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HOPF BIFURCATION AND GLOBAL PERIODIC SOLUTION
IN A DELAYED STAGE-STRUCTURED
PREY-PREDATOR SYSTEM

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Abstract: A prey-predator model with stage-structured for predator and time delay is considered. The characteristic equations of the equilibrium points are analyzed, by applying the theorem of Hopf bifurcation, the conditions for the positive equilibrium occurring local Hopf bifurcation are given. The conditions for the existence of global Hopf bifurcation of the system are obtained. Finally, numerical simulation and brief conclusion are given.

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Key Words: prey-predator system, stage-structured, time delay, Hopf bifurcation, global periodic solution

1. Introduction

In population dynamics, time delays play an important role, which can make the system's equilibrium points loss of its stability and bifurcate various periodic solutions, even chaos. Recently, S.Y. Li and X.G. Xue [1, 2] investigated the local Hopf bifurcation of a three-stage-structured prey-predator system, they took time delay τ as the bifurcation parameter, shown that, the positive equilibrium loses its stability and a local Hpof bifurcation occurring when the

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delay τ passes through the first critical value τ_0.

In delayed population systems, are there existence of large-scale periodic solutions (global Hopf bifurcation) when τ far away from the first Hopf bifurcation critical values τ_0? C.J. Sun, M.A. Han and Y.P. Lin [3] studied the global Hopf bifurcation of a delayed logistic model by using a the result due to Wu [4]. The global periodic solutions of a delayed predator-prey system have considered by X.P. Yan and W.T. Li [5]. In this paper, we consider the Hopf bifurcation and global periodic solution of following stage-structured prey-predator system with hunting delay

$$\begin{cases} x_1'(t) = \alpha x_2(t) - x_1(t)[\gamma - \eta x_1(t) - Ey(t-\tau)], \\ x_2'(t) = \Omega x_1(t) - \theta x_2(t), \\ x_3'(t) = ax_2(t) - bx_3(t) - cx_3^2(t), \\ y'(t) = y(t)(kEx_1(t) - d - fy(t)), \end{cases} \tag{1}$$

where $\gamma = \gamma_1 + \Omega, \theta = \theta_1 + a$. $x_i(t)(i = 1, 2, 3)$ are the densities of immature preys, mature preys and old preys at time t, $y(t)$ is the density of predator at time t, respectively. All of the parameters are positive, α is the birth rate of mature prey population, and γ_1, θ_1, b are the death rate of immature, mature and old prey population, respectively. Ω and a are the maturity rate and ageing rate of the prey population, respectively. η, c and f are the density dependent coefficients of immature and old prey population, predator population, respectively. $k(0 < k < 1)$ is the rate of conversing prey into predator and E is the predation coefficient. τ is the hunting delay for predator population. The initial conditions for (1) are:

$$x_i(t) = \phi_i(i) \geq 0(i = 1, 2, 3), y(t) = \varphi(t) \geq 0, t \in [-\tau, 0].$$

2. Local Hopf Bifurcation

Obviously, (1) has two boundary equilibrium $E_0 = (0, 0, 0, 0)$ and $E_1(x_1, x_2, x_3, 0)$ (if condition $C_1 : \alpha\Omega - \gamma\theta > 0$ holds), and a unique positive equilibrium $E_2(x_1^*, x_2^*, x_3^*, y^*)$ (if condition $C_2 : kEx_1^* - d > 0$ holds), where

$$x_1 = \frac{\alpha\Omega - \gamma\theta}{\eta\theta}, x_2 = \frac{\Omega}{\theta}x_1, x_3 = \frac{\sqrt{b^2 + 4acx_2} - b}{2c} = f(x_2),$$

$$x_1^* = \frac{f(\alpha\Omega - \gamma\theta) + dE\theta}{\theta(kE^2 + \eta f)}, x_2^* = \frac{\Omega}{\theta}x_1^*, x_3^* = f(x_2^*), y^* = \frac{kEx_1^* - d}{f}.$$

