International Journal of Pure and Applied Mathematics
Volume 95 No. 4 2014, 593-596
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v95i4.12
ijpam.eu

AMNM PROPERTY ON VARIATION SEQUENCE SPACES

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Abstract: In this paper, we will show that the spaces of p-bounded variation sequences are AMNM.

AMS Subject Classification: 47B37, 47A25
Key Words: bounded variation space, AMNM

1. Introduction

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ϕ, l_∞ and c_0 denote the set of all finite, bounded and null sequences. We write $l_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$, and $bv = \{x \in \omega : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty\}$ for the set of all sequences of bounded variation and extend the definition to real $p \geq 1$ by putting $bv_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p < \infty\}$ so that $bv_1 = bv$. The set bv_p also arise from the sets l_p, that is a sequence x is in bv_p, if and only if the sequence $(x_k - x_{k-1})_{k=0}^{\infty}$ is in l_p. It is this concept rather than the first one plays an important role in our studies.

By e and $e^{(n)}$ $(n \in N_0)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, 2, ...$ and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ whenever $k \neq n$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section. A sequence $(b^{(n)})_{n=0}^{\infty}$ in

Received: June 14, 2014

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a linear metric space X is called Schuader basis if, for every $x \in X$ there is a unique sequence (λ_n) of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b^{(n)}$. An FK space is a complete linear metric sequence space with the property that convergence implies coordinatewise convergence. A BK space is a normed FK space. An FK space X containing ϕ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is $x = \lim_{n\to\infty} x^{[n]}$.

If A is a Banach algebra, then the set of all linear functionals on A is denoted by A^* and the set of all its nonzero multiplicative functionals is denoted by $\hat{A}$. If $\varphi \in A^*$, then define $\check{\varphi}(a,b) = \varphi(ab) - \varphi(a)\varphi(b)$ for all $a, b \in A$. If $\delta \in \mathbb{R}^+$, we say that φ is $\delta-$multiplicative, whenever $\|\check{\varphi}\| \leq \delta$. Also for each $\varphi \in A^*$ define $d(\varphi) = \inf\{\|\varphi - \psi\| : \psi \in \hat{A} \cup \{0\}\}$. Also, we say that A is an algebra in which approximately multiplicative functionals are near multiplicative functionals, or A is $AMNM$ for short, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $d(\varphi) < \varepsilon$ whenever φ is a $\delta-$multiplicative linear functional.

B. E. Johnson has shown that various Banach algebras are $AMNM$ and some of them fail to be $AMNM$ ([1]). Also, this property is still unknown for some Banach algebras such as H^∞ and Douglas algebras. In this paper we will show that bv_p is $AMNM$. For this topics on this sources see [1–4].

2. Main Results

In this section we investigate bv_p as a Banach algebra that is $AMNM$. It has been proved that bv_p is a Banach space with BK property.

Lemma 2.1. *(i) If $x = (x_k)_{k=0}^{\infty} \in bv_p$, then x is a bounded sequence.*

(ii) If $x = (x_k)_{k=0}^{\infty}$, $y = (y_k)_{k=0}^{\infty} \in bv_p$, then $xy = (x_k y_k)_{k=0}^{\infty} \in bv_p$.

(iii) The space bv_p is a commutative Banach algebra.

Proof. (i) Since $x_k = x_0 + x_1 - x_0 + x_2 - x_1 + ... + x_k - x_{k-1}$ for all integers $k \geq 0$, we get

$$\begin{aligned} |x_k|^p &= |x_0 + (x_1 - x_0) + (x_2 - x_1) + ... + (x_k - x_{k-1})|^p \\ &\leq 2^p \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p = 2^p \|x\|_{bv_p}^p, \end{aligned}$$

Thus $\sup_k |x_k| < \infty$.

(ii) Note that

$$\begin{aligned}\sum_{k=0}^{\infty}|x_k y_k - x_{k-1}y_{k-1}|^p &= \sum_{k=0}^{\infty}|x_k(y_k-y_{k-1})+y_{k-1}(x_k-x_{k-1})|^p \\ &\leq \{\|x\|_\infty\|y\|_{bv_p}+\|y\|_\infty\|x\|_{bv_p}\}^p,\end{aligned}$$

So by part (i) it follows that $xy \in bv_p$.

