \( Y \) be a geometrical integral projective curve defined over \( \mathbb{F}_q \) and with only seminormal singularity. Let \( u : C \to Y \) be the normalization map. Let \( \Delta_Y \) be the degree of the polynomial \( Z_Y(t)/Z_C(t) \). Assume \( 2\Delta_Y \leq \#(C(\mathbb{F}_{q^2})) - \#(C(\mathbb{F}_q)) \). We have \( \#(Y(\mathbb{F}_q)) \leq \#(C(\mathbb{F}_{q^2})) \) and equality holds if and only if \( Y \) is isomorphic to \( C_{[q,2]} \) over \( \mathbb{F}_q \).

**Proof.** We have \( \#(Y(\mathbb{F}_q)) \leq \#(C(\mathbb{F}_q)) + \Delta_Y \) and equality holds if and only if each inverse root of \( Z_Y(t)/Z_C(t) \) is equal to \(-1\). Hence we may assume \( 2\Delta_Y = \#(C(\mathbb{F}_{q^2})) - \#(C(\mathbb{F}_q)) \). Since \( Y \) has only seminormal singularities, \( u \) is
unramified. Since $u$ is unramified, we have $\sharp(u^{-1}(P)) \geq 2$ for all $P \in \text{Sing}(Y)$. Hence Lemma 1 gives that the fibers of $u$ are the fibers of the normalization map $C \to C_{[q,2]}$. Since $Y$ and $C_{[q,2]}$ are seminormal, we get that they are isomorphic. They are isomorphic over $\mathbb{F}_q$, because $u$ is defined over $\mathbb{F}_q$ and the seminormalization is defined over $\mathbb{F}_q$. \hfill $\square$

**Remark 3.** In the case $C \cong \mathbb{P}^1$ Propositions 3 and 4 are partial answers to a question raised in [5], Remark 2.5. Examples 1, 2 and Lemma 2 show that we need to add some conditions on the curve $Y$, not only to fix the normalization $\mathbb{P}^1$ and assume $\Delta_Y \leq (q^2 - q)/2$.

**Example 1.** Fix a geometrically integral projective curve $A$ defined over $\mathbb{F}_q$ and $P \in A(\mathbb{F}_q)$. Now we define a geometrically integral curve $Y$ defined over $\mathbb{F}_q$ and a morphism $v : A \to Y$ defined over $\mathbb{F}_q$, such that $v \setminus A \setminus \{P\}$ is an isomorphism onto $Y \setminus v(P)$, but $v$ is not an isomorphism. Notice that for each such pair $(Y, v)$ we would have $p_a(Y) > p_a(A)$ and that for every integer $t \geq 1$ $v$ induces a bijection $A(\mathbb{F}_q^t) \to Y(\mathbb{F}_q^t)$. To define $Y$ and $v$ it is sufficient to define them in a neighborhood of $P$ in $A$ and the glue to it the identity map $A \setminus \{P\} \to A \setminus \{P\}$. Fix an embedding $j : A \hookrightarrow \mathbb{P}^r$, $r \geq 3$, and take a projection of $j(A)$ into $\mathbb{P}^2$ from an $(r - 3)$-dimensional linear subspace not containing $j(P)$, but intersecting the Zariski tangent space of $j(A)$ at $j(P)$.

