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STRUCTURE OF NEAR-RINGS SATISFYING CERTAIN POLYNOMIAL IDENTITIES

Abdelkarim Boua[1 §], Ahmed A.M. Kamal[2]

[1]Department of Mathematics
Faculty of Sciences of Agadir
Ibn Zohr University
P.O. Box 8106, Agadir, MOROCCO

[2]Department of Mathematics
Faculty of Sciences
Cairo University, Giza, EGYPT

[2]King Saud University
College of Science
Department of Mathematics
P.O. Box 2455, Riyadh, 11451, KINGDOM OF SAUDI ARABIA

Abstract: In this paper, we will introduce the concept of two-sided α-generalized derivation in prime near-rings as it was outlined by the author N. Argac in [1]. Thereafter, we will generalize the same results proved by many authors (see [2], [4] and [5]) in the case of derivations, semiderivations and generalized derivations. Furthermore, we will give examples to demonstrate that the restrictions imposed on the hypothesis of various results are not superfluous.

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[§]Correspondence author

1. Introduction

A right near-ring is a set N with two operations $+$ and $.$ such that $(N,+)$ is a group (not necessarily abelian) and $(N,.)$ is a semigroup satisfying the right distributive law $(x+y)z = x.z + y.z$ for all $x, y, z \in N$. A right near-ring N is called zero symmetric if $x.0 = 0$ for all $x \in N$ (recalling that A right distributivity yields $0.x = 0$. A near-ring N is said to be prime if for all $x, y \in N$, $xNy = \{0\}$ implies $x = 0$ or $y = 0$. A nonempty subset I of N will be called a semigroup ideal if $IN \subseteq I$ and $NI \subseteq I$. A nonempty subset I of N is called stable by the additive law, if for any $x, y \in I$, $x + y \in I$ An additive mapping $\sigma : N \to N$ is called an involution if $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for all $x, y \in N$. An additive mapping $d : N \longrightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$. An additive mapping $F : N \longrightarrow N$ is said to be a generalized derivation on N if there exists a derivation $d : N \longrightarrow N$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in N$. An additive mapping $d : N \to N$ is called an (α, β)-derivation if there exist functions $\alpha, \beta : N \to N$ such that $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in N$. An additive mapping $d : N \to N$ is called a two-sided α-derivation if d is an $(\alpha, 1)$-derivation as well as $(1, \alpha)$-derivation.

Now we introduce the notion of two-sided α-generalized derivation of a near-ring N as follows. An additive mapping $F : N \to N$ is called a (α, β)-generalized derivation if there exist functions $\alpha, \beta : N \to N$ and a (α, β)-derivation such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in N$. An additive mapping $F : N \to N$ is called a two-sided α-derivation if F is an $(\alpha, 1)$-generalized derivation as well as $(1, \alpha)$-generalized derivation. For $\alpha = 1$, a two-sided α-derivation (resp. two-sided α-generalized derivation) is of course just a derivation (resp. generalized derivation). Recently, many authors have studied commutativity of prime and semiprime near-rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations, and generalized derivations acting on appropriate subsets of the near-rings. In the present paper, we would like to study the structure of a prime near-ring having a $(\sigma, 1)$-generalized derivation where σ is an involution or a two-sided α-generalized derivation where α is an homomorphism which satisfies suitable algebraic properties on semigroup ideals of N.

2. Some Preliminaries

We begin with several lemmas, most of which are known. Those for which neither a proof nor a precise citation is given are to be found in [3], [4] and [6]. Unless it is stated otherwise, it will be assumed that N is a zero-symmetric near ring and d is a two-sided α-derivation of N associated with an endomorphism α.

Lemma 1. [3, Lemma 1.3(1), Lemma 1.5 and Lemma 1.4(3)] *Let N be a prime near-ring and I be a nonzero semigroup ideal of N.*

(1) *If $Ia = \{0\}$ and $a \in N$, then $a = 0$.*

(2) *If $Z(N)$ contains a nonzero semigroup ideal I, then N is a commutative ring.*

(3) *If $x, y \in N$ and $xIy = \{0\}$, then $x = 0$ or $y = 0$.*

Lemma 2. [6, Theorem 2] *Let N be a prime near-ring and I be a nonzero semigroup ideal of N. If I stable by the additive law and admits an involution σ, then N is a ring.*