The characteristic equation about the positive equilibrium E_2 is given by

$$\det \begin{pmatrix} a_{11} & \alpha & 0 & -Ex_1^*e^{-\lambda\tau} \\ \Omega & -\theta-\lambda & 0 & 0 \\ 0 & a & -b-2cx_3^*-\lambda & 0 \\ kEy^* & 0 & 0 & -fy^*-\lambda \end{pmatrix} = 0,$$

where $a_{11} = -\gamma - 2\eta x_1^* - Ey^* - \lambda$. Namely, $(\lambda + b + 2cx_3^*)D(\lambda, \tau) = 0$. $\lambda_4 = -b - 2cx_3^* < 0, \lambda_i(i=1,2,3)$ are the roots of the characteristic equation

$$D(\lambda, \tau) = M(\lambda) + N(\lambda)e^{-\lambda\tau} = 0, \tag{2}$$

where

$$M(\lambda) = \lambda^3 + m_2\lambda^2 + m_1\lambda + m_0, N(\lambda) = n_1\lambda + n_0$$
$$m_2 = \gamma + 2\eta x_1^* + Ey^* + \theta + fy^*,$$
$$m_1 = \theta(\eta x_1^* + fy^*) + fy^*(\gamma + 2\eta x_1^* + Ey^*),$$
$$m_0 = f\eta\theta x_1^*y^*, n_1 = kE^2x_1^*y^*, n_0 = kE^2\theta x_1^*y^*.$$

When $\tau = 0$, (2) becomes

$$\lambda^3 + m_2\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0. \tag{3}$$

Note that, if $C_3 : kE^2 < f\eta$ hold true, then $m_2(m_1 + n_1) - (m_0 + n_0) > 0$. By Routh-Hurwits criterion, all roots of (3) have negative real parts. Then, the equilibrium E_2 is local stable.

Suppose $\lambda = i\omega(\omega > 0)$ is a root of (2) and separating the real and imaginary parts, we have

$$\begin{cases} m_2\omega^2 - m_0 = (n_0 - n_2\omega^2)\cos\omega\tau + n_1\omega\sin\omega\tau, \\ \omega^3 - m_1\omega = n_1\omega\cos\omega\tau - (n_0 - n_2\omega^2)\sin\omega\tau, \end{cases} \tag{4}$$

From (4), one can get that

$$(n_0 - n_2\omega^2)^2 + n_1^2\omega^2 = (m_2\omega^2 - m_0)^2 + (\omega^3 - m_1\omega)^2. \tag{5}$$

Namely

$$F(\omega) = \omega^6 + p\omega^4 + q\omega^2 + r = 0, \tag{6}$$

where $p = m_2^2 - 2m_1 - n_2^2 > 0, q = m_1^2 + 2n_2n_0 - n_1^2 - 2m_2m_0,$

$$r = m_0^2 - n_0^2 = x_1^*y^*\theta(m_0 + n_0)(kE^2 - f\eta). \tag{7}$$

If $C_3 : kE^2 < f\eta$ and $q > 0$ hold, we know that (6) has a unique positive root ω_0. From (4), we have

$$\cos\omega_0\tau = \frac{(m_2\omega_0^2 - m_0)(n_0 - n_2\omega_0^2) + n_1\omega_0(\omega_0^3 - m_1\omega_0)}{(n_0 - n_2\omega_0^2)^2 + (n_1\omega_0)^2},$$

Thus

$$\tau_j = \frac{1}{\omega_0} \cos^{-1}\left[\frac{(m_2\omega_0^2 - m_0)(n_0 - n_2\omega_0^2) + n_1\omega_0(\omega_0^3 - m_1\omega_0)}{(n_0 - n_2\omega_0^2)^2 + (n_1\omega_0)^2}\right] \\ + \frac{2j\pi}{\omega_0}, \ j = 0, 1, 2, \cdots \tag{8}$$

Let $\lambda(\tau) = v(\tau) + \mathrm{i}\omega(\tau)$ be the roots of (2) such that when $\tau = \tau_j$ satisfying $v(\tau_j) = 0$ and $\omega(\tau_n) = \omega_0$. Since ω_0 is the unique positive root of (6), then, at each $\tau = \tau_j(j = 0, 1, 2, \cdots)$ straightforward calculation gives [6]

$$\mathrm{sign}\left[\mathrm{d}F(\omega)/\mathrm{d}\omega|_{\omega=\omega_0}\right] = 2\omega_0(3\omega_0^4 + 2p\omega_0^2 + q) > 0.$$

According to the Hopf bifurcation theorem for functional differential equations [7], we have the following result.

