International Journal of Pure and Applied Mathematics
Volume 95 No. 2 2014, 233-243
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v95i2.9
ijpam.eu

POSTULATION OF GENUS 1 PROJECTIVE CURVES WITH LINES AS THEIR IRREDUCIBLE COMPONENTS

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Abstract: Fix lines $L_1, \dots, L_d \subset \mathbb{P}^n$, $d \geq 3$, such that $A := L_1 \cup \cdots \cup L_d$ is a nodal curve of degree d and $L_i \cap L_j \neq \emptyset$ if and only if either $|i-j| \leq 1$ or $(i,j) \in \{(1,d),(d,1)\}$. We say that A is a genus 1 loop of degree d. We prove that general genus 1 loops of degree d have maximal rank, i.e. good postulations, if either $n \geq 4$ or $n = 3$ and most d. We also study the postulation of general disjoint unions of such curves.

AMS Subject Classification: 14N05, 15A69
Key Words: unions of lines, genus one, reducible curve, postulation, Hilbert function

1. Introduction

A closed subscheme $A \subset \mathbb{P}^n$ is said to have *maximal rank* if for every integer t either $h^0(\mathcal{I}_A(t)) = 0$ or $h^1(\mathcal{I}_A(t)) = 0$.

A degree d *tree* is a connected, reduced and nodal curve $A \subset \mathbb{P}^n$ with $\deg(A) = d$, $p_a(A) = 0$ and whose irreducible components are lines. An irreducible component L of a tree A is called a *final line* if either $d = 1$ (in this case $A = L$) or $L \cap \overline{A \setminus L}$ is a single point. A degree d tree A is a *bamboo* if there is an ordering $A_1, \dots, A_d$ such that $A_i \cap A_j \neq \emptyset$ if and only if $|i-j| \leq 1$. A genus 1 loop of lines of $\mathbb{P}^n$, is a reduced, nodal and connected curve $C \subset \mathbb{P}^n$ such that $d := \deg(C) \geq 3$, each irreducible component of C is a line and there is an

Received: April 21, 2014

ordering $A_1, \dots, A_d$ of the irreducible components of C such that $A_i \cap A_j \neq \emptyset$ if and only if either $|i-j| = 1$ if $\{i,j\} = \{1,d\}$ (these curves are called *closed bamboos* in [3]). A tree of degree ≥ 2 is a bamboo if and only if it has only two final lines.

Theorem 1. *For all integers $d > n \ge 4$ there is a genus 1 loop of lines $A \subset \mathbb{P}^n$ with maximal rank and non-degenerate.*

We only give a partial result when $n = 3$ (see Proposition 3). For disjoint unions of bamboos and genus 1 loops we obrain the following result.

Proposition 1. *There is an integer $k_0 > 0$ with the following property. Fix any integer $k \ge k_0$ and any $(a,b) \in \mathbb{N}^2$ such that $1 \le a+b \le k$. Fix any integer $n \ge 4$ and positive integers d_j, $1 \le j \le a+b$ such that $d_j \ge 3$ for all $j > a$, and $k(d_1 + \cdots + d_{a+b}) + a \le \binom{n+k}{n}$. Let $X \subset \mathbb{P}^n$ be a general union of a bamboo of degree $d_1, \dots, d_a$ and b genus 1 loops of degree $d_{a+1}, \dots, d_{a+b}$. Then $h^1(\mathcal{I}_X(k)) = 0$. If we also have $(k-1)(d_1 + \cdots + d_{a+b}) + a \ge \binom{n+k-1}{n}$, then we may find X with the additional property that $h^0(\mathcal{I}_X(k-1)) = 0$.*

2. The proof

Fix $e \in \{0,1\}$. Let $X \subset \mathbb{P}^n$ be a closed subscheme such that $\dim(X) \le 1$, and $h^1(\mathcal{O}_X(e)) = 0$. Fix any integer $k \ge e+1$ such that $h^1(\mathcal{I}_X(k)) = 0$. By Castelnuovo-Mumford's lemma we have $h^1(\mathcal{I}_X(t)) = 0$ for all $t > k$. Assume for the moment $h^1(\mathcal{O}_X(e-1)) > 0$. The minimal integer $x > e$ such that $h^1(\mathcal{I}_X(x)) = 0$ is called *the critical value* of X.

For any reduced algebraic set $E \subset \mathbb{P}^n$ the residual scheme $\mathrm{Res}_H(E)$ of E with respect to H is the union of the irreducible components of E not contained in H.

For all integers $n \ge 3$ and $k \ge 1$ define the integers $r_{n,k}$ and $q_{n,k}$ by the relations

$$kr_{n,k} + 1 + q_{n,k} = \binom{n+k}{n}, \quad 0 \le q_{n,k} \le k-1 \tag{1}$$

From (1) we get

$$r_{n,k-1} + k(r_{n,k} - r_{n,k-1}) + q_{n,k} - q_{n,k-1} = \binom{n+k-1}{n-1} \tag{2}$$

We use the following assertion $R(n,k)$ which is a small modification of Assertion $H(n,k)$ ([1], pages 205), and which seems to be more useful for extending [1] to other situations.

