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STRONGLY SEMICOMMUTATIVE RINGS RELATIVE TO A MONOID

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Abstract: For a monoid M, we introduce strongly semicommutative rings relative to M, which are a generalization of strongly semicommutative rings, and investigates its properties. We show that every reduced ring is strongly M-semicommutative for any unique product monoid M. Also it is shown that for a monoid M and an ideal I of R. If I is a reduced ring and R/I is strongly M-semicommutative, then R is strongly M-semicommutative.

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1. Introduction

Throughout this article, R and M denote an associative ring with identity and a monoid, respectively. Recall that a ring is reduced if it has no nonzero nilpotent elements. Lambek [9] called a ring R symmetric provided $abc = 0$ implies

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$acb = 0$ for $a, b, c \in R$. Habeb [10] called a ring R zero commutative if R satisfies the condition: $ab = 0$ implies $ba = 0$ for $a, b \in R$, while Cohn [14] used the term reversible for what is called zero commutative. A generalization of a reversible ring is a semicommutative ring. A ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Historically, some of the earliest results known to us about semicommutative rings (although not so called at the time) was due to Shin [6]. He proved that (i) R is semicommutative if and only if $r_R(a)$ is an ideal of R where $r_R(a) = \{b \in R \mid ab = 0\}$ [6, Lemma 1.2], (ii) every reduced ring is symmetric [6, Lemma 1.1] (but the converse does not hold [3, Example II.5]), and (iii) any symmetric ring is semicommutative but the converse does not hold ([6, Proposition 1.4 and Example 5.4(a)]). Semicommutative rings were also studied under the name zero insertive by Habeb [10]. In [12], Kim and Lee showed that polynomial rings over reversible rings need not be reversible. In [7], Yang and Liu introduced the notation of strongly reversible. A ring R is called strongly reversible, if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. All reduced rings are strongly reversible but converse is not true. Another generalization of a reduced ring is an Armendariz ring. Rege and Chhawchharia [11] called a ring R Armendariz if whenever any polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j.(The converse is always true). In [19], Z. Liu studied a generalization of Armendariz rings, which is called M-Armendariz rings, where M is monoid. A ring R is called M-Armendariz if whenever $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n, \beta = b_1h_1 + b_2h_2 + \cdots + b_mg_m \in R[M]$, satisfy $\alpha\beta = 0$, then $a_ib_j = 0$, for each i, j,where $g_i, h_j \in M$. In [1], a ring R is called strongly M-reversible, whenever $\alpha\beta = 0$ implies $\beta\alpha = 0$, where $\alpha, \beta \in R[M]$. Properties, examples and counterexamples of semicommutative rings were given in Huh, Lee and Smoktunowicz [2], Kim and Lee [12], Liu [20] and Yang [8]. In [18, Corollary 2.3], it was claimed that all semicommutative rings are McCoy. However, Hiranos claim assumed that if R is semicommutative then $R[x]$ is semicommutative, but this was later shown to be false in [2, Example 2]. Moreover, Nielsen [15], gave an example to show that a semicommutative ring R need not be right McCoy, also he proved that the polynomial ring $R[x]$ over it actually is not semicommutative. In [17], a ring R is called strongly semicommutative if whenever polynomials $f(x), g(x)$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)R[x]g(x) = 0$.

Recall that a monoid M is called a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoid is quite large and important (see [5, 4]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has no nonunity element of finite order.

Motivated by the results of Z. Liu [19], Y. Gang and D.R. Juan [17] and T. K. Kwak, Y. Lee and S. J. Yun [16], we investigate a generalization of strongly semicommutative rings which we call strongly M-semicommutative rings.

2. Main Results

For a monid M, e will always stand for the identity of M. If R is a ring, then $R[M]$ denotes the monoid ring over R. In this article we introduce the concept of a strongly M-semicommutative ring and investigate its properties. We start with the following definition.

Definition 2.1. *A ring R is called strongly M-semicommutative, if whenever elements $\alpha = a_1g_1+a_2g_2+\cdots+a_ng_n, \beta = b_1h_1+b_2h_2+\cdots+b_mh_m \in R[M]$, with $g_i, h_j \in M$ satisfy $\alpha\beta = 0$, then $\alpha R[M]\beta = 0$.*

Let $M = (N \cup \{0\}, +)$. Then a ring R is strongly M-semicommutative if and only if R is strongly semicommutative.