Theorem 1.   *If $C_3 : kE^2 < f\eta$ and $q > 0$ hold, then there exists τ_0, when $\tau \in [0, \tau_0)$ the positive E_2 of (1) is asymptotically stable and unstable when $\tau > \tau_0$; system (1) can undergo a local Hopf bifurcation near the positive equilibrium E_2 when $\tau = \tau_j(j = 0, 1, 2, \cdots)$.*

3. Global Hopf Bifurcation

In this section, we study the existence of global Hopf bifurcations. The method we used here is based on the global Hopf bifurcating theorem for general functional differential equations introduced by Wu [4]. For convenience, we write system (1) as the following form

$$z'(t) = F(z_t, \tau) \tag{9}$$

where $z(t) = (x_1(t), x_2(t), x_3(t), y(t))$, $z_t(\theta) = z(t + \theta) \in ([-\tau, 0], \mathbb{R}^4)$.

Define $X = C([-\tau, 0], \mathbb{R}^4)$, $\Sigma = Cl\{(z(t), \tau, T) \in X \times \mathbb{R} \times \mathbb{R}^+, z(t) \text{ is } T\text{-periodic solution of (1)}\}$, $N = \{(z(t), \tau, T), F(z(t), \tau, T) = 0\}$.

Let $l_{(E_2, \tau_j, 2\pi/\omega_0)}$ be the connected component $(E_2, \tau_j, 2\pi/\omega_0)$ in Σ, where τ_j and ω_0 defined in (8).

Lemas 2.   *All solution of (1) are positive and uniformly bounded.*

Proof. It is true that,

$$x_1'(t)|_{x_1(t)=0} = \alpha x_2(t) \geq 0,$$
$$x_2'(t)|_{x_2(t)=0} = \Omega x_1(t) \geq 0, x_3'(t)|_{x_3(t)=0} = ax_2(t) \geq 0,$$
$$y(t) = \varphi(0)\exp\left\{\int_0^t [kEx_1(s) - d - fy(s)]\mathrm{d}s\right\} \geq 0.$$

Let $V(t) = \sum_{i=1}^{3} x_i(t)$, for $0 < \varepsilon < |\theta - \alpha - a|$, we have

$$V'(t)|_{(1)} + \varepsilon V(t) \leq -(\gamma - \Omega - \varepsilon + \eta x_1(t))x_1(t) - (\theta - \alpha)x_2(t)$$
$$+ (a + \varepsilon)x_2(t) - (b + cx_3(t) - \varepsilon)x_3(t).$$

There exist $C > 0$ such that $V'(t)|_{(1)} + \varepsilon V(t) \leq C$, then $\lim_{t \to +\infty} V(t) \leq C/\varepsilon := M_1^*$. Therefore, exist $T_1 > 0$ and constant $M_1 > M_1^*$ such that $x_i(t) < M_1 (i = 1, 2, 3)$ for all $t > T_1$. From the fourth equation of system (1), we have $y'(t) \leq y(t)(kEM_1 - d - fy(t))$. Then, there exist $T_2 > T_1$ and positive constant M_2 such that $y(t) \leq (kEM_1 - d)/f := M_2$ for all $t > T_2$. $\square$

Lemas 3. If condition C_2 holds, then (1) has no nontrivial periodic solutions of τ.

Proof. Assume (1) has a nontrivial periodic solution of period τ, then the following differential equations

$$\begin{cases} x_1'(t) = \alpha x_2(t) - (\gamma + \eta x_1(t) + Ey(t))x_1(t), \\ x_2'(t) = \Omega x_1(t) - \theta x_2(t), \\ x_3' = ax_2(t) - bx_3(t) - cx_3^2(t), \\ y'(t) = y(t)(kEx_1(t) - d - fy(t)), \end{cases} \tag{10}$$

have periodic solutions. Let $V(t) = \sum_{i=1}^{3} c_i x_i(t) + c_4 y(t)$, where $c_i (i = 1, 2, 3, 4)$ are positive constants to be determined, it follows that

$$\begin{aligned} V'(t)|_{(10)} &= \sum_{i=1}^{3} c_i x_i'(t) + c_4 y'(t) \\ &= (c_2 \Omega - c_1 \gamma)x_1(t) - c_1 \eta x_1^2(t) + (c_1 \alpha - c_2 \theta + c_3 a)x_2(t) \\ &\quad - c_3(b + cx_3(t))x_3(t) + Ex_1(t)y(t)(c_4 k - c_1) \\ &\quad - c_4 y(t)(d + fy(t)). \end{aligned}$$