Assertion $R(n,k)$, $n \ge 4$, $k \ge 1$: There is a disjoint union $A \subset \mathbb{P}^n$ of $q_{n,k}+k+1$ disjoint bamboos such that $h^i(\mathcal{I}_A(k)) = 0$, $i = 0,1$, and every connected component of A has degree ≥ 2.

Proposition 2. *$R(n,k)$ is true for all $n \ge 4$ and $k \ge 2$.*

Proof. Fix a hyperplane $H \subset \mathbb{P}^n$. The case $k = 1$ is obvious (it would be false for $n = 3$). First assume $k = 2$. We have $q_{4,2} = 0$, $r_{4,2} = 7$, $q_{5,2} = 0$, $r_{5,2} = 10$ and hence $r_{n,2} \ge 2(3+q_{n,2})$ for $n = 4,5$. The inequality $r_{n,2}-1 \ge 2(q_{n,2}+3)$ is true for all $n \ge 6$. Every tree with degree ≤ 3 is a bamboo. Hence $R(4,2)$ means that a general union in $\mathbb{P}^4$ of 3 reducible conics has maximal rank; this is true by Assertion $H(2,4)$ of [2] for the data $s = 3$, $d_1 = d_2 = d_3 = 2$. For $n \ge 5$ we prove $R(n,2)$ using a solution $T \subset H$ of $R(n-1,2)$ and a curve $Y \subset \mathbb{P}^n$ which is a linearly normal bamboo of degree n with either $Y \cap T$ a point (case $q_{n,2} = q_{n-1,2}$) or $Y \cap T = \emptyset$ (case $q_{n,2} = 1$ and $q_{n-1,0} = 0$) or linking two connected final lines of two different connected components of T, i.e., calling Y' and Y'' the final lines of Y, we assume that $Y' \cap H$ and $Y'' \cap H$ are smooth points of T belonging to final lines of different connected components of T (case $q_{n,2} = 0$ and $q_{n-1,0} = 1$).

Now assume $k \ge 3$ and that $R(n,k-1)$ is true. Take a solution $Y \subset H$ of $R(n,k-1)$. Without losing generality we may assume that $Y \cap H$ is a general union of $\deg(Y)$ points of H. We have $k(r_{n,k} - r_{n,k-1} - 1) \le \binom{n+k-1}{n-1}$ by (2). In all cases $T \subset H$ will be a disjoint union of bamboos and $\deg(T) = r_{n,k} - r_{n,k-1} - 1$. In all cases by (2) we will obtain $h^0(\mathcal{O}_{(Y\cap H)\cup T}(k)) = \binom{n+k-1}{n-1}$. Hence $h^0(H, \mathcal{I}_{(Y\cap H)\cup T}(k)) = h^1(\mathcal{I}_{(Y\cap H)\cup T}(k))$ and $h^0(H,\mathcal{I}_T(k)) = h^1(H,\mathcal{I}_T(k)) + \sharp(Y\cap H) - \sharp(Y \cap T)$. Since $Y \cap H \setminus Y \cap T$ is general in H, it is sufficient to prove that $h^1(H,\mathcal{I}_T(k)) = 0$.

In [1], Lemmas 8, 9, 10, we proved $R(4,k)$ (not just $H(4,k)$) for $k = 2,3,4$. Hence if $n = 4$ we could take $k \ge 5$.

(a) First assume $q_{n,k}+k+1 > q_{n,k-1}+k$, i.e. $q_{n,k} \ge q_{n,k-1}$. In this case we take $T \subset H$ with $a := q_{n,k} - q_{n,k-1} + 1$ connected components and $T \cap (Y \cap H) = \emptyset$. We have $a \le q_{n,k}+1 \le k$.

Claim 1: We claim that $r_{n,k} - r_{n,k-1} \ge 2a$, unless $(n,k) = (4,6)$.

Proof of Claim 1: Since $a \le k$, we may quote [1], Lemma 22 (which gives the stronger inequality $r_{n,k} - r_{n,k-1} \ge 4k-2$) if either $n \ge 6$ or $n = 5$ and $k \ge 10$ or $n = 4$ and $k \ge 40$. For $n = 5$ and $k \le 10$, see Remark 1. Now assume $n = 4$ and $5 \le k \le 39$. Recall that $a = q_{4,k} - q_{4,k-1} + 1 \le k$. We have $r_{4,k} - r_{4,k-1} \ge 2k$ for all $k \ge 12$, while all pairs $(r_{4,k}, q_{4,k})$ are listed in [1], Remark 8, for $k \le 15$.

Claim 2: We have $r_{n-1,k} \geq r_{n,k} - r_{n-1,k} + a - 1$ with strict inequality if either $n \geq 5$ or $k \geq 4$.

Proof of Claim 2: By (2) it is sufficient to prove that $r_{n,k-1} \geq 1 + k(a-1)$. Since $a \leq k$, it is sufficient to check that $1 + (k-1)k(k-1) \leq \binom{n+k-1}{n}$ for all $n \geq 4$ and $k \geq 3$. For the strict inequality we need $1 + (k-1)k^2 \leq \binom{n+k-1}{n}$, which is true if either $k \geq 4$ or $k = 3$ and $n \geq 5$.

Assume for the moment the existence of a bamboo $T' \subset H$ such that $\deg(T') = r_{n,k} - r_{n-1,k} + a - 1$ and $h^1(\mathcal{I}_{T'}(k)) = 0$.