Lemma 3. *Let N be a prime near-ring. If N admits a two-sided α-derivation d, then $d(x)\alpha(y) + xd(y) = xd(y) + d(x)\alpha(y)$ for all $x, y \in N$.*

Proof. The proof is the same of the proof of Proposition 1 of [7]. □

Lemma 4. *Let N be a prime near-ring and d is a two-sided α-derivation where α is an homomorphism, then*

(1) *$x(d(y)\alpha(z) + yd(z)) = xd(y)\alpha(z) + xyd(z)$ for all $x, y, z \in N$.*

(2) *If $ad(x) = 0$ for all $x \in N$ and $a \in N$, then $a = 0$ or $d = 0$.*

Proof. (1) From the sample calculation of $d((xy)z) = d(x(yz))$, we obtain the required result.
(2) Assume that $ad(x) = 0$ for $a \in N$ and for all $x \in N$. According to lemma 3, we have

$$\begin{aligned} 0 &= ad(xy) \\ &= a(d(x)\alpha(y) + xd(y)) \\ &= ad(x)\alpha(y) + axd(y) \\ &= axd(y) \text{ for all } x, y \in N. \end{aligned}$$

Hence, $aNd(y) = \{0\}$ for all $x, y \in N$. By primeness of N, we conclude that $a = 0$ or $d = 0$, whereby the lemma is proved. □

The following lemma is very useful in the sequel and it is a special case of Theorem 1 and Theorem 3 of [4].

Lemma 5. *Let N be a prime near-ring admitting a generalized derivation F associated with a nonzero derivation d.*

(1) [4, Theorem 1] *If $F([x, y]) = 0$ for all $x, y \in N$, then N is a commutative ring.*

(2) *If $F(x \circ y) = 0$ for all $x, y \in N$, then N is a commutative ring of characteristic 2.*

Proof. (2) The proof of N is a commutative ring is the same as of the proof of Theorem 3 of [4]. For all $x, y \in N$, $F(x \circ y) = 0$ implies that $F(xy)+F(xy) = 0$ for all $x, y \in N$. Replacing y by yz, $z \in N$, we get

$$\begin{aligned} 0 &= F(xyz) + F(xyz) \\ &= F(xy)z + xyd(z) + F(xy)z + xyd(z) \\ &= (F(xy) + F(xy))z + (2x)yd(z) \\ &= (2x)yd(z) \ \text{ for all } \ x, y, z \in N. \end{aligned}$$

The primeness implies that $2N = \{0\}$. □

Lemma 6. [3, Lemma 1.2(i), (ii)] *Let N be a prime near-ring and $z \in Z(N) - \{0\}$. Then z is not a zero divisor. Moreover, if $z + z \in Z(N)$, then $(N, +)$ is abelian.*

We conclude with the following useful remark.

Remark 1. Let N be a prime near-ring admitting a two-sided α-generalized derivation F associated with a two-sided α-derivation d. If $d \neq 0$, then $F \neq 0$. So it is sufficient to write $d \neq 0$ for both cases ($d \neq 0$ and $F \neq 0$).

3. Some Results Involving $(\sigma, 1)$-Generalized Derivations

This Section is devoted to studying the structure of a zero symmetric prime right near-ring N admitting a nonzero $(\sigma, 1)$-generalized derivation F associated with a $(\sigma, 1)$-derivation d where σ is an involution. More precisely, we will prove the following.

Theorem 1. *Let N be a prime near-ring and I be a nonzero semigroup ideal of N stable by the additive law. If I admits a nonzero $(\sigma, 1)$-generalized derivation where σ is an involution of I, then N is a commutative ring.*

Proof. By the hypothesis given and by Lemma 2, we get N is a ring. So

$$\begin{aligned} d(x(yz)) &= d(x)\sigma(yz) + xd(yz) \\ &= d(x)\sigma(z)\sigma(y) + xd(y)\sigma(z) + xyd(z) \quad \text{for all } x, y, z \in I. \end{aligned}$$

In another way

$$\begin{aligned} d((xy)z) &= d(xy)\sigma(z) + xyd(z) \\ &= (d(x)\sigma(y) + xd(y))\sigma(z) + xyd(z) \\ &= d(x)\sigma(y)\sigma(z) + xd(y)\sigma(z) + xyd(z) \quad \text{for all } x, y, z \in I. \end{aligned}$$