Let $c_1 = 1, c_2 = \gamma/\Omega, c_3 = (\alpha\Omega - \gamma\theta)/(a\Omega), c_4 = 1/k$, then

$$V'(t) = -c_1 \eta x_1^2(t) - c_3(b + cx_3(t))x_3(t) - c_4 y(t)(d + fy(t)) \leq 0.$$

Applying Barbalat's lemma [8], we conclude that

$$\lim_{t \to \infty} (x_1(t), x_2(t), x_3(t), y(t)) = E_2 = (x_1^*, x_2^*, x_3^*, y^*),$$

which contradicts the fact that (1) has periodic solutions. $\square$

Theorem 4. Suppose C_2, C_3 and $q > 0$ hold, then for each $\tau > \tau_j (j = 0, 1, 2, \cdots)$, (1) has at least $j + 1$ periodic solutions, where τ_j is defined in (8).
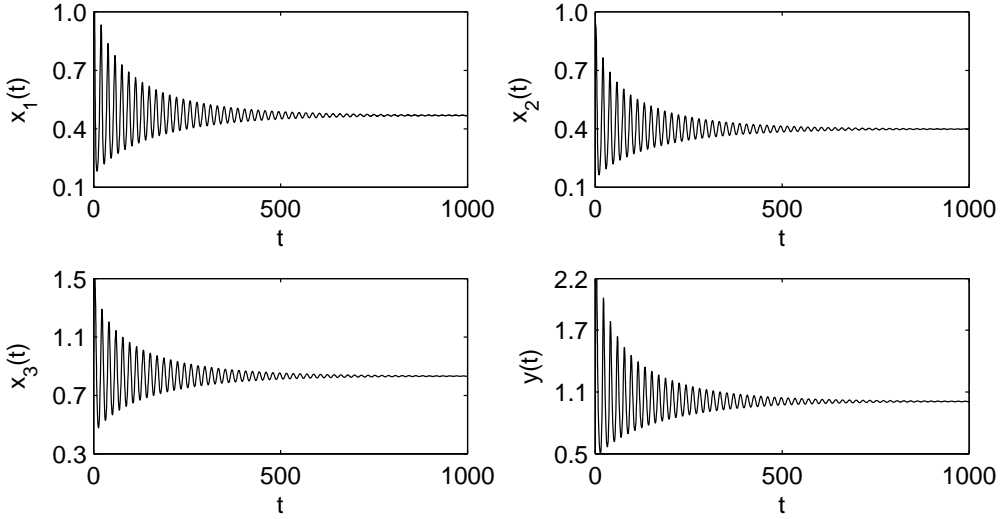
Figure 1: The time-series plot show that positive equilibrium point E_2 of system (1) is asymptotically stable for $\tau = 2.3 < \tau_0$.

Proof. The characteristic equation of (1) at the positive equilibrium E_2 is given by

$$\det(\Delta(E_2, \tau, p)(\lambda)) = (\lambda + b + 2cx_3^*)(M(\lambda) + N(\lambda)e^{-\lambda\tau}).$$

The equilibrium E_2 is local stable when $\tau = 0$, it can be verified that $(E_2, \tau_j, 2\pi/\omega_0)$ are isolated centers. Let

$$\Omega_{\varepsilon, 2\pi/\omega_0} = \{(\mu, p) : 0 < \mu < \varepsilon, |p - 2\pi/\omega_0| < \varepsilon\}$$

Clearly, $|\tau - \tau_j| \le \delta$ and $(\mu, p) \in \partial\Omega_\varepsilon$, then the necessary and sufficient conditions for $\Delta(E_2, \tau, p)(\mu + i2\pi/p) = 0$ are $\mu = 0, \tau = \tau_j$ and $p = 2\pi/\omega_0$. Define

$$H^{\pm}(E_2, \tau_j, 2\pi/\omega_0)(\mu, p) = \Delta(E_2, \tau_j \pm \delta, p)(\mu \pm i2\pi/p)$$