(iii) Since bv_p is a BK space and convergence implies coordinate wise convergence, thus the right multiplications $x \mapsto yx$ and the left multiplications $x \mapsto xy$ are continuous. Now, clearly $\|xy\|_{bv_p} \leq \|x\|_{bv_p}\|y\|_{bv_p}$. □

Lemma 2.2. (see [4]) *Define the sequence* $b^{(k)} = \{b_n^{(k)}\}_{k\in\mathbb{N}}$ *of the elements of the space* bv_p*, by* $b_n^{(k)} = 1$ *for all* $n \geq k$ *and otherwise* 0*. Hence the sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* bv_p *and any* $x \in bv_p$ *has a unique representation of the form* $x = \sum_k \lambda_k b^{(k)}$*, where* $\lambda_k = x_k - x_{k-1}$ *for all* $k \in \mathbb{N}$*.*

Definition 2.3. Let $1 < q < \infty$. Define the space d_q consisting of all sequences $a = (a_k)_{k=0}^{\infty}$ normed by $\|a\|_{d_q} = (\sum_k |\sum_{j=k}^{\infty} a_j|^q)^{\frac{1}{q}} < \infty$.

Theorem 2.4. *The space* bv_p^* *is isometrically isomorphic to* d_q*.*

Proof. Define the transformation $A : bv_p^* \to d_q$ by $f \mapsto Af = (f_0, f_1, f_2, ...)$ where $f_j = f(e^{(j)})$ for each $j \in \mathbb{N}$. Therefore we obtain that $\|f\| = (\sum_{k=0}^{\infty}|g_k|^q)^{\frac{1}{q}}$ where $g_k = \sum_{j=k}^{\infty} f_j$ for each $k \in \mathbb{N}$. Hence A is an isometric isomorphism. □

Theorem 2.5. *If* $1 < p < \infty$*, then* bv_p *is AMNM.*

Proof. Let $1 < p < \infty$ and $\varepsilon \in (0,1)$, set $\delta = \frac{\varepsilon^2}{16}$ and consider $f \in bv_p* = d_q$ with $\|\check{f}\| < \delta$. If $\|f\| \leq \varepsilon$, then $d(f) \leq \varepsilon$. Therefore, suppose that $\|f\| > \varepsilon$. For each subset E of $\mathbb{N}_0$, let $\alpha(E) = [\sum_{k\in E}|\sum_{j=k}^{\infty} f_j|^q]^{\frac{1}{q}}$. For any subsets E and F of $\mathbb{N}_0$, if $E \cap F = \phi$, then either $\alpha(E) \leq \frac{\varepsilon}{4}$ or $\alpha(F) \leq \frac{\varepsilon}{4}$. So if $E \subseteq \mathbb{N}_0$, we have that either $\alpha(E) \leq \frac{\varepsilon}{4}$ or $\alpha(E) \geq \frac{3\varepsilon}{4}$. Note that, if $E, F \subseteq \mathbb{N}_0$ with $\alpha(E) \leq \frac{\varepsilon}{4}$ and $\alpha(F) \leq \frac{\varepsilon}{4}$, then $\alpha(E \bigcup F) \leq \frac{\varepsilon}{4}$. Since $\|f\| > \varepsilon$, there exists a positive integer m such that $\alpha(m) \geq \frac{3\varepsilon}{4}$, then $\alpha(\mathbb{N}) \leq \frac{\varepsilon}{4}$. Let $b^{(m)}$ be the $m-$th element of a shuader basis, $(b^{(k)})_{k=0}^{\infty}$. We have $|\check{f}(b^{(m)}, b^{(m)})| = |f(b^{(m)}.b^{(m)}) - f(b^{(m)}).f(b^{(m)})| = |f(b^{(m)}) - f(b^{(m)})^2| = |g_m - g_m^2|$, where $g_m = \sum_{j=m}^{\infty} f_j$. Since $\|\check{f}\| < \delta$, we get $|g_m - g_m^2| < \delta$. So either $|g_m| \leq 2\delta$ or $|1 - g_m| \leq 2\delta$. But $\alpha(m) = |g_m| \geq \frac{3\varepsilon}{4}$, thus $|1 - g_m| \leq 2\delta$. Now if ψ is the evaluation functional defined by $\psi(a) = a_m$, then $\|f - \psi\|_{bv_p^*} = [\alpha(\mathbb{N})^q + |g_m - 1|^q]^{\frac{1}{q}} \leq \frac{\varepsilon}{4} + 2\delta < \varepsilon$. Thus, indeed $d(f) < \varepsilon$ and the proof is complete. □

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