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IMPROVED BOUNDS FOR THE POISSON-BINOMIAL RELATIVE ERROR

K. Teerapabolarn[1 §], A. Boondirek[2]
[1,2]Department of Mathematics
Faculty of Science
Burapha University
Chonburi, 20131, THAILAND

Abstract: In this article, new uniform and non-uniform bounds on two forms of the relative error of the binomial cumulative distribution function with parameters n and p and the Poisson cumulative distribution function with mean $\lambda = np$ are determined. These are sharper than those reported in Teerapabolarn [14, 15].

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Key Words: cumulative distribution function, Poisson approximation, relative error, Stein-Chen method, w-function associated with the binomial random variable

1. Introduction

Let a non-negative integer-valued random variable X have the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0,1)$ and probability function $p_X(x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, ..., n$, where $q = 1 - p$ and the mean and variance of X are $\mu = np$ and $\sigma^2 = npq$, respectively. In particular case, $n = 1$, the random variable is the Bernoulli random variable with parameter p. In addition, from the experimental point of view, the random variable X can be thought of as the number of successes in a sequence of n independent Bernoulli trials,

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[§]Correspondence author

where each trial results in the success or the failure with probabilities p and q. It is well-known that if the number of trials $n \to \infty$ and the probability of success $p \to 0$ while $\lambda = np$ remains a constant $(0 < \lambda < \infty)$ then $\binom{n}{x} p^x q^{n-x} \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x = 0, 1, ..., n$, which is a Poisson limit theorem. Therefore the Poisson distribution with mean $\lambda = np$ can be used as an approximation of the binomial distribution with parameters n and p when n is sufficiently large and p is sufficiently small. In the past, there have been a lot of studies related to Poisson approximation of binomial distribution. For example, in the case of pointwise approximation was examined by Anderson and Samuels [1], Feller [6] and Johnson et al. [8] also [3] and [2]. In the case of cumulative probability approximation, Anderson and Samuels [1] provided that

$$\mathbb{P}_\lambda(x_0) - \mathbb{B}_{n,p}(x_0) \begin{cases} > 0 & \text{if } \ x_0 \le \dfrac{\lambda n}{n+1}, \\ < 0 & \text{if } \ x_0 \ge \lambda, \end{cases} \tag{1.1}$$

where $\mathbb{P}_\lambda(x_0) = \sum_{k=0}^{x_0} \frac{e^{-\lambda}\lambda^k}{k!}$ and $\mathbb{B}_{n,p}(x_0) = \sum_{k=0}^{x_0} \binom{n}{k} p^k q^{n-k}$ are the Poisson and binomial cumulative distribution functions at $x_0 \in \{0, 1, ..., n\}$, respectively. Ivchenko [7] gave the asymptotic relation on the ratio of the binomial and Poisson cumulative distribution functions

$$\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} = 1 + o(1), \tag{1.2}$$

which is fulfilled uniformly in $x_0 < \lambda$. Very similar criteria for measuring the accuracy of Poisson approximation were used by Teerapabolarn [12], who applying the Stein-Chen method gave both non-uniform and uniform bounds for the relation error of the considered distributions. His results presented in [12] are as follows:

$$\left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \right| \le \frac{p(e^\lambda - 1)\Delta(x_0)}{x_0 + 1}, \ x_0 = 0, 1, ..., n, \tag{1.3}$$

where

$$\Delta(x_0) = \begin{cases} e^{-\lambda} q^{-n} & \text{if } x_0 < \lambda, \\ 1 & \text{if } x_0 \ge \lambda, \end{cases} \tag{1.4}$$

and he also gave a non-uniform bound for the another form of the relative error of two such cumulative distribution functions,

$$\left| \frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1 \right| \le \frac{(e^\lambda - 1)p}{x_0 + 1}, \ x_0 = 0, 1, ..., n. \tag{1.5}$$

and those from [13] are of the form:

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\frac{(1-e^{-\lambda})(1-q^n)}{nq^n} \tag{1.6}$$

and

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right|\le\frac{(e^{\lambda}-1)(1-q^n)}{n}. \tag{1.7}$$