Then, we have the transversal number

$$\begin{aligned}
\gamma(E_2, \tau_j, 2\pi/\omega_0) = {} & \deg_B(H^-(E_2, \tau_j, 2\pi/\omega_0), \Omega_{\varepsilon, 2\pi/\omega_0}) \\
& - \deg_B(H^+(E_2, \tau_j, 2\pi/\omega_0), \Omega_{\varepsilon, 2\pi/\omega_0}) = -1
\end{aligned}$$

By Theorem 3.3 of Wu [4], we conclude that the connected component $l_{(E_2, \tau_j, 2\pi/\omega_0)}$ through $(E_2, \tau_j, 2\pi/\omega_0)$ in Σ is nonempty. Meanwhile, we have

$$\sum_{(z(t), \tau, p) \,\in\, l_{(E_2, \tau_j, 2\pi/\omega_0)}} \gamma(z(t), \tau, p) < 0$$
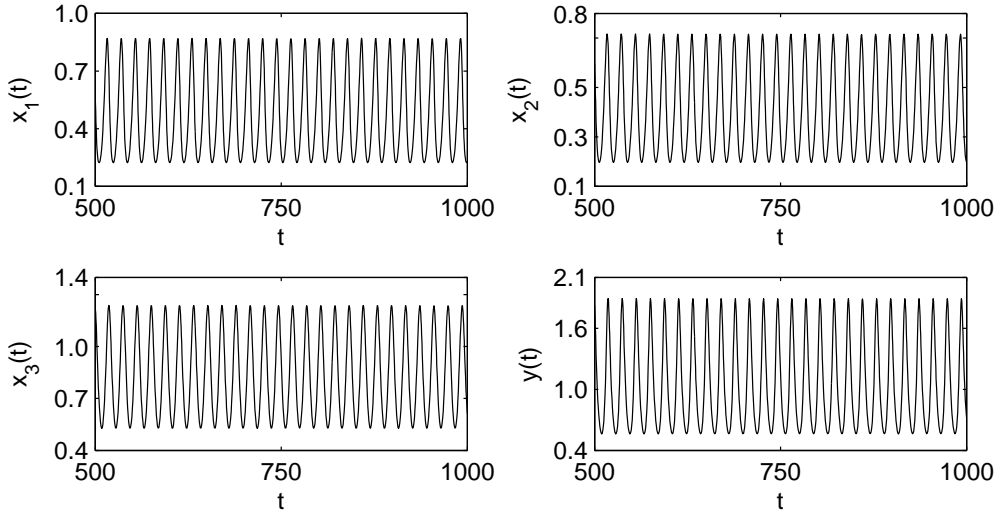
Figure 2: The time-series plot show that the local Hopf bifurcation for $\tau = 2.5 > \tau_0$ near positive equilibrium point E_2.

and hence $l_{(E_2,\tau_j,2\pi/\omega_0)}$ is unbounded.

Lemma 1 implies that the projection of $l_{(E_2,\tau_j,2\pi/\omega_0)}$ onto the z-space is bounded. From the definition of τ_j, we have $2\pi/\omega_0 < \tau_j$ for $j \geq 1$. Lemma 2 implies that the projection of $l_{(E_2,\tau_j,2\pi/\omega_0)}$ onto the τ-space is bounded. Assume the projection of $l_{(E_2,\tau_j,2\pi/\omega_0)}$ onto the τ-space $(0,\tau^*)$ and $\tau^* > \tau_j$. Applying Lemma 2, we know $(z(t),\tau,p) \in l_{(E_2,\tau_j,2\pi/\omega_0)}$ implies $p < \tau_j$. This shows that in order for $l_{(E_2,\tau_j,2\pi/\omega_0)}$ to be unbounded, its projection onto the τ-space must be unbounded. Consequently, the projection of $l_{(E_2,\tau_j,2\pi/\omega_0)}$ onto the τ-space includes $[\tau_j, +\infty)$. This shows that, for each τ_j, (1) has $j + 1$ nonconstant periodic solutions. $\square$