**Example 2.** Fix a geometrically integral projective curve $A$ defined over $\mathbb{F}_q$ and any point $P \in A_{\text{reg}}(\mathbb{F}_q)$. Let $t$ be the minimal integer $t \geq 1$ such that $P \in A(\mathbb{F}_q^t)$, i.e. let $t$ be the cardinality of the orbit of $P$ by the action of the Frobenius. We assume $t \geq 2$, because the case $t = 1$ is covered by Example 1. Hence the orbit of $P$ by the action of the Frobenius $F_q$ has order $t$ (it is $\{P, F_q(P), \ldots, F_q^{t-1}(P)\}$). Let $Y$ denote the only curve and $v : A \to Y$ the only morphism obtained in the following way. We fix a bijection of sets $v : A \to Y$ and use it to define a topology on the set $Y$. Now we define $Y$ as a ringed space. On $Y \setminus u(\{P, F_q(P), \ldots, F_q^{t-1}(P)\})$ we assume that $v$ is an isomorphism of local ringed spaces. For each $Q \in \{P, F_q(P), \ldots, F_q^{t-1}(P)\}$ we impose that $O_{Y,Q}$ is the local ring of a unibranch singular point and that $v$ is the normalization map (it may be done using the method of Example 1 with $Q$ instead of $P$ and $\mathbb{F}_q^t$ instead of $\mathbb{F}_q$). We need to do the construction simultaneously over all $Q \in \{P, F_q(P), \ldots, F_q^{t-1}(P)\}$ and in such a way that the morphism is defined over $\mathbb{F}_q$. As in Example 1 it is sufficient to define $v|U$, where $U$ is a neighborhood of $\{P, F_q(P), \ldots, F_q^{t-1}(P)\}$. There is an embedding $j : A \to \mathbb{P}^r$, $r \geq t + 2$, such that the $t$ lines $T_Q(j(A), Q \in \{P, F_q(P), \ldots, F_q^{t-1}(P)\})$, are linearly independent. Since $j$ is defined over $\mathbb{F}_q$ the Frobenius $F_q$ of $\mathbb{P}^r$ acts on $j(A)$ and on the tangent developable of $A$. Since
$j(P)$ is defined over $\mathbb{F}_q^t$. Hence $T_{j(P)}j(A)(\mathbb{F}_q^t) \setminus j(P)$ has $(q^{t+1} - 1)/(q - 1) - 1$ elements. Fix any $O \in T_{j(P)}j(A)(\mathbb{F}_q^t) \setminus j(P)$. For each $x \in \{1, \ldots, t-1\}$, $F_q^x(O) \in T_{j(F_q^x(P))}j(A)(\mathbb{F}_q^t) \setminus j(F_q^x(P))$. Since the $t$ tangent lines are linearly independent, the linear space $E := \langle \{O, F_q^x(O), \ldots, F_q^{t-1}(O)\} \rangle$ has dimension $t-1$. Since $E$ is $F_q$-invariant, it is defined over $\mathbb{F}_q$. Let $\pi : \mathbb{P}^r \setminus E \to \mathbb{P}^{r-t}$ denote the linear projection from $E$. Since $E$ is defined over $\mathbb{F}_q$, $\pi$ is defined over $\mathbb{F}_q$. Hence the integral projective curve $T := \pi(j(A) \setminus E \cap j(A)) \subset \mathbb{P}^{r-t}$ is defined over $\mathbb{F}_q$. Since the $t$ tangent lines are linearly independent and $O \neq j(P)$, we have $E \cap \{j(P), \ldots, j(F_q^{t-1}(P))\} = \emptyset$. Hence $E \cap j(U) = \emptyset$ for a sufficiently small neighborhood $U$ of $\{P, F_q(P), \ldots, F_q^{t-1}(P)\}$. Assume for the moment that $\pi|j(A) \setminus j(A) \cap E$ is birational onto its image. Since $\pi|j(A) \setminus j(A) \cap E$ is birational onto its image, it is separable. Hence only finitely many points of $j(A_{\text{reg}})$ have a tangent line intersecting $E$. Restricting if necessary $U \subseteq A_{\text{reg}}$ we may assume that for no other point $Q \in j(U)(\mathbb{F}_q)$ the Zariski tangent space $T_{j(Q)}j(A)$ intersects $E$. Since $\pi|j(A) \setminus j(A) \cap E$ is birational onto its image, it is generically injective. Hence restricting $U \subseteq A_{\text{reg}}$ we may assume that $\pi|j(U)$ is injective and an isomorphism outside $\{j(P), j(F_q(P)), \ldots, j(F_q^{t-1}(P))\}$. At these points the curve $T$ has a cusp, but perhaps not an ordinary cusp, i.e. it is a unibranch singular point. Hence to conclude the example it is sufficient to find $j$ such that $\pi|j(A) \setminus j(A) \cap E$ is birational onto its image. We take as $j$ is a linearly normal embedding of degree $d > \max\{2p_a(A) - 2, p_a(A) + t\}$. Since $d > \max\{2p_a(A) - 2, p_a(A) + t\}$, Riemann-Roch gives $r = d - p_a(A)$. Assume that $\pi|j(A) \setminus j(A) \cap E$ is not birational onto its image and call $x \geq 2$ its degree. Thus $\deg(T) \leq d/x \leq d/2$. Since $j(A)$ spans $\mathbb{P}^r$, $T$ spans $\mathbb{P}^{r-t}$. Hence $\deg(T) \geq r-t = d - p_a(A) - t$. Hence $\deg(T) \geq 2(d - p_a(A) - t)$, contradicting our assumption $d > p_a(A) + t$.