By Claim 1 if $(n,k) \neq (4,6)$ we may take $T \subseteq T'$ with $\deg(T) = r_{n,k} - r_{n-1,k}$, T with a connected components and such that each connected component of T has degree at least two. The first part of Claim 2 shows that T' exists if $n \geq 5$, by [1], Theorem 1. The second part of Claim 2 gives the existence of T' by [1], Theorem 3, if $n = 4$ and $k \notin \{3, 6\}$. If $(n,k) = (4,3)$, then we use that $r_{4,3} = 11$, $q_{4,3} = 1$, $r_{4,2} = 7$, $q_{4,2} = 0$ and $r_{3,3} = 6$.

Now assume $(n,k) = (4,6)$. We have $r_{4,4} = 17$, $q_{4,4} = 1$, $r_{4,5} = 25$, $q_{4,5} = 0$, $r_{4,6} = 34$ and $q_{4,6} = 5$. We start with a disjoint union $Y_1 \subset \mathbb{P}^4$ of two bamboos such that $\deg(Y_1) = r_{4,4}$ and $h^i(\mathcal{I}_{Y_1}(4)) = 0$, $i = 0, 1$. Let $F' \subset H$ be a union of 8 disjoint lines such that $h^i(\mathcal{I}_{Y_1 \cup F'}(5))) = 0$, $i = 0, 1$, $Y_1 \cup F'$ is a union of two bamboos of degree ≥ 2 (two lines of F' meets final lines of different components of Y_1; we have $h^1(H, \mathcal{I}_{F'}(5)) = 0$ by [4], because $6 \times 8 \leq 56 = \binom{8}{3}$).

(b) Now assume $q_{n,k} + k + 1 = q_{n,k-1} + k$. Let $T \subset H$ be a general bamboo of degree $r_{n,k} - r_{n,k-1} - 1$ containing exactly one point of $Y \cap H$, this point belonging to a final line of H. In this case obviously every connected component of $Y \cup T$ has degree ≥ 2. Hence to check this case we only need the numerical inequalities used in [1] (even when $n = 4$).

(c) Now assume $q_{n,k} + k + 1 < q_{n,k-1} + k$ and set $s := q_{n,k-1} - q_{n,k} - 1$. We have $s \leq k-2$. In this case T has $s-1$ connected components, $\sharp((Y \cap H) \cap T) = s$, each point of $(Y \cap H) \cap T$ is contained in a final line of Y and each connected component of T links two different connected components of Y. In this case we do not need any assumption on the degrees of the connected components of T. The numerical inequalities are checked in [1]. □

Remark 1. We have $\binom{7}{5} = 21$, $r_{5,2} = 10$, $q_{5,2} = 0$, $\binom{8}{5} = 56$, $r_{5,3} = 18$, $q_{5,3} = 1$, $\binom{9}{5} = 126$, $r_{5,4} = 31$, $q_{5,4} = 1$, $\binom{10}{5} = 252$, $r_{5,5} = 50$, $q_{5,5} = 1$, $\binom{11}{5} = 462$, $r_{5,6} = 76$, $q_{5,6} = 5$, $\binom{12}{5} = 792$, $r_{5,7} = 113$, $q_{5,7} = 0$, $\binom{13}{5} = 1287$, $r_{5,8} = 160$, $q_{5,8} = 6$, $\binom{14}{5} = 2002$, $r_{5,9} = 222$, $q_{5,9} = 3$, $\binom{15}{5} = 3003$, $r_{5,10} = 300$, $q_{5,10} = 2$.

Proof of Theorem 1 : Let $A(n,d)$ be the set of all genus 1 chains in $\mathbb{P}^n$. Fix the integers $d > n \geq 4$. Every linearly normal element of $A(n, n+1)$ is arithmetically Cohen-Macaulay by Castelnuovo-Mumford's lemma and so it has maximal rank. Hence we may assume $d \geq n+2$. The critical value of $A(n, d)$ is the minimal integer $k \geq 2$ such that $kd \leq \binom{n+k}{n}$. Let k be the critical value of $A(n, d)$. Since $A(n, d)$ is irreducible, it is sufficient to prove the existence of $A, B \in A(n, d)$ such that $h^1(\mathcal{I}_A(k)) = 0$ and $h^0(\mathcal{I}_B(k-1)) = 0$. In steps (a) and (b) we prove the existence of A, while in step (c) we prove the existence of B. Fix a hyperplane $H \subset \mathbb{P}^n$. If $q_{n,k} \leq k-2$, then $r_{n,k-1} < d \leq r_{n,k}$. If $q_{n,k} = k-1$, then $r_{n,k-1} < d \leq r_{n,k}+1$ and if $d = r_{n,k}$ the existence of A is equivalent to $h^i(\mathcal{I}_C(k)) = 0$, $i = 0, 1$, for a general $C \in A(n, k)$. Set $s := q_{n,k-1} + k$.