Lemma 2.2. *[1, Lemma 1]. Let M be a u.p.-monoid and R a reduced, then $R[M]$ is reduced.*

Lemma 2.3. *[16, Remark 2.4(3)]. Strongly reversible rings are clearly strongly semicommutative, but the converse does not hold.*

Proposition 2.4. *Let M be a u.p.-monoid and R a reduced ring. If R is strongly M-reversible ring then R is strongly M-semicommutative.*

Proof. Suppose $\alpha = a_1g_1 + \cdots + a_ng_n$ and $\beta = b_1h_1 + \cdots + b_mh_m \in R[M]$, with $a_i, b_j \in R$ and $g_i, h_j \in M$ for all i, j. Let $\alpha\beta = 0$. Then $\beta\alpha = 0$, since R is strongly M-reversible and $\beta\alpha\gamma = 0$ for any $\gamma \in R[M]$, so $\alpha\gamma\beta = 0$, since $R[M]$ is reduced by Lemma 2.2. Hence R is strongly M-semicommutative. □

Proposition 2.5. *Let M be a u.p.-monoid and R a reduced. Then R is strongly M-semicommutative.*

Proof. Let M be a u.p.-monoid and R a reduced ring. Then by [1, Proposion 1], R is strongly M-reversible. Thus, R is strongly M-semicommutative by Proposition 2.4. □

Let $(M, \leq)$ be an ordered monoid. If for any $g, g', h \in M, g < g'$ implies that $gh < g'h$ and $hg < hg'$, then $(M, \leq)$ is called a strictly ordered monoid.

Corollary 2.6. *Let M be strictly totally ordered monoid and R a reduced ring. Then R is strongly M-semicommutative.*

Lemma 2.7. *The class of strongly M-semicommutative is closed under subrings and direct products.*

Proposition 2.8. *Let M be a commutative, cancellative monoid and N and ideal of M. If R is strongly N-semicommutative ring, then R is strongly M-semicommutative ring.*

Proof. Suppose that $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ and $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R[M]$, such that $\alpha\beta = 0$. Take $g \in N$, then $gg_1, gg_2, \cdots, gg_n, h_1g, h_2g, \cdots, h_mg \in N$ and $gg_i \neq gg_j$ and $h_ig \neq h_jg$ when $i \neq j$. So

$$\alpha_1\beta_1 = (\sum_{i=1}^{n} a_igg_i)(\sum_{j=1}^{m} b_jh_jg) = 0.$$

Since R is strongly N-semicommutative so $\alpha_1\gamma_1\beta_1 = 0$ for any $\gamma_1 \in R[N]$. Thus, $\alpha\gamma\beta = 0$, for any $\gamma \in R[M]$. Therefore R is strongly M-semicommutative. □

Theorem 2.9. *For a monoid M. Suppose that R/I is strongly M-semicommutative for some ideal I of a ring R. If I is a reduced ring then R is strongly M-semicommutative.*

Proof. Let $\alpha\beta = 0$ with $\alpha, \beta \in R[M]$. Then we have $\alpha R\beta \subseteq I[M]$ and $\beta I\alpha = 0$, since $\beta I\alpha \subseteq I[M], (\beta I\alpha)^2 = 0$ and $I[M]$ is reduced. Thus, $(\alpha R\beta I)^2 = \alpha R\beta I\alpha R\beta I = 0$ and so $\alpha R\beta I = 0$. Hence $(\alpha R\beta)^2 \subseteq \alpha R\beta I = 0$ since $\alpha R\beta \subseteq I[M]$. Then $\alpha R\beta = 0$ since $I[M]$ is reduced. Therefore R is strongly M-semicommutative. □

Recall that an element u of a ring R is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

Proposition 2.10. *Let R be a ring and Δ be a multiplicative monoid in R consisting of central regular elements. Then R is strongly M-semicommutative if and only if so is $\Delta^{-1}R$.*

Proof. ($\Leftarrow$) This is obvious since R is a subring of $\Delta^{-1}R$.