The bounds presented above were then improved by the same author in [14, 15] to the sharper ones:

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| \le \max\left\{e^{-\lambda}q^{-n}-1,\frac{1-(1+\lambda)e^{-\lambda}}{nq^n}\min\left(1,\frac{2(1-q^n)}{\lambda}\right)\right\} \tag{1.8}$$

and

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right| \le \max\left\{1-e^{\lambda}q^n,\frac{e^{\lambda}-\lambda-1}{n}\min\left(1,\frac{2(1-q^n)}{\lambda}\right)\right\}, \tag{1.9}$$

for the uniform case and

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\frac{2(e^{\lambda}-\lambda-1)\Delta(x_0)}{n(x_0+1)},\ x_0=0,1,...,n \tag{1.10}$$

and

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right|\le\frac{2(e^{\lambda}-\lambda-1)}{n(x_0+1)},\ x_0=0,1,...,n \tag{1.11}$$

in the non-uniform one.

The aim of this article is further improvement of the latter bounds. It will be achieved by using the Stein-Chen method and the w-function associated with the binomial random variable and the obtained results presented in Sections 2 and 3. In Section 4, some numerical examples are provided to show the goodness of new bounds. Concluding remarks are presented in the last section.

2. Method

In this study, we use as our main tools the Stein-Chen method and the w-function associated with the binomial random variable.

2.1. The w-Function

The w-functions were studied by many authors, among others by Cacoullos and Papathanasiou [4], Papathanasiou and Utev [10], and Majsnerowska [9]. The following lemma presents another form of the w-function associated with the binomial random variable given in [9] and [10].

Lemma 2.1. *Let $w(X)$ be the w-function associated with the binomial random variable X, then*

$$w(x) = \frac{(n-x)p}{\sigma^2}, \ x = 0, 1, ..., n, \tag{2.1}$$

where $\sigma^2 = npq$.

The next relation stated by Cacoullos and Papathanasiou [4] is crucial for obtaining our main results.

If a non-negative integer-valued random variable Y has probability mass function $p_Y(y) > 0$ for every $y \in \mathcal{S}(Y)$, the support of Y, and $\sigma^2 = Var(Y)$ is finite, then

$$E[(Y-\mu)f(Y)] = \sigma^2 E[w(Y)\Delta f(Y)], \tag{2.2}$$

for any function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ for which $E|w(Y)\Delta f(Y)| < \infty$, where $\mu = E(Y)$ and $\Delta f(y) = f(y+1) - f(y)$.

2.2. The Stein-Chen Method

The classical Stein method introduced by Stein in [11] was developed for Poisson case by Chen [5]. The resulted version is referred to as the Stein-Chen method. Following Teerapabolarn [12], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda > 0$ is of the form

$$h_{x_0}(x) - \mathbb{P}_\lambda(x_0) = \lambda f_{x_0}(x+1) - x f_{x_0}(x), \tag{2.3}$$

where $x_0, x \in \mathbb{N} \cup \{0\}$ and function $h_{x_0} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is defined by

$$h_{x_0}(x) = \begin{cases} 1 & \text{if } \ x \le x_0, \\ 0 & \text{if } \ x > x_0 \end{cases}$$

and

$$f_{x_0}(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x-1)[1-\mathbb{P}_\lambda(x_0)]] & \text{if } x \le x_0, \\ (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(x-1)]] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.4}$$

Lemma 2.2. *For $x_0, x \in \mathbb{N}$, let $\check{\lambda} = \left\lfloor \frac{n\lambda}{n+1} \right\rfloor + 1$ and $\hat{\lambda} = \lceil \lambda \rceil$, we have*
1. For $x_0 > \frac{n\lambda}{n+1}$,

$$\sup_{x\ge 1} |\Delta f_{x_0}(x)| \le \frac{(x_0+2)\mathbb{P}_\lambda(x_0)}{(x_0+1)(x_0+2-\lambda)} \tag{2.5}$$

and

$$\sup_{x\ge 1} |\Delta f_{x_0}(x)| \le \frac{(\check{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\check{\lambda}+2-\lambda} \min\left\{\frac{1}{\check{\lambda}+1}, \frac{1}{x}\right\}. \tag{2.6}$$