(a) In this step we assume $d \geq r_{n,k-1} + s$. Fix a curve Y satisfying $R(n, k-1)$. Y has degree $r_{n,k-1} - 1$ and s connected components, all of them with two distinct final lines. For a general Y we may assume that $Y \cap H$ is a general subset of H with cardinality $\deg(Y)$. Let $Y_1, \ldots, Y_s$, be the connected components of Y. A general bamboo $U \subset H$ passes through $\deg(U)-1$ general points of H. Hence there is a disjoint union $T \subset H$ of s bamboos such that $\sharp(T \cap U) = s+1$ and $T \cup Y$ is a genus 1 loop (take a suitable ordering of the $2s$ points of H which are the intersection of H of a final line of Y). Since $d \geq r_{n,k-1} + s$, we have $d - s - 1 \geq \deg(U)$. Hence we may take T with the additional property that $\deg(T) = d - \deg(U)$. To conclude using the Horace Lemma ([5], or [1], Lemma 1) (also called the Castelnuovo's inequalities) it is sufficient to prove that $h^1(H, \mathcal{I}_T(k)) = 0$. There is a bamboo $T' \subset H$ such that $T \subseteq T'$ and $\deg(T') = \deg(T) + s - 1$. For every integer $t \geq 2$ the restriction map $H^0(\mathcal{O}_{T'}(t)) \to H^0(\mathcal{O}_T(t))$ is surjective. Hence it is sufficient to prove that $h^1(H, \mathcal{I}_{T'}(k)) = 0$. First assume $n \geq 5$. In this case by [1], Theorem 1, it is sufficient to check that $\deg(T') \leq r_{n-1,k}$. Since $d \leq r_{n,k}+1$, it is sufficient to check that $r_{n,k} - r_{n,k-1} + 2 + s - 1 \leq r_{n-1,k}$. By (2) it is sufficient to check that $k(s+1) \leq r_{n,k-1}$. Hence by (2) it is sufficient to prove that $k^2(s+1) + 1 \leq \binom{n+k-1}{n}$. Since $s \leq 2k-2$, it is sufficient to check that $k^2(2k-1) + 1 \leq \binom{n+k-1}{n}$. This is true if $k \geq 3$ and $n \geq 5$.

(a1) Now assume $n = 4$. We first assume $k \equiv 1, 5 \pmod 6$. In this case $q_{3,k} = 0$ and $h^i(H, \mathcal{I}_U(k)) = 0$, $i = 0, 1$, for a general disjoint union $U \subset H = \mathbb{P}^3$ of $k+1$ bamboos such that $\deg(U) = r_{3,k} - 1$. We may take as T a union of some of the lines of U if $r_{4,k} - r_{4,k-1} \leq r_{3,k} - 1 - s$, i.e. $r_{4,k} - r_{4,k-1} \leq r_{3,k} - 1 - q_{4,k-1}$. Assume $r_{4,k} - r_{4,k-1} \geq r_{3,k} - q_{4,k-1}$. Since $q_{3,k} = 0$, we have $kr_{3,k} = \binom{k+3}{3} - 1$. Hence from (2) we get

$$r_{4,k-1} + q_{4,k} - (k+1)q_{4,k-1} \leq 1 \tag{3}$$

Since $q_{4,k-1} \leq k-2$, $q_{4,k} \geq 0$ and $(k-1)r_{4,k-1} \leq \binom{k+3}{4}$, for large k, say $k \geq 24$, we get a contradiction. For low k we see the explicit values of $r_{4,t}$ and $q_{4,t}$. Only the case $k = 7$, $k = 9$, $k = 16$, have relatively high $q_{3,k-1}$ and the middle one has $k \equiv 3 \pmod 6$. Assume $k = 7$. We have $r_{4,6} = 34$, $q_{4,7} = 0$ and $q_{4,6} = 5$ and hence (3) is satisfied. In this case Y has degree 33 and 12 connected components, all of them of degree at least two, while T has degree $r_{4,7} - r_{4,6} + 1 = 14$. We may add a union $T \subset H$ of 2 reducible conics and 10 lines so that $Y \cup T$ is a genus 1 loop. Since $Y \cap H$ is general in H, T may be considered as a general union of 2 reducible conics and 10 lines. To conclude the proof of this case it is sufficient to prove that $h^1(H, \mathcal{I}_T(7)) = 0$. Fix a smooth quadric surface $Q \subset H$. Let $G \subset H$ be a general union of 9 lines. We have $h^1(\mathcal{I}_G(5)) = 0$ ([4]) and $G \cap Q$ is a general union of 18 points. Let $R \subset Q$ be a union of 5 lines of type $(0,1)$ on Q, exactly two of them containing a point $G \cap Q$. By Horace lemma to prove that $h^1(H, \mathcal{I}_{E \cup R}(7)) = 0$ (and hence by the semicontinuity theorem to prove that $h^1(H, \mathcal{I}_T(7)) = 0$), it is sufficient to prove that $h^1(Q, \mathcal{I}_{(Q \cap E) \cup R}(7)) = 0$, i.e. $h^1(Q, \mathcal{I}_{(Q \cap E) \setminus E \cap R}(7,2)) = 0$. This is true, because $h^0(Q, \mathcal{O}_Q(7,2)) = 24$ and $Q \cap E \setminus E \cap R$ is a general union of 16 points of Q.

Assume $k = 16$. We have $r_{4,15} = 257$ and $q_{4,15} = 10$ and hence (3) is not satisfied.