($\Rightarrow$) Suppose that R is strongly M-semicommutative. Let $\phi\psi = 0$, for $\phi = u^{-1}\alpha$ and $\psi = v^{-1}\beta \in (\Delta^{-1}R)[M]$ where u, v are regular and $\alpha, \beta \in R[M]$. Since Δ is contained in the center of R then we have $0 = \phi\psi =$

$u^{-1}\alpha v^{-1}\beta = (u^{-1}v^{-1})\alpha\beta = (uv)^{-1}\alpha\beta$ and so $\alpha\beta = 0$. Since R is strongly M-semicommutative, $\alpha R\beta = 0$ and $\alpha s^{-1}R\beta = 0$ for any regular element s. This implies $\phi\Delta^{-1}R\beta = 0$ and therefore $\Delta^{-1}R$ is strongly M-semicommutative. □

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers, denote it by $R[x; x^{-1}]$.

Corollary 2.11. *Let M be a monoid, for a ring R. $R[x]$ is strongly M-semicommutative if and only if $R[x; x^{-1}]$ is strongly M-semicommutative.*

Proposition 2.12. *Let M be a monoid, and R be a ring, e central idempotent of R. Then the following statements are equivalent:*

1. *R is strongly M-semicommutative.*

2. *eR and $(1-e)R$ are strongly M-semicommutative.*

Proof. (1)⇔(2) This is straightforward since subrings and finite direct products of strongly M-semicommutative rings are strongly M-semicommutative by Lemma 2.7. □

Rege-Chhawchhaaria showed that commutative (hence semicommutative) rings need not to be Armendariz in [11, Example 3.2]. Conversely Huh, Lee and Smoktunowicz [2], gave a ring which is Armendariz but not semicommutative. However we have the following result.

Proposition 2.13. *For a monoid M. Let R be an M-Armendariz. If R is a semicommutative ring, then R is strongly M-semicommutative.*

Proof. Suppose that $\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j \in R[M]$ satisfy $\alpha\beta = 0$. Then since R is M-Armendariz, each $a_i b_j$ is zero, additionally R is semicommutative, therefore $a_i r b_j = 0$ for any element r in R for all i, j. Now it is easy to check that $\alpha\gamma\beta = 0$ for any $\gamma \in R[M]$. □

Since reversible rings are semicommutative, the following corollary is clear.

Corollary 2.14. *Let M be a monoid and R be an M-Armendariz ring. If R is a reversible ring, then R is strongly M-semicommutative ring.*

Lemma 2.15. *Let M be a strictly totally ordered monoid and R a reduced ring. Then $R[M]$ is reduced.*

Proof. Suppose that $\alpha = a_1g_1 + \cdots + a_ng_n \in R[M]$ is such that $\alpha^2 = 0$. Not loss the generality we assume that $g_1 < g_2 < \cdots < g_n$. Then from $\alpha^2 = 0$ it follows that

$$a_1^2g_1^2 + a_1a_2g_1g_2 + a_2a_1g_2g_1 + \cdots + a_n^2g_ng_n = 0.$$

Suppose $g_1g_1 = g_ig_j$ for some $1 \leq i, j \leq n$. Then $g_1 \leq g_i$, $g_1 \leq g_j$. If $g_1 < g_i$, then $g_1g_1 < g_ig_1 \leq g_ig_j = g_1g_1$, a contradiction. Thus, $g_1 = g_i$. Similarly $g_1 = g_j$. Thus, we have $a_1^2 = 0$ and so $a_1 = 0$ since R is reduced. Now $\alpha = a_2g_2 + \cdots + a_ng_n$. By analogy with above proof, we have $a_2 = 0, \cdots$, $a_n = 0$. Thus $\alpha = 0$. This means that $R[M]$ is reduced. □

Lemma 2.16. *[19, Corollary 1.2] Let M be a strictly totally ordered monoid and R a reduced ring. Then R is M-Armendariz.*

Proposition 2.17. *Let M be a strictly totally ordered monoid and R a reduced ring. If R is strongly M-smeicommutative, then $S_3(R)$ is strongly M-smeicommutative.*

Proof. For $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S_3(R)$.