2. For $x_0 \ge \lambda$,

$$\sup_{x\ge 1} |\Delta f_{x_0}(x)| \le \frac{(\hat{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\hat{\lambda}+2-\lambda} \min\left\{\frac{1}{\hat{\lambda}+1}, \frac{1}{x}\right\}. \tag{2.7}$$

Proof. 1. For $x \le x_0$, it follows from [15] that

$$\begin{aligned} \Delta f_{x_0}(x) &\le \frac{\mathbb{P}_\lambda(x_0)}{x_0+1}\left\{1 + \frac{\lambda}{x_0+2} + \frac{\lambda^2}{(x_0+2)(x_0+3)} + \cdots\right\} \\ &\le \frac{\mathbb{P}_\lambda(x_0)}{x_0+1}\left\{1 + \frac{\lambda}{x_0+2} + \left(\frac{\lambda}{x_0+2}\right)^2 + \cdots\right\} \\ &= \frac{(x_0+2)\mathbb{P}_\lambda(x_0)}{(x_0+1)(x_0+2-\lambda)}, \end{aligned} \tag{2.8}$$

which also implies that

$$\Delta f_{x_0}(x) \le \frac{(\check{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\check{\lambda}+2-\lambda} \min\left\{\frac{1}{\check{\lambda}+1}, \frac{1}{x}\right\}. \tag{2.9}$$

For $x > x_0$, by following [15],

$0 < -\Delta f_{x_0}(x)$

$$\leq \frac{\mathbb{P}_\lambda(x_0)}{x}\left\{\frac{1}{x+1}+\frac{2\lambda}{(x+1)(x+2)}+\frac{3\lambda^2}{(x+1)(x+2)(x+3)}+\cdots\right\} \quad (2.10)$$

$$\leq \frac{\mathbb{P}_\lambda(x_0)}{x_0+1}\left\{\frac{1}{x_0+2}+\frac{2\lambda}{(x_0+2)(x_0+3)}+\frac{3\lambda^2}{(x_0+2)(x_0+3)(x_0+4)}+\cdots\right\}$$

$$\leq \frac{\mathbb{P}_\lambda(x_0)}{x_0+1}\left\{1+\frac{\lambda}{x_0+2}+\left(\frac{\lambda}{x_0+2}\right)^2+\cdots\right\}$$

$$= \frac{(x_0+2)\mathbb{P}_\lambda(x_0)}{(x_0+1)(x_0+2-\lambda)}, \quad (2.11)$$

and by (2.10) and (2.11), we can obtain

$$-\Delta f_{x_0}(x) \leq \frac{(\check{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\check{\lambda}+2-\lambda}\min\left\{\frac{1}{\check{\lambda}+1}, \frac{1}{x}\right\}. \quad (2.12)$$

Hence, the inequality (2.5) is obtained from (2.8) and (2.11) and the inequality (2.6) follows from (2.9) and (2.12).

2. The arguments derived in the proof of (2.6) lead to the result in (2.7).□

The next lemma is obtained from Teerapabolarn [12].

Lemma 2.3. *For* $x_0 \in \{0, 1, ..., n\}$*, the following relation holds:*

$$\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} \leq e^{-\lambda}q^{-n}. \quad (2.13)$$

3. Main Results

We now present the main results of the study, i.e. new new non-uniform and uniform bounds on two forms of the relative error of the binomial and Poisson cumulative distribution functions.

Theorem 3.1. *For* $x_0 \in \{0, ..., n\}$*, the following inequality holds:*

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| \leq \begin{cases} e^{-\lambda}q^{-n}\min\left\{1-e^{\lambda}q^n, \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}\right\} & \text{if } x_0 \leq \frac{n\lambda}{n+1}, \\ \frac{\Delta(x_0)}{x_0+1}\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda-\lambda-1)}{n}\right\} & \text{if } x_0 > \frac{n\lambda}{n+1}, \end{cases} \quad (3.1)$$

where $\Delta(x_0)$ *is defined in* (1.4).