Now assume $k \equiv 0, 2, 3, 4 \pmod 6$. By [1], Assertion A(k), we have $h^i(H, \mathcal{I}_W(k)) = 0$, $i = 0, 1$, for a general disjoint union $W \subset H = \mathbb{P}^3$ of $1 + q_{3,k}$ bamboos such that $\deg(W) = r_{3,k}$. In this case to take as T a union of some of the irreducible components of W we need to check the inequality $r_{4,k} - r_{4,k-1} \leq r_{3,k} - s + 1 + q_{3,k} = r_{3,k} - k + 1 - q_{4,k-1} + q_{3,k}$. Assume $r_{4,k} - r_{4,k-1} \geq r_{3,k} - (k-2) - q_{4,k-1} + q_{3,k}$. Since $kr_{3,k} = \binom{k+3}{3} - 1 - q_{3,k}$, (2) gives

$$r_{4,k-1} - (k-2)k + q_{4,k} - (k+1)q_{4,k-1} \leq 1 - (k-1)q_{3,k} \tag{4}$$

For all $m \in \mathbb{N}$ we have $q_{3,6m+2} = 3m+1$, $q_{3,6m+3} = 2m+1$, $q_{3,6m+4} = 3m+2$ and $q_{3,6m+6} = 5m+5$. Hence $q_{3,9} = 3$. Since $r_{4,8} = 61$, $q_{4,9} = 3$ and $q_{4,8} = 6$, (4) is satisfied if $k = 9$. In this case $\deg(Y) = 60$ and Y has 15 connected components of degree at least two. We have $\deg(T) = r_{4,9} - r_{4,8} + 1 = 20$. We take as T a disjoint union of 5 reducible conics and 10 lines. For general Y the curve T may be considered as a general union of 5 reducible conics and 10 lines. We only need to check that $h^1(H, \mathcal{I}_T(9)) = 0$. Fix a smooth quadric surface $Q \subset H$. Let $E \subset H$ be a general union of 15 lines. We have $h^i(\mathcal{I}_E(7)) = 0$, $i = 0, 1$, by [4] and $E \cap Q$ is a general union of 30 points of Q. Let $J \subset Q$ be a disjoint union of 5 lines in the same ruling of Q (say of type $(1,0)$), each of them containing a point of $E \cap Q$. Since $E \cap Q \setminus J \cap E$ is a general union of 25 points of Q, we have $h^1(Q, \mathcal{I}_{J\cup(E\cap Q)}(9)) = h^1(Q, \mathcal{I}_{E\cap Q\setminus E\cap J}(9,4)) = 0$, Horace Lemma gives $h^1(\mathcal{I}_{E\cup F}(9)) = 0$. Hence $h^1(H, \mathcal{I}_T(9)) = 0$.

(b) In this step we assume $d < r_{n,k-1} - 1 + s$. Set $e := r_{n,k-1} + s - d$. We take a union $Y' \subset Y$ of some of the irreducible components of Y' with $\deg(Y') = \deg(Y) - e$, Y' with s connected components and no connected component of Y' is a line (this is possible, because $\deg(Y) = r_{n,k-1} - 1 \geq 2s + e$). Since $k \geq 3$, the restriction map $H^0(\mathcal{O}_Y(k-1)) \to H^0(\mathcal{O}_{Y'}(k-1))$ is surjective. Hence $h^1(\mathcal{I}_{Y'}(k-1)) = 0$. Then we continue as in step (a) using Y' instead of Y and taking as T a disjoint union of s lines (which has maximal rank by [4]); we may avoid the more complicated proof given in [1], because no connected component of Y is a line.

(c) In this step we prove the existence of B. Set $x := r_{n,k-1}$ if $q_{n,k-1} = 0$ and $x := r_{n,k-1} + 1$ if $q_{n,k-1} > 0$. By [1], Theorem 1, if $n \geq 5$ there is a bamboo $G \subset \mathbb{P}^n$ such that $\deg(G) = x$ and $h^0(\mathcal{I}_G(k-1)) = 0$. If $d > x$, then there is a genus 1 loop $U \supset G$ with $\deg(U) = d$. Since $h^0(\mathcal{I}_G(k-1)) = 0$, we have $h^0(\mathcal{I}_U(k-1)) = 0$. Since $d > r_{n,k-1}$, it only remains the case $d = r_{n,k-1} + 1$ and $q_{n,k-1} > 0$ (at least if $n \geq 5$). We may assume $k \geq 3$, because for all $x \geq n$ a general $X \in A(n,x)$ spans $\mathbb{P}^n$. Set $f := k - 1 + q_{n,k-2}$. Take $E \in A(n, r_{n,k-2} - 1)$ satisfying $R(n, k-2)$. We have $h^0(\mathcal{I}_E(k-2)) = 0$. We may assume that $E \cap H$ is a general subset of H with cardinality $\deg(E)$. We need to prove the existence of a disjoint union $Z \subset H$ of f bamboos such that $\deg(Z) = d - r_{n,k-2} + 1$, $\sharp(E \cap H) \cap Z) = f + 1$, $E \cup Z$ is a genus 1 loop and $h^0(H, \mathcal{I}_{(E\cap H)\cup Z}(k-1)) = 0$ (the last equality would prove the existence of B by the semicontinuity theorem because the Horace lemma would give $h^0(\mathcal{I}_{E\cup Z}(k-1)) = 0$). We need $\deg(Z) \geq f$, i.e. (since $d \geq x$) $r_{n,k-1} - r_{n,k-2} + 2 \geq f$. Since $f \leq 2k - 4$, we have $\deg(Z) \geq f$ if $r_{n,k-1} - r_{n,k-2} \geq 2k - 6$; this is true if either $n \geq 6$ or $n = 5$ and $k \geq 11$ or $n = 4$ and $n = 4$ and $k \geq 41$ by [1], Lemma 22; if $n = 5$ and $3 \leq k \leq 11$, then use Remark 1. See below for the case $n = 4$ and $k \leq 40$.