Comparing the two last expressions, we obtain

$$d(x)\sigma(y)\sigma(z) = d(x)\sigma(z)\sigma(y) \quad \text{for all } x, y, z \in I. \tag{1}$$

By using N is a ring, the equation (1) can be rewritten as

$$d(x)[\sigma(y), \sigma(z)] = 0 \quad \text{for all } x, y, z \in I. \tag{2}$$

By definition of F and the same techniques introduced previously, we arrive at

$$F(x)[\sigma(y), \sigma(z)] = 0 \quad \text{for all } x, y, z \in I. \tag{3}$$

Taking tx instead of x in (3), we have

$$F(t)\sigma(x)[\sigma(y), \sigma(z)] + td(x)[\sigma(y), \sigma(z)] = 0 \quad \text{for all } x, y, z, t \in I. \tag{4}$$

Using (2), then (4) becomes

$$F(t)\sigma(x)[\sigma(y), \sigma(z)] = 0 \quad \text{for all } x, y, z, t \in I. \tag{5}$$

Applying σ again, we get

$$[z, y]x\sigma(F(t)) = 0 \quad \text{for all } x, y, z, t \in I. \tag{6}$$

This means that

$$[z, y]I\sigma(F(t)) = \{0\} \quad \text{for all } y, z, t \in I.$$

Using Lemma 1(3), σ is an involution and $F \neq 0$, the above expression becomes $[z, y] = 0$ for all $y, z \in I$. Substituting yt for y where $t \in N$. The upshot is that $y[z, t] = 0$ for all $y, z \in I, t \in N$, which implies that $I[z, t] = 0$ for all $z \in I, t \in N$. From Lemma 1(1) and Lemma 1(2), we conclude that N is a commutative ring. □

Corollary 1. *Let N be a prime near-ring. If N admits a nonzero $(\sigma, 1)$-generalized derivation where σ is an involution of N, then N is a commutative ring.*

Corollary 2. *Let N be a prime near-ring and σ is an involution of N. If N admits a nonzero $(\sigma, 1)$-derivation, then N is a commutative ring.*

Example 1. Let N be the noncommutative prime ring $M_2(\mathbb{Z}_2)$ and d be the inner derivation induced by the element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Clearly that d is a nonzero $(1, 1)$-derivation associated with $\sigma = 1$. Therefore, the condition "σ is an involution" in Theorem 1 is not superfluous.

The following example shows the necessity of the primeness in Theorem 1.

Example 2. Let S be a non-abelian right near-ring. We define N, I and $\sigma : N \to N$ by:

$$N = I = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}, \; \sigma \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$$

$$F \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \; d \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that N is not prime near-ring and σ is an involution of N. Also, it is clear that F is a $(\sigma, 1)$-generalized derivation associated with a $(\sigma, 1)$-derivation d, but N is not a ring.

4. Two-Sided α-Generalized Derivations Acting on Prime Near-Rings

In this section, F will denote a two-sided α-generalized derivation associated with a nonzero two-sided α-derivation d, where α is an endomorphism of N. The main purpose of this paragraph is to generalize several results du to some authors (see [2], [4] and [5]) concerning the study of commutativity of prime right near-ring satisfying certain differential identities. For more details see the following results.

Theorem 2. *Let N be a prime near-ring. If N admits a two-sided α-generalized derivation F associated with a nonzero two-sided α-derivation d, where α is an endomorphism of N, then the following statements are equivalent:*

(1) $F([u,v]) = 0$ *for all* $u, v \in N$,

(2) N *is a commutative ring.*

Proof. It is obvious that (2) implies (1).
(1) $\Rightarrow$ (2) Suppose that

$$F([u,v]) = 0 \ \text{ for all } \ u, v \in N. \tag{7}$$

By definition of F, we get

$$F(u)v + \alpha(u)d(v) = F(u)\alpha(v) + ud(v) \ \text{ for all } \ u, v \in N. \tag{8}$$

Replacing u by $[x,y]$ in (8) and applying (7), we obtain

$$\alpha([x,y])d(v) = [x,y]d(v) \ \text{ for all } \ x, y, v \in N,$$

or equivalently,

$$\big(\alpha([x,y]) - [x,y]\big)d(v) = 0 \ \text{ for all } \ x, y, v \in N. \tag{9}$$