Proof. For $x_0 \leq \frac{n\lambda}{n+1}$, by combining the inequalities in (1.1) and (2.13), we have that $0 < \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \leq e^{-\lambda}q^{-n} - 1$ or $\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq e^{-\lambda}q^{-n} - 1$, which yields the first bound. For the second one, not that it is the result in (1.10) applied to considered x_0. Therefore, we obtain we obtain the first inequality of (3.1).

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq e^{-\lambda}q^{-n}\min\left\{1 - e^\lambda q^n, \frac{2(e^\lambda - \lambda - 1)}{n(x_0+1)}\right\}. \tag{3.2}$$

For $x_0 > \frac{n\lambda}{n+1}$, substituting x by X and taking expectation in (2.3) yields

$$\begin{aligned}\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0) &= E[\lambda f(X+1) - Xf(X)]\\ &= \lambda E[f(X+1)] - E[(X-\mu)f(X)] - \mu E[f(X)]\\ &= \lambda E[\Delta f(X)] - E[(X-\mu)f(X)],\end{aligned}$$

where $f = f_{x_0}$ is defined in (2.4). Because $E|w(X)\Delta f(X)| < \infty$, we have by (2.2),

$$\begin{aligned}|\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)| &= |\lambda E[\Delta f(X)] - \sigma^2\mathbb{E}[w(X)\Delta f(X)]|\\ &\leq E\{|\lambda - \sigma^2 w(X)||\Delta f(X)|\}\\ &= E\{|\lambda - (n-X)p||\Delta f(X)|\} \quad \text{(by Lemma 2.1)} \qquad (3.3)\\ &\leq \sup_{x\geq 1}|\Delta f(x)|\, E(X)p\\ &\leq \frac{(x_0+2)\mathbb{P}_\lambda(x_0)\lambda p}{(x_0+1)(x_0+2-\lambda)} \quad \text{(by (2.5))} \qquad (3.4)\end{aligned}$$

and dividing the inequality (3.4) by $\mathbb{B}_{n,p}(x_0)$, we obtain

$$\begin{aligned}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| &\leq \frac{(x_0+2)\mathbb{P}_\lambda(x_0)\lambda p}{(x_0+1)(x_0+2-\lambda)\mathbb{B}_{n,p}(x_0)}\\ &\leq \frac{(x_0+2)\lambda p\Delta(x_0)}{(x_0+1)(x_0+2-\lambda)} \quad \text{(by (1.1))}, \qquad (3.5)\end{aligned}$$

which gives the first bound. The second one follows immediately from (1.10), that is, $\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq \frac{2(e^\lambda-\lambda-1)\Delta(x_0)}{n(x_0+1)}$. Thus, we also obtain

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq \frac{\Delta(x_0)}{x_0+1}\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda - \lambda - 1)}{n}\right\}. \tag{3.6}$$

Hence, by (3.2) and (3.6), the inequality (3.1) holds. □

Corollary 3.1. *For $x_0 \in \{0,...,n\}$, then*

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \leq \begin{cases} \min\left\{1 - e^\lambda q^n, \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}\right\} & \text{if } x_0 \leq \frac{n\lambda}{n+1}, \\ \frac{1}{x_0+1}\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda-\lambda-1)}{n}\right\} & \text{if } x_0 > \frac{n\lambda}{n+1}. \end{cases} \tag{3.7}$$

Proof. If $x_0 \leq \frac{n\lambda}{n+1}$, using the same inequalities in (1.1) and (2.13), we have that $0 < 1 - \frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} \leq 1 - e^\lambda q^n$ or $\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \leq 1 - e^\lambda q^n$. For $\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \leq \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}$, which together with (1.11) gives the bounds in the first case of (3.7).

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \leq \min\left\{1 - e^\lambda q^n, \frac{2(e^\lambda - \lambda - 1)}{n(x_0+1)}\right\}. \tag{3.8}$$

For $x_0 > \frac{n\lambda}{n+1}$, i.e. for the second case of (3.7), the inequality follows from dividing the inequality (3.4) by $\mathbb{P}_\lambda(x_0)$ and using (1.11).