We complete the proof by adapting the proof of [13, proposition 2] or [12, proposition 1.2]. It is easy to see that there exists an isomorphism of rings $S_3(R)[M] \rightarrow S_3(R[M])$ define by:

$$\sum_{i=1}^{n} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} g_i \rightarrow \begin{pmatrix} \sum_{i=1}^{n} a_ig_i & \sum_{i=1}^{n} b_ig_i & \sum_{i=1}^{n} c_ig_i \\ 0 & \sum_{i=1}^{n} a_ig_i & \sum_{i=1}^{n} d_ig_i \\ 0 & 0 & \sum_{i=1}^{n} a_ig_i \end{pmatrix}.$$

Suppose that $\alpha = A_1g_1 + A_2g_2 + \cdots + A_ng_n$, and $\beta = B_1h_1 + B_2h_2 + \cdots + B_mh_m \in S_3(R)[M]$ are such that $\alpha\beta = 0$, where $A_i, B_j \in S_3(R)$. We claim $\alpha S_3(R)[M]\beta = 0$ for each i, j. Assume that

$$A_i = \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix}, B_j = \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \sum_{i=1}^{n} a_ig_i & \sum_{i=1}^{n} b_ig_i & \sum_{i=1}^{n} c_ig_i \\ 0 & \sum_{i=1}^{n} a_ig_i & \sum_{i=1}^{n} d_ig_i \\ 0 & 0 & \sum_{i=1}^{n} a_ig_i \end{pmatrix}$$

$$\times \begin{pmatrix} \sum_{j=1}^{m} a'_j h_j & \sum_{j=1}^{m} b'_j h_j & \sum_{j=1}^{m} c'_j h_j \\ 0 & \sum_{j=1}^{m} a'_j h_j & \sum_{j=1}^{m} d'_j h_j \\ 0 & 0 & \sum_{j=1}^{m} a'_j h_j \end{pmatrix} = 0.$$

Thus

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} a'_j h_j) = 0, \tag{1}$$

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j) = 0, \tag{2}$$

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j) + (\sum_{i=1}^{n} c_i g_i)(\sum_{j=1}^{m} a'_j h_j) = 0, \tag{3}$$

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} d'_j h_j) + (\sum_{i=1}^{n} d_i g_i)(\sum_{j=1}^{m} a'_j h_j) = 0. \tag{4}$$

By Lemma 2.16 R is M-Armendariz, from Eq. (1), we have $a_i a'_j = 0$ for all i, j. Thus $(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{n} a'_j h_j) = 0$. Then we have

$$(\sum_{i=1}^{n} a_i g_i) R[M] (\sum_{j=1}^{n} a'_j h_j) = 0$$

since R is strongly M-semicommutative. If we multiply Eq. (2) on the right side by $(\sum_{j=1}^{m} a'_j h_j)$, then

$$0 = ((\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j))(\sum_{j=1}^{m} a'_j h_j) = (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j)^2,$$

and so $(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} a'_j h_j) = 0$ for all i, j since $R[M]$ is reduced by Lemma 2.15 and $(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b'_j h_j) = 0$. Similarly, from Eq. (4), we have

$$(\sum_{i=1}^{n} d_i g_i)(\sum_{j=1}^{m} a'_j h_j) = 0$$

and

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} d'_j h_j) = 0$$

for all i, j.

Also, if we multiply the Eq. (3), on the right side by $\sum_{j=1}^{m} a'_j h_j$ then

$$0 = ((\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j) + (\sum_{i=1}^{n} c_i g_i)(\sum_{j=1}^{m} a'_j h_j))(\sum_{j=1}^{m} a'_j h_j)$$
$$= (\sum_{i=1}^{n} c_i g_i)(\sum_{j=1}^{m} a'_j h_j)^2$$

implies $(\sum_{i=1}^{n} c_i g_i)(\sum_{j=1}^{m} a'_j h_j) = 0$ for all i, j since $R[M]$ is reduced by Lemma 2.15 and