Since $d \neq 0$, according to Lemma 4(2), (9) implies that

$$\alpha([x,y]) = [x,y] \ \text{ for all } \ x, y \in N. \tag{10}$$

Taking $[x,y]$ instead of v in (8) and invoking (10), we arrive at

$$\alpha(u)d([x,y]) = ud([x,y]) \ \text{ for all } \ x, y, u \in N. \tag{11}$$

Since α is an endomorphism of N. Replacing u by ut in (11) and using the above argument, we arrive at

$$\alpha(u)td([x,y]) = utd([x,y]) \ \text{ for all } \ x, t, y, u \in N$$

which leads to

$$\big(\alpha(u) - u\big)Nd([x,y]) = \{0\} \ \text{ for all } \ x, y, u \in N. \tag{12}$$

Since N is prime, then (12) implies

$$\alpha = id_N \ \text{ or } \ d([x,y]) = 0 \ \text{ for all } \ x, y \in N. \tag{13}$$

(a) If $\alpha = id_N$, then F becomes a generalized derivation, in this case, Lemma 5(1) assures that N is a commutative ring.
(b) If $d([u,v]) = 0$ for all $u, v \in N$. Replacing u by uv, it is obvious to see that

$$\begin{aligned} 0 &= d([uv,v]) \\ &= d([u,v]v) \\ &= d([u,v])\alpha(v) + [u,v]d(v) \\ &= [u,v]d(v) \text{ for all } u,v \in N \end{aligned}$$

Thus,

$$[u,v]d(v) = 0 \text{ for all } u,v \in N \tag{14}$$

substituting ur for u in (14) and using (14), we arrive at

$$[u,v]rd(v) = 0 \text{ for all } r,u,v \in N$$

which implies that

$$[u,v]Nd(v) = \{0\} \text{ for all } u,v \in N$$

By 3-primeness of N, we find that

$$v \in Z(N) \text{ or } d(v) = 0 \text{ for all } v \in N. \tag{15}$$

If there is an element $v_0 \in N$ such that $v_0 \in Z(N)$, then by Lemma 3, we have

$$\begin{aligned} d(v_0)\alpha(u) + v_0 d(u) &= d(v_0 u) \\ &= d(uv_0) \\ &= \alpha(u)d(v_0) + d(u)v_0 \text{ for all } u \in N, \end{aligned}$$

the upshot is reduced to $d(v_0)\alpha(u) = \alpha(u)d(v_0)$ for all $u \in N$. On the other hand, if $v \notin Z(N)$, then equation (15) implies $d(v) = 0$ which shows that $d(v)\alpha(u) = \alpha(u)d(v)$ for all $u \in N$. So we have the following equation

$$d(v)\alpha(u) = \alpha(u)d(v) \text{ for all } u,v \in N. \tag{16}$$

Since

$$\begin{aligned} d(u)v + \alpha(u)d(v) &= d(uv) \\ &= d(vu) \\ &= vd(u) + d(v)\alpha(u) \text{ for all } u,v \in N \end{aligned}$$

by (16), the above expression gives

$$d(u)v = vd(u) \text{ for all } u, v \in N. \tag{17}$$

Replacing u by $d(u)t$ in (17) and by application of (17), we arrive at

$$d^2(u)\alpha(t)v = vd^2(u)\alpha(t) \text{ for all } u, v, t \in N.$$

Expanding the last equation and using (17) again, we get

$$[\alpha(t), v]Nd^2(u) = \{0\} \text{ for all } u, v, t \in N. \tag{18}$$

Since N is prime, then (18) implies that

$$d^2(u) = 0 \text{ or } \alpha(t) \in Z(N) \text{ for all } u, t \in N. \tag{19}$$

(1) If $d^2(u) = 0$ for all $u \in N$. By definition of d, we have

$$d(u)v + \alpha(u)d(v) = d(u)\alpha(v) + ud(v) \text{ for all } u, v \in N. \tag{20}$$

Replace u by $d(u)$ in (20), to get

$$\alpha(d(u))d(v) = d(u)d(v) \text{ for all } u, v \in N$$

which reduces to

$$(\alpha(d(u)) - d(u))d(v) = 0 \text{ for all } u, v \in N. \tag{21}$$

Since $d \neq 0$, by application of Lemma 4(2), (21) gives $\alpha(d(u)) = d(u)$ for all $u \in N$. In this case, putting uv instead of u, we have

$$\alpha(d(u)v + \alpha(u)d(v)) = d(u)\alpha(v) + ud(v) \text{ for all } u, v \in N$$

this means that,

$$d(u)\alpha(v) + \alpha^2(u)d(v) = d(u)\alpha(v) + ud(v) \text{ for all } u, v \in N.$$