By (2) for $k' := k - 1$ to prove that $h^0(H, \mathcal{I}_{(E\cap H)\cup Z}(k-1)) = 0$ it is sufficient to prove that $h^1(H, \mathcal{I}_Z(k-1)) = 0$. Take $Z' \supseteq Z$ with Z' union of Z and $q_{n,k-2}$ lines and Z' union of k disjoint bamboos. Since the restriction map $H^0(\mathcal{O}_{Z'}(k-1)) \to H^0(\mathcal{O}_Z(k))$ is surjective, it is sufficient to prove that $h^1(\mathcal{I}_{Z'}(k-1)) = 0$. If $n = 5$ by $H''(n-1, k-1)$ of [1] it is sufficient to prove that $\deg(Z') \leq r_{n-1,k-1} - 2$, i.e. $r_{n,k-1} - r_{n,k-2} \leq r_{n-1,k-1} - 3 - q_{n,k-2}$. Assume $r_{n,k-1} - r_{n,k-2} \geq r_{n-1,k-1} - 2 - q_{n,k-2}$. From (2) for the integer $k' := k - 1$ we get $r_{n,k-2} + (k-1)r_{n-1,k-1} - 2k + 2 + q_{n,k-1} - (k-2)q_{n,k-2} \leq \binom{n+k-2}{n-1}$. Since $(k-1)r_{n-1,k-1} = \binom{n+k-2}{n-1} - q_{n-1,k-1} \geq \binom{n+k-2}{n-1} - k + 2$ and $q_{n,k-1} \geq 0$, we get $r_{n,k-2} - (k-2)q_{n,k-2} - 3k \leq 0$. Since $q_{n,k-2} \leq k - 3$ and $(k-2)r_{n,k-2} \geq \binom{n+k-2}{n} - k - 2$, we get a contradiction.

Now assume $n = 4$. We work as in step (a1), using $k-1$ instead of k. $\square$

Remark 2. Fix positive integer t, x such that there is a degree x bamboo Y with $h^0(\mathcal{I}_Y(t)) = 0$. Then for all $d \ge x+1$ we have $h^0(\mathcal{O}_X(t)) = 0$ for a a general $X \in A(3,t)$.

Addendum to step (iii) at page 213 of [1]: A few lines are missing explaining why this case is similar to the one solved in step (i) (my fault!) (see also step (a1) of the proof of Proposition 3). To get W when $k-1 \equiv 1,0,5,4 \pmod 6$ we start as in steps (a), (d), (e), (f) with a solution of $A(k-7)$, then we arrive at a curve A with $h^0(\mathcal{I}_A(k-2)) = 0$, $\deg(A) = r_{3,k-2}$, A a disjoint union of $r_{3,k} - r_{3,k-2}$ bamboos and a disjoint union $J \subset Q$ of $r_{3,k} - r_{3,k-2} + 1$ lines so that $A \cup J$ is a bamboo of degree $r_{3,k} + 1$. Now assume $k - 1 = 6m + 2$ for some positive integer m. We have $r_{3,6m+2} - r_{3,6m} = 4m + 3$. We use $A(6m-2)$ to get a pair (A, T), where A is a disjoint union of $4m+3$ bamboos, $\deg(A) = r_{3,6m}+1$, T is a union of $4m+3$ lines and $A \cup T$ is a bamboo of degree $r_{3,6m+2} + 1$ with $h^0(\mathcal{I}_{A\cup T}(6m+2)) = 0$.

Proposition 3. *Fix an integer $d \ge 151$ such that $d \ne r_{3,k}$ for any k. Then a general element of $A(3,d)$ has maximal rank.*

Proof. Let k be the critical value of the space curves of degree d and genus 1. Since $d > 150 = r_{3,27}$, we have $k \ge 28$ and $q_{3,k} + 1 < k$. Hence k is the critical value also for the curves of degree d and genus zero. Our assumption gives $r_{3,k-1} < d < r_{3,k}$.

To prove the existence of $A \in A(3,d)$ with $h^1(\mathcal{I}_A(k)) = 0$ it is not sufficient (as in the case of bamboos) to do the case $d = r_{3,k} - 1$. In all cases as in the proof of the existence of B only at the end of the construction we get a genus 1 loop (before them we only have disjoint unions of bamboos) and if in the last step we add the union J of x lines of type $(0,1)$ of Q, then in the in the next to last step we construct a degree $d - x$ curve which is a disjoint union of x bamboos).

(a) In this step we assume $d = r_{3,k} - 1$.