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j) = 0. \tag{5}$$

Multiplying Eq. (5) on the left side by

$$(\sum_{i=1}^{n} a_i g_i)$$

then

$$(\sum_{i=1}^{n} a_i g_i)((\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j) + (\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j)) = (\sum_{i=1}^{n} a_i g_i)^2(\sum_{j=1}^{m} c'_j h_j),$$

and so

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} c'_j h_j) = 0$$

for all i, j since $R[M]$ is reduced by Lemma 2.15 and $(\sum_{i=1}^{n} b_i g_i)(\sum_{j=1}^{m} d'_j h_j) = 0$ for all i, j. Hence, these yields that

$$(\sum_{i=1}^{n} b_i g_i)R[M](\sum_{j=1}^{m} a'_j h_j) = 0,$$
$$(\sum_{i=1}^{n} a_i g_i)R[M](\sum_{j=1}^{m} b'_j h_j) = 0,$$
$$(\sum_{i=1}^{n} d_i g_i)R[M](\sum_{j=1}^{m} a'_j h_j) = 0,$$
$$(\sum_{i=1}^{n} a_i g_i)R[M](\sum_{j=1}^{m} d'_j h_j) = 0,$$

$$(\sum_{i=1}^{n} c_i g_i)R[M](\sum_{j=1}^{m} a'_j h_j) = 0,$$

$$(\sum_{i=1}^{n} a_i g_i)R[M](\sum_{j=1}^{m} c'_j h_j) = 0$$

and

$$(\sum_{i=1}^{n} b_i g_i)R[M](\sum_{j=1}^{m} d'_j h_j) = 0$$

since R is strongly M-semicommutative.

Thus

$$\begin{pmatrix} \sum_{i=1}^{n} a_i g_i & \sum_{i=1}^{n} b_i g_i & \sum_{i=1}^{n} c_i g_i \\ 0 & \sum_{i=1}^{n} a_i g_i & \sum_{i=1}^{n} d_i g_i \\ 0 & 0 & \sum_{i=1}^{n} a_i g_i \end{pmatrix} \begin{pmatrix} rl & tl & sl \\ 0 & rl & ul \\ 0 & 0 & rl \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{m} a'_j h_j & \sum_{j=1}^{m} b'_j h_j & \sum_{j=1}^{m} c'_j h_j \\ 0 & \sum_{j=1}^{m} a'_j h_j & \sum_{j=1}^{m} d'_j h_j \\ 0 & 0 & \sum_{j=1}^{m} a'_j h_j \end{pmatrix} = 0$$

for any $\gamma = Cl \in S_3(R)[M]$ where $l \in M$, and

$$C = \begin{pmatrix} r & t & s \\ 0 & r & u \\ 0 & 0 & r \end{pmatrix} \in S_3(R),$$

and so $\alpha S_3(R)[M]\beta = 0$. Therefore $S_3(R)$ is strongly M-semicommutative. □

Let R be a ring and let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

where n is a positive integer ≥ 2. Based on Proposition 2.17, one may suspect that S_n may also be strongly M-semicommutative for $n \geq 4$. But the following example eliminates the possibility.

Example 2.18. *Let M be a strictly totally ordered monoid and R a reduced ring. Take $e \neq g \in M$. Let $\alpha = e_{12}e + (e_{12} - e_{13})g$ and $\beta = e_{34}e + (e_{24} + e_{34})g$ be in $S_n(R)[M]$ where e_{ij}'s are the matrix units in $S_n(R)(n \geq 4)$. Then $\alpha\beta = 0$, but $\alpha\gamma\beta \neq 0$, where $\gamma = e_{13}e + (e_{13} + e_{23})g \in S_n(R)[M]$. Thus $S_n(R)[M]$ is not strongly M-semicommutative $(n \geq 4)$.*

The converse of Proposition 2.4 is not true in general by Proposition 2.17 as follows:

Example 2.19. *Let M be a strictly totally ordered monoid and R a reduced ring. By Proposition 2.17 $S_3(R)$ is strongly M-smeicommutative. However, $\alpha\beta = 0$, where $\alpha = e_{13}e + e_{23}g, \beta = e_{12}e + e_{13}g \in S_3(R)[M]$, with $e \neq g \in M$. But $\beta\alpha \neq 0$. So $S_3(R)$ is not strongly M-reversible.*

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R, m \in M$ and the usual matrix operations are used.

Proposition 2.20. *Let M be a strictly totally ordered monoid and R a reduced ring. If R is strongly M-smeicommutative, then $T(R, R)$ is strongly M-smeicommutative.*

Proof. Note that $T(R, R)$ is isomorphic to the ring

$$\left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in R \right\}.$$

Now the result follows from Proposition 2.17 and from the fact that every subring of strongly M-smeicommutative ring is also strongly M-smeicommutative. □

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