And therefore,

$$(\alpha^2(u) - u)d(v) = 0 \text{ for all } u, v \in N. \tag{22}$$

By Lemma 4(2) and $d \neq 0$, we obtain $\alpha^2 = id_N$. In this case, replacing u by $x\alpha(t)$ and v by x in (17), we get

$$d(x)tx = xd(x)t \text{ for all } x, t \in N$$

which implies that

$$[x,t]Nd(x) = \{0\} \ \text{ for all } \ x,t \in N.$$

In view of the primeness of N, the above expression yields that

$$d(x) = 0 \text{ or } x \in Z(N) \ \text{ for all } \ x \in N. \tag{23}$$

If there exists an element $x_0 \in N$ such that $x_0 \in Z(N)$, then replacing u by $x_0\alpha(t)$ in (17), we arrive at

$$[v,t]Nd(x_0) = \{0\} \ \text{ for all } \ v,t \in N.$$

By the primeness of N, the last expression gives

$$d(x_0) = 0 \text{ or } v \in Z(N) \ \text{ for all } \ v \in N. \tag{24}$$

Since $d \neq 0$, by (23) and (24) we arrive at $N \subseteq Z(N)$, according to Lemma 1 (2), we conclude that N is a commutative ring.
(2) If $\alpha(t) \in Z(N)$. Substituting ut for u in (17), we obtain

$$d(u)tv + \alpha(u)d(t)v = vd(u)t + v\alpha(u)d(t) \ \text{ for all } \ u,v,t \in N$$

which is reduced to

$$[v,t]Nd(u) = \{0\} \ \text{ for all } \ u,v,t \in N.$$

By the primeness of N and $d \neq 0$, we arrive at $N \subseteq Z(N)$ and by Lemma 1(2), we find that N is a commutative ring, which complete the required proof. □

If we replace the product $[x,y]$ by $x \circ y$ in Theorem 2, then N is a commutative ring of characteristic 2. In fact, we obtain the following result:

Theorem 3. *Let N be a prime near-ring. If N admits a two-sided α-generalized derivation F associated with a nonzero two-sided α-derivation d, where α is a surjective endomorphism of N, such that $F(u \circ v) = 0$ for all $u,v \in N$, then N is a commutative ring of characteristic 2.*

Proof. Assume that

$$F(u \circ v) = 0 \ \text{ for all } \ u,v \in N. \tag{25}$$

Replacing u by $x \circ y$ in (8) and from (25) it follows that

$$\big(\alpha(x \circ y) - (x \circ y)\big)d(v) = 0 \ \text{ for all } \ x,y,v \in N. \tag{26}$$

By Lemma 4(2) and $d \neq 0$, we get

$$\alpha(x \circ y) = x \circ y \text{ for all } x, y \in N. \tag{27}$$

Substituting $x \circ y$ for v in (18) and using (27), we obtain

$$\alpha(u)d(x \circ y) = ud(x \circ y) \text{ for all } x, y, u \in N. \tag{28}$$

Taking ut instead of x in (28) and by application of (28), we find that

$$\big(\alpha(u) - u\big)Nd(x \circ y) = \{0\} \text{ for all } x, y, u \in N \tag{29}$$

which, by virtue of the primeness of N, proves that

$$\alpha = id_N \text{ or } d(x \circ y) = 0 \text{ for all } x, y \in N.$$

(a) If $\alpha = id_N$, then by Lemma 5(2) we get N is a commutative ring of characteristic 2.
(b) If $d(x \circ y) = 0$ for all $x, y \in N$. Using the same techniques as we have introduced in the proof of precedent Theorem in the case where $[u, v]$ is replaced by $u \circ v$, one can easily see that

$$uvd(v) = -vud(v) \text{ for all } u, v \in N. \tag{30}$$

Taking ut instead of u in (30) and using (30), we find that

$$\big(-vu + uv\big)Nd(-v) = \{0\} \text{ for all } u, v \in N. \tag{31}$$

By the primeness on N, we get

$$v \in Z(N) \text{ or } d(v) = 0 \text{ for all } v \in N. \tag{32}$$

(1) If there is an element $v \in N$ such that $v \in Z(N)$, then $d(vu) = d(uv)$ for all $u \in N$ which implies by using Lemma 3 that

$$\begin{aligned} d(v)\alpha(u) + vd(u) &= d(u)v + \alpha(u)d(v) \\ &= \alpha(u)d(v) + d(u)v \text{ for all } u \in N. \end{aligned}$$