(a1) First assume $k \equiv 0,1,2,4 \pmod 6$. We start as in steps (a), (d), (e), (f) with a solution F of $A(k-4)$, i.e. $\deg(F) = r_{3,k-4}$ and F a disjoint union of $1+q_{3,k-4}$ bamboos. We add in Q a suitable union M of $r_{3,k-2}-r_{3,k-4}-1$ lines of type $(1,0)$ so that $F \cup M$ is a disjoint of $r_{3,k} - r_{3,k-2}$ bamboos. We may deform $F \cup M$ to a disjoint union E of $r_{3,k} - r_{3,k-2}$ bamboos, $\deg(E) = r_{3,k-2} - 1$ such that there is a disjoint union $J \subset Q$ of $r_{3,k} - r_{3,k-2}$ lines of type $(0,1)$ with $E \cup J$ a genus 1 loop of degree $r_{3,k} - 1$. If this is possible, then $h^1(\mathcal{I}_{E\cup J}(k)) = 0$, because $\sharp(E \cap Q \setminus E \cap J) = 2r_{3,k-2} - 2 - 2(r_{3,k} - r_{3,k-2}) \le h^0(Q, \mathcal{O}_Q(k, k - r_{3,k} + r_{3,k-2}))$.

To get J from F and M we need to take a very particular Q after fixing F, because we have $r_{3,k} - r_{3,k-2} > r_{3,k-2} - r_{3,k-4} - 1$. Fix $y \in \{0, 1, 2, 3\}$. We take Q containing y disjoint secant lines of F, call them $L_1, \dots, L_y$, and call $(0, 1)$ the ruling of Q containing them. We assume that the union of F and $L_1, \dots L_y$ is a disjoint union of $1 + q_{3,k-4} - y$ bamboos (in particular each L_i intersects final lines of different connected components of F). Using M we may deform $A \cup M$ so that the lines L_i are y secant lines of F, while the lines in M contribute to the existence of $\deg(M)$ secant lines to E (as in the proof of [1], Lemma 5). We need $r_{3,k-2} - r_{3,k-4} - 1 + y = r_{3,k} - r_{3,k-2}$. If $k \equiv 0 \pmod 6$, then we take $y = 2$. If $k \equiv 4 \pmod 6$, then we take $y = 2$. If $k \equiv 2 \pmod 6$, then we take $y = 3$. If $k \equiv 1 \pmod 6$, then we take $y = 3$.

(a2) Now assume $k \equiv 3 \pmod 6$ and $k \geq 27$. Write $k = 6m + 9$ with $m \geq 2$. We start with a solution F of $B(6m + 5)$ ([1], page 211, i.e. with a disjoint union F of $4m+12$ bamboos, with $\deg(F) = r_{3,6m+5} - 1$ and $h^1(\mathcal{I}_F(6m+5)) = 0$. We use as E a disjoint union of $r_{3,k} - r_{3,k-2} = 4m + 7$ bamboos and $\deg(E) = r_{3,k-2} - 1$.

(a3) Now assume $k \equiv 5 \pmod 6$. Write $k = 6m + 5$ for some $m \in \mathbb{N}$. We use $A(6m + 1)$ to construct a triple (A, Q, T) such that A is a disjoint union of $4m + 6$ bamboos, $\deg(A) = r_{3,6m+3} - 1$, Q ia a smooth quadric surface, $h^1(\mathcal{I}_A(6m + 3)) = 0$, $T \subset Q$ is a disjoint union of $4m + 6$ lines and $A \cup T$ is a genus 1 loop.

(b) Now assume $r_{3,k-1} < d \leq r_{3,k} - 2$. We take $F' \subset F$ with $\deg(F) - \deg(F') = r_{3,k-1} - d$ and F and F' with the same number of connected components. Then from F' we obtain E' and J. □

Proof of Proposition 1: Let $Z(n; a, b; d_1, \cdots, d_{a+b})$ be the set of all curves $X \subset \mathbb{P}^n$ which are the disjoint union of a bamboos of degree $d_1, \dots, d_a$ and b genus 1 loops of degree $d_{a+1}, \dots, d_{a+b}$. Since $Z(n; a, b; d_1, \cdots, d_{a+b})$ is irreducible, for the second part it is sufficient to find $X, W \in Z(n; a, b; d_1, \cdots, d_{a+b})$ with $h^1(\mathcal{I}_X(k)) = 0$ and $h^0(\mathcal{I}_W(k-1)) = 0$. Set $z := q_{n,k-1} + k - a$ if $(a, b, q_{n,k-1}) \neq (k, 0, 0)$ and $z := 1$ if $(a, b, q_{n,k-1}) = (k, 0, 0)$. Let $H \subset \mathbb{P}^n$ be a hyperplane. Notice that $d_1 + \cdots + d_{a+b} \leq r_{n,k} - 1$. In steps (a) and (b) we prove the existence of X.