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \leq \frac{1}{x_0+1}\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda - \lambda - 1)}{n}\right\}. \tag{3.9}$$

Hence, the result in is obtained from (3.8) and (3.9). □

The following corollary is an immediately consequence of the Theorem 3.1 and Corollary 3.1.

Corollary 3.2. *We have the following relation:*

$$\sup_{0 \leq x_0 \leq \frac{n\lambda}{n+1}} \left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq e^{-\lambda} q^{-n} - 1 \tag{3.10}$$

and

$$\sup_{0 \leq x_0 \leq \frac{n\lambda}{n+1}} \left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \leq 1 - e^\lambda q^n. \tag{3.11}$$

Corollary 3.3. *The following inequalities hold:*

$$\sup_{\lambda \leq x_0 \leq n} \left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq \begin{cases} \frac{e^\lambda-\lambda-1}{n}\min\left\{1, \frac{2(1-q^n)}{\lambda}\right\} & \text{if } \lambda \leq 1, \\ \frac{(\hat{\lambda}+2)p}{\hat{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\hat{\lambda}+1}, 1 - q^n\right\} & \text{if } \lambda > 1, \end{cases} \tag{3.12}$$

and

$$\sup_{\lambda\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right|\le\begin{cases}\frac{e^\lambda-\lambda-1}{n}\min\left\{1,\frac{2(1-q^n)}{\lambda}\right\} & \text{if } \lambda\le 1,\\ \frac{(\hat{\lambda}+2)p}{\hat{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\hat{\lambda}+1},1-q^n\right\} & \text{if } \lambda>1.\end{cases}\tag{3.13}$$

Proof. For $x_0\ge\lambda$, the bound in (3.12) and (3.13) are the same bound. Then it suffices to show that (3.12) holds. For $\lambda\le 1$, Teerapabolarn [14] showed that

$$\sup_{\lambda\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\frac{e^\lambda-\lambda-1}{n}\min\left\{1,\frac{2(1-q^n)}{\lambda}\right\}.\tag{3.14}$$

For $\lambda>1$, from (3.3), we have

$$\begin{aligned}|\mathbb{B}_{n,p}(x_0)-\mathbb{P}_\lambda(x_0)|&\le\sum_{x=1}^{n}xp|\Delta f(x)|p_X(x)\\&\le\sum_{x=1}^{n}\frac{(\hat{\lambda}+2)\mathbb{P}_\lambda(x_0)xp_X(x)p}{\hat{\lambda}+2-\lambda}\min\left\{\frac{1}{\hat{\lambda}+1},\frac{1}{x}\right\}\ \text{(by (2.7))}\\&=\frac{(\hat{\lambda}+2)\mathbb{P}_\lambda(x_0)p}{\hat{\lambda}+2-\lambda}\sum_{x=1}^{n}xp_X(x)\min\left\{\frac{1}{\hat{\lambda}+1},\frac{1}{x}\right\}\\&=\frac{(\hat{\lambda}+2)\mathbb{P}_\lambda(x_0)p}{\hat{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\hat{\lambda}+1},1-q^n\right\}.\end{aligned}$$

Dividing the last inequality by $\mathbb{B}_{n,p}(x_0)$, we obtain

$$\sup_{\lambda\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\frac{(\hat{\lambda}+2)p}{\hat{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\hat{\lambda}+1},1-q^n\right\}.\tag{3.15}$$

Therefore, from (3.14) and (3.15), the inequality (3.12) holds. □

Theorem 3.2. *For* $\delta(\lambda)=\begin{cases}e^{-\lambda}q^{-n} & \text{if } \check{\lambda}<\hat{\lambda},\\ 1 & \text{if } \check{\lambda}=\hat{\lambda}.\end{cases}$, *we have*

1. If $\check{\lambda}=1$, *then*

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\max\left\{e^{-\lambda}q^{-n}-1,\frac{(e^\lambda-\lambda-1)\delta(\lambda)}{n}\times\min\left\{1,\frac{2(1-q^n)}{\lambda}\right\}\right\}.\tag{3.16}$$

2. If $\check{\lambda} > 1$, then

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| \le \max\left\{e^{-\lambda}q^{-n}-1, \frac{(\check{\lambda}+2)p\delta(\lambda)}{\check{\lambda}+2-\lambda} \times \min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}\right\}. \tag{3.17}$$

Proof. 1. The inequality (3.16) directly follows from the result in [14].