Thus, $v \in Z(N)$ implies $d(v)\alpha(u) = \alpha(u)d(v)$ for all $u \in N$. α is surjective implies $d(v) \in Z(N)$. Therefore, equation (32) becomes $d(v) \in Z(N)$ for all $v \in N$ which means that $d(N) \subseteq Z(N)$. $d \neq 0$ implies the existence of an element $a \in N$ such that $d(a)$ is not a zero divisor element in N by Lemma 6. Using d is $(\alpha, 1)$-derivation and Lemma 3, we have

$$d(xy) = d(x)\alpha(y) + xd(y)$$

$$= \quad xd(y) + d(x)\alpha(y) \text{ for all } x, y \in N.$$

Multiplying $d(xy)$ by $\alpha(y)$ in the right and the left respectively and using $d(N) \subseteq Z(N)$, we get

$$\begin{aligned} d(xy)\alpha(y) &= (xd(y) + d(x)\alpha(y))\alpha(y) \\ &= xd(y)\alpha(y) + d(x)\alpha(y)\alpha(y) \\ &= x\alpha(y)d(y) + \alpha(y)\alpha(y)d(x). \end{aligned}$$

By using Lemma 3 and Lemma 4(1),

$$\begin{aligned} \alpha(y)d(xy) &= \alpha(y)(xd(y) + d(x)\alpha(y)) \\ &= \alpha(y)xd(y) + \alpha(y)d(x)\alpha(y) \\ &= \alpha(y)xd(y) + \alpha(y)\alpha(y)d(x). \end{aligned}$$

So $x\alpha(y)d(y) = \alpha(y)xd(y)$, which means that $(x\alpha(y) - \alpha(y)x)d(y) = 0$ for all $x, y \in N$. Since $d(a)$ is not a right zero divisor in N, we get

$$x\alpha(a) = \alpha(a)x \text{ for all } x \in N \tag{33}$$

Again, multiplying $d(xy)$ by $\alpha(a)$ in the right and the left respectively and using equation (33), we have $(\alpha(a)\alpha(y) - \alpha(y)\alpha(a))d(x) = 0$ for all $x, y \in N$. Using $d(a)$ is not a right zero divisor in N, we get

$$\alpha(a)\alpha(y) = \alpha(y)\alpha(a) \text{ for all } y \in N. \tag{34}$$

Now, multiplying $d(xa)$ by $\alpha(z)$ in the right and the left respectively and using equation (34), it follows that

$$x\alpha(z) = \alpha(z)x \text{ for all } x, z \in N. \tag{35}$$

Since α is surjective, we conclude that N is a commutative near-ring. We have $0 \neq d(a) \in Z(N)$ and $d(a) + d(a) = d(a + a) \in Z(N)$. Thus, $(N, +)$ is an abelian group by Lemma 6. Therefore, N is a commutative ring.
The proof of N is of characteristic 2 is the same as the proof of Lemma 5(2). □

Example 3. Let $N = M_2(\mathbb{Z}_5)$ and d be the inner derivation induced by the element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then N is a non-commutative prime ring and $d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}$. Let $x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $[x, y] = x \circ y =$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $d([x,y]) = d(x \circ y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If we take $\alpha = id_N$ the identity automorphism of N, then d is a nonzero two-sided α-generalized derivation associated with a nonzero two-sided α-derivation d. Therefore, this example shows that the condition "$F([u,v]) = 0$ for all $u, v \in N$" in Theorem 2 and the condition "$F(u \circ v) = 0$ for all $u, v \in N$" in Theorem 3 are not superfluous.

The following example shows that the primeness hypothesis in Theorem 2 and Theorem 3 cannot be omitted.

Example 4. Let S be a non-abelian right near-ring and

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}$$

Defining the maps $F, d, \alpha : N \to N$ as the following:

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & 2x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is clear that N is not prime right near-ring admitting a non-zero two-sided α-generalized derivation F associated with a nonzero two-sided α-derivation. Moreover, it is easy to verify that F satisfy the properties:

(i) $F[A,B] = 0$ **(ii)** $F(A \circ B) = 0$ for all $A, B \in N$.

However, N is not a commutative ring.

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