(a) First assume $d_1 + \cdots + d_{a+b} \geq r_{n,k-1} - 1 + z$. Take a solution Y of $R(n, k-1)$. We have $\deg(Y) = r_{n,k-1} - 1$, $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$, each connected component of Y is a bamboo and Y has $k + q_{n,k-1}$ connected components. We may also assume that $Y \cap H$ is a general subset of H with cardinality $r_{n,k-1} - 1$. Since each connected component of Y has degree at least two, for every integer $e \geq z$ there is a degree e curve $E \subset H$ which is a disjoint union of z bamboos and such that $Y \cup E$ is a disjoint union of a bamboos and b genus 1 loops. Since $d_1 + \cdots + d_{a+b} - \deg(Y) \geq z$, we may take $e = d_1 + \cdots + d_{a+b} - \deg(Y)$. By the Horace Lemma it is sufficient to prove that $h^1(H, \mathcal{I}_{(Y \cap H) \cup E}(k)) = 0$. Since $k(d_1 + \cdots + d_a) + a \leq \binom{n+k}{n}$, $h^0(\mathcal{O}_Y(k-1)) = \binom{n+k-1}{n-1}$ and $\chi(\mathcal{O}_{Y \cup E}) = \chi(\mathcal{O}_Y) + \chi(\mathcal{O}_Y) - \sharp(Y \cap E)$. we have $h^0(\mathcal{O}_{(Y \cap H) \cup E}(k)) \leq \binom{n+k-1}{n-1}$. Since $Y \cap H$ is general in H, it is sufficient to prove that $h^1(H, \mathcal{I}_E(k)) = 0$. The curve E has z connected components, each of them being a bamboo. Hence there is a bamboo E' such that $E \subsetneq E' \subset E$ and $\deg(E') = \deg(E) + z - 1$. For large k (say $k \geq k_1$ with k_1 the same for all $n \geq 4$), we have $\deg(E) + z - 1 \leq r_{n-1,k} - k$, because $\deg(Y) - \deg(Y \cap E) \geq (z+k)k$. Hence $h^1(H, \mathcal{I}_{E'}(k)) = 0$ ([1], Theorem 1, and [1], either E(3,k) or Theorem 3). Hence $h^1(H, \mathcal{I}_E(k)) = 0$.

(b) Now assume $d_1 + \cdots + d_{a+b} < r_{n,k-1} - 1 + z$. Assume for the moment $d_1 + \cdots + d_{a+b} - z \geq 2(k + q_{n,k-1})$. Take a solution Y of $R(n, k-1)$. For every integer y with $2k + 2q_{n,k-1} \leq y \leq r_{n,k-1} - 1$ there is a curve $Y' \subset Y$ such that Y' has $k + q_{3,k-1}$ connected components and each connected of Y' has degree ≥ 2. We take Y' with $\deg(Y) = d_1 + \cdots + d_{a+b} - z$. As in step (a) we may take a union $E \subset H$ of z bamboo such that $Y \cup E \in Z(n; a, b; d_1, \cdots, d_{a+b})$. Since $h^0(\mathcal{O}_{Y'}(k-1)) \leq \binom{n+k-1}{n-1}$, we get $h^0(\mathcal{O}_{(Y' \cap H) \cup E}(k)) \leq \binom{m+k-1}{n-1}$. We conclude as in step (a). Since $Y' \cap H$ is general in H and $h^1(H, \mathcal{I}_E(k)) = 0$ by step (a), we are done. Now assume $d_1 + \cdots + d_{a+b} - z \leq -1 + 2(k + q_{n,k-1})$. We get $d_1 + \cdots + d_{a+b} \leq 6k$ and this case is trivial (e.g., take $k_0 \geq 20$ and do first the case $n = 4$ and then use induction on n; alternatively, we have an upper bound on $d_1 + \cdots + d_{a+b}$ which does not depend on n and hence we may use the same numerical upper bounds simultaneously for all $n \geq 4$).

(c) The proof of the existence of Y is similar, using $k-1$ instead of k (if an integer k_0' works for the existence of X using only that $\deg(Y) - \sharp(Y \cap E) \leq 3k^2$ for all $k \geq k_0'$, then the integer $k_0' + 1$ works for the existence of W).

(d) In this step we check the existence of B.

(d1) First assume $d = r_{3,k-1} + 1$. In this case we follow the Addendum with the following modifications. Assume $k - 1 \equiv 1, 0, 5, 4 \pmod 6$. We start as in steps (a), (d), (e), (f) of [1], proof of Theorem 3, with a solution F of $A(k-5)$, then we arrive at a curve E with $h^0(\mathcal{I}_E(k-3)) = 0$, $\deg(E) = r_{3,k-3}$, E a disjoint union of $r_{3,k} - r_{3,k-2} + 1$ bamboos. We also get the existence of E with the following strong property: there is disjoint union $J \subset Q$ of $r_{3,k} - r_{3,k-2} + 1$ lines so that $A \cup J$ is a genus 1 chain of degree $r_{3,k} + 1$. Now assume $k - 1 = 6m + 2$ for some positive integer m. We use $A(6m-2)$ to get a pair (A, T), where A is a disjoint union of $4m+3$ bamboos, $\deg(A) = r_{3,6m} + 1$, T is a union of $4m+3$ lines and $A \cup T$ is a genus 1 loop of degree $r_{3,6m+2} + 1$

with $h^0(\mathcal{I}_{B\cup T}(6m+2)) = 0$.

(d2) Now assume $r_{3,k-1} + 2 \leq d < r_{3,k}$. Take F as in step (d1) and add to it a bamboo of degree $d - r_{3,k-1} + 2$ (call F' this union) so that F and F' have the same number of connected components, each of them being a bamboo. Use F' instead of F and get E', A' and J' as in step (d1), but with $\deg(E') = \deg(E) + d - r_{3,k-1} - 1$, $\deg(J') = \deg(J)$ and $\deg(A') = \deg(A) + d - r_{3,k-1} - 1$ □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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