2. For $x_0 > \frac{n\lambda}{n+1}$, using (2.6) and the arguments derived in the proof of (3.12), we have

$$|\mathbb{B}_{n,p}(x_0)-\mathbb{P}_\lambda(x_0)| \le \frac{(\check{\lambda}+2)\mathbb{P}_\lambda(x_0)p}{\check{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}.$$

Dividing the inequality by $\mathbb{B}_{n,p}(x_0)$, we get

$$\begin{aligned}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| &\le \frac{(\check{\lambda}+2)p\mathbb{P}_\lambda(x_0)}{(\check{\lambda}+2-\lambda)\mathbb{B}_{n,p}(x_0)}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\} \\ &\le \frac{(\check{\lambda}+2)p\delta(\lambda)}{\check{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}\end{aligned}$$

which gives

$$\sup_{\frac{n\lambda}{n+1}<x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| \le \frac{(\check{\lambda}+2)p\delta(\lambda)}{\check{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}. \tag{3.18}$$

Therefore, by combining (3.10) and (3.18), we have (3.17). □

The following corollary is a consequence of the Theorem 3.2.

Corollary 3.4. *We have the following inequalities.*

1. If $\check{\lambda} = 1$, then

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right| \le \max\left\{1-e^{\lambda}q^n, \frac{e^{\lambda}-\lambda-1}{n}\min\left\{1, \frac{2(1-q^n)}{\lambda}\right\}\right\}.$$

2. If $\check{\lambda} > 1$, then

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right| \le \max\left\{1-e^{\lambda}q^n, \frac{(\check{\lambda}+2)p}{\check{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}\right\}.$$

Remark. 1. Let us consider the results in Theorem 3.1, Corollary 3.1, Theorem 3.2 and Corollary 3.4. Note that, if p or λ is small, then all bounds presented in the study approach zero. It indicates that the results in approximating the binomial cumulative distribution function by the Poisson cumulative distribution function are more accurate when p or λ is small.

2. Because

$\min\left\{1 - e^{\lambda}q^n, \frac{2(e^{\lambda}-\lambda-1)}{n(x_0+1)}\right\} \leq \frac{2(e^{\lambda}-\lambda-1)}{n(x_0+1)}$ for $x_0 \leq \frac{n\lambda}{n+1}$ and

$\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^{\lambda}-\lambda-1)}{n}\right\} \leq \frac{2(e^{\lambda}-\lambda-1)}{n}$ for $x_0 > \frac{n\lambda}{n+1}$ and

$\max\left\{1 - e^{\lambda}q^n, \frac{(\check{\lambda}+2)p}{\check{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}\right\}$

$\leq \max\left\{1 - e^{\lambda}q^n, \frac{e^{\lambda}-\lambda-1}{n}\min\left\{1, \frac{2(1-q^n)}{\lambda}\right\}\right\}$ for $\check{\lambda} > 1$.

Hence the bounds given in Theorem 3.1 and Corollary 3.1 are sharper than those in (1.10) and (1.11), and for $\check{\lambda} > 1$, the bounds in Theorem 3.2 and Corollary 3.4 are sharper than the bounds in (1.8) and (1.9).

4. Conclusion

In this study, the uniform and non-uniform bounds in Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.4 provide new general criteria for measuring the accuracy in approximating a binomial cumulative distribution with parameter n and p by the Poisson cumulative distribution with mean $\lambda = np$. With the bounds, it is pointed out that each result in the theorems and corollaries gives a good Poisson approximation when p is small, and all bounds obtained in this study are sharper than those reported in Teerapabolarn [14, 15].

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