4. Numerical Simulation

Let $\alpha = 2.8, \gamma_1 = 0.2, \Omega = 0.85, \eta = 0.15, E = 1.25, \theta_1 = 0.2, a = 0.8, b = 0.3, c = 0.1, k = 0.6, d = 0.15, f = 0.2$ and the initial point $X(0) = (1,1,1,1)$. System (1) has unique positive equilibrium point $E_2 = (0.47, 0.40, 0.83, 1.01)$. We evaluate $p = 11.80, q = 0.29, r = -0.20, \omega_0 = 0.3417, \tau_0 = 2.4084$ and $\tau_j = 2.4084 + j5.854\pi (j = 0,1,2,\cdots)$. The the positive equilibrium point E_2 is asymptotically stable when $\tau = 2.3 < \tau_0$ (Fig. 1). When $\tau = 2.5 > \tau_0$,
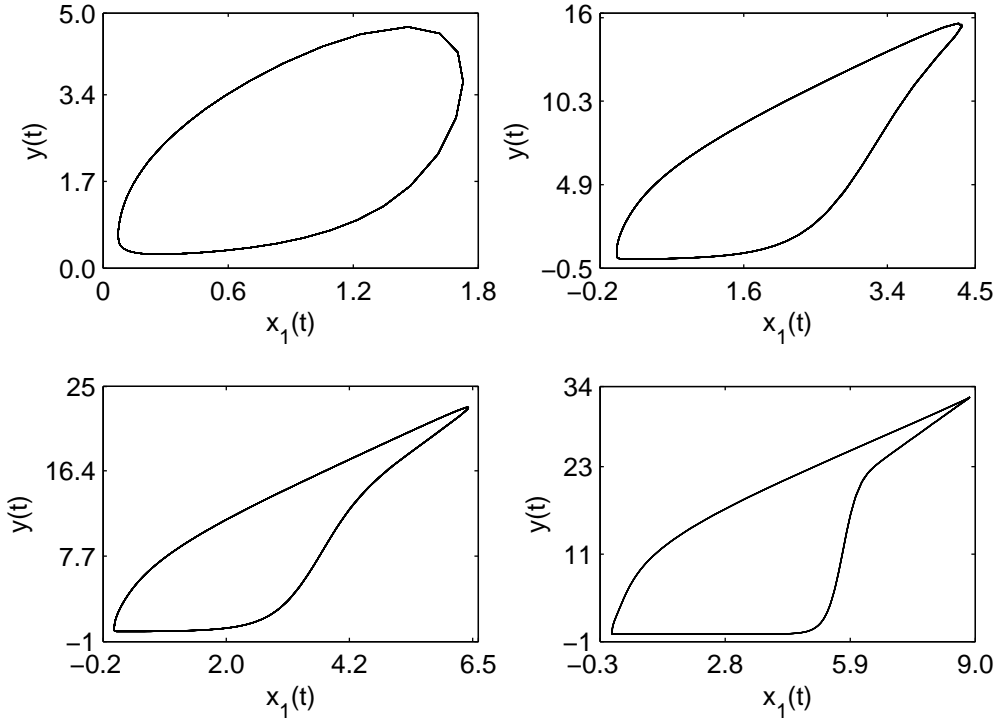
Figure 3: The large period oscillation (global Hopf bifurcation) of system (1) when $\tau = 3, 5, 7, 25$.

the positive equilibrium E_2 is unstable and the local Hopf bifurcation occurring around the positive E_2 are shown (Fig. 2). The simulations consistently show global existence of periodic solution (global Hopf bifurcation): existence of large amplitude periodic solutions for values of τ far away from τ_j, where $\tau = 3, 5, 7, 25$ and the periodic solutions are stable. From Fig. 3, we know that the period of the periodic solutions are increasing when τ 3 to 25. These are the slowly oscillating periodic solutions for (1). But, too large time delay would make the population to be die out, because the population very close to zero as time delay increase to some critical value.

5. Conclusion

In this paper, we studied a delayed stage-structured prey-predator system and analyzed the stability of the positive equilibrium, obtained the conditions of

the positive equilibrium occurring local Hopf bifurcation and global Hopf bifurcation (global periodic solution).

Numerical examples by time-series plot, shown that the system considered local asymptotically stable and stable local Hopf bifurcation periodic solutions. And the populations can be coexistence with large periodic fluctuating under some conditions which caused by the large time delay far away from τ_0, and the amplitudes of period oscillatory are increasing as time delays increased. In particular, we observe that the solutions of the delayed system could arbitrary close to zero when the delay tend to some critical values, which means that the prey or predator would tend to extinction. These are very interesting in mathematics and biology.

Acknowledgements

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