**Lemma 2.** Fix an integer $y > 0$. Let $A$ be a geometrically integral projective curve defined over $\mathbb{F}_q$. Assume $A_{\text{reg}}(\mathbb{F}_q) \neq \emptyset$ and fix $P \in A_{\text{reg}}(\mathbb{F}_q)$. Then there are a geometrically integral projective curve $Y$ and a morphism $u : A \to Y$ defined over $\mathbb{F}_q$ such that $u$ induces an isomorphism of $A \setminus \{P\}$ onto $Y \setminus u(P)$ and $p_a(Y) = p_a(A) + y$.

**Proof.** Let $m$ be the maximal ideal of the local ring $\mathcal{O}_{A,P}$. By assumption $\mathcal{O}_{A,P}/m \cong \mathbb{F}_q$ and $\mathbb{F}_q \cdot 1 \subset \mathcal{O}_{A,P}$. Hence the $\mathbb{F}_q$-vector space $\mathcal{O}_{A,P}$ is the direct sum of its subspaces $\mathbb{F}_q \cdot 1$ and $m$. Set $\mathcal{O}_{Y,u(P)} := \mathbb{F}_q \cdot 1 + m^{y+1} \subset \mathcal{O}_{A,P}$. It is easy to check that $\mathcal{O}_{Y,u(P)}$ is a local ring with $m^{y+1}$ as its maximal ideal. Since $P \in A_{\text{reg}}$, $\mathcal{O}_{A,P}$ is a DVR. Hence $m/m^{y+1}$ is a $\mathbb{F}_q$-vector space of dimension $y$. We take as $Y$ the same topological space as $Y$, but with $\mathcal{O}_{Y,u(P)}$ at the point $u(P)$ associated to $P$ instead of $\mathcal{O}_{A,P}$. With this definition of $u$ we have
\[ \dim_{\mathbb{F}_q}(u_*(\mathcal{O}_A)/\mathcal{O}_Y) = y. \text{ Hence } p_a(Y) = p_a(A) + y. \]  

**Remark 4.** Fix \( q, C \) and an integer \( n \geq 2 \). Here we explain one way to check the existence of the curve \( C_{[q,n]} \). We obtain \( C_{[q,n]} \) in finitely many steps each of them similar to the one described in Example 2. We use \( z \) steps, where \( z \) is the number of orbits of \( F_q \) in \( C(\mathbb{F}_{q^n}) \setminus C(\mathbb{F}_q) \). At each of the steps we glue together one of these orbits. We do not need any notion of gluing, except that set-theoretically in each step one of these orbits is sent to a single point and for all other points the map is an isomorphism. Fix \( Q \in C(\mathbb{F}_{q^n}) \setminus C(\mathbb{F}_q) \) and assume \( \text{ord}(Q, q) = t | n \). Hence \( \{Q, F_q(Q), \ldots, F_{t-1}^q(Q)\} \) is the orbit of \( Q \) for the action of \( F_q \). Call \( A \) the geometrically integral curve arising in the steps at which we want to glue this orbit. Hence there is a geometrically integral projective \( A \) curve defined over \( \mathbb{F}_q \) with \( C \) as its normalization (call \( u : C \to A \)) and \( u(Q) \in A_{\text{reg}} \) (in the previous steps (if any) the maps where isomorphism at each point of \( \{Q, F_q(Q), \ldots, F_{t-1}^q(Q)\} \)). Set \( P := u(Q) \). Since \( u \) is defined over \( \mathbb{F}_q \) and \( u \) is an isomorphism in a neighborhood of \( u^{-1}(\{P, F_q(P), \ldots, F_{t-1}^q(P)\}) \), we have \( \{P, F_q(P), \ldots, F_{t-1}^q(P)\} \subset A_{\text{reg}} \) and these \( t \) points are distinct. Hence \( P \in A_{\text{reg}}(\mathbb{F}_{q^t}) \) and \( P \in A_{\text{reg}}(\mathbb{F}_q) \) if and only if \( t | y \). As in Example 2 we get several curves \( Y \) and morphism \( v : A \to Y \) defined over \( \mathbb{F}_q \), sending \( \{P, F_q(P), \ldots, F_{t-1}^q(P)\} \) to a single point, \( O \), of \( Y \) and induces an isomorphism of \( A \setminus \{P, F_q(P), \ldots, F_{t-1}^q(P)\} \) onto \( Y \setminus \{O\} \). Let \( A_1 \) be the seminormalization of \( Y \) in \( A \). Then we use \( A_1 \) instead of \( A \). After \( z \) steps we get \( C_{[q,n]} \). To get \( C_{[q,n]} \) we use the existence of the seminormalization. The result does not depend from the order of the gluing. Hence \( C_{[q,n]} \) depends only from \( q, C \) and \( n \). Hence the curves \( \mathbb{P}^1_{[q,n]} \) depends only from \( q \) and \( n \).

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**References**


