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SYMMETRIC TENSOR RANK WITH RESPECT TO CURVES

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Abstract: Let $\nu : d : \mathbb{P}^m \to \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, be the order d Veronese embedding. For any $P \in \mathbb{P}^r$ let $c(P)$ (resp. $cc(P)$, resp. $ic(P)$) be the minimal degree of a reduced (resp. reduced and connected, resp. integral) curve $C \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(C) \rangle$. We study these invariants when P has border rank ≤ 4.

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1. Ranks with Respect to Curves

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety of dimension $m \geq 2$. For each $P \in \mathbb{P}^r$ let $c_X(P)$ (resp. $cc_X(P)$, resp. $ic_X(P)$) be the minimal degree of a reduced (resp. reduced and connected, resp. integral) curve $C \subset X$ such that $P \in \langle C \rangle$, where $\langle\ \rangle$ denote the linear span). Obvious examples shows that it is better to consider reduced curves, not just locally Cohen-Macaulay curves. Obviously $ic_X(P) \geq cc_X(P) \geq c_X(P)$ and $c_X(P)$ is at least the minimal de-

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gree of an integral curve contained in X. However in the most important case of X-rank, the symmetric tensor, this is integer we are looking at, but rather these number divided by d. Fix integers $m \ge 1$ and $d \ge 1$. Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, denote the order d Veronese embedded of $\mathbb{P}^m$, i.e. the embedding of $\mathbb{P}^m$ by the complete linear system. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$. Now assume $m \ge 2$. For any $P \in \mathbb{P}^r$ set $c_{m,d}(P) := c_{X_{m,d}}(P)/d$, $cc_{m,d}(P) := cc_{X_{m,d}}(P)/d$, and $ic_{m,d}(P) := ic_{X_{m,d}}(P)/d$ The integer $c_{m,d}(P)$ (resp. $cc_{m,d}(P)$, resp. $ic_{m,d}(P)$) is the minimal degree of a reduced (resp. reduced and connected, resp. integral) curve $C \subset \mathbb{P}^m$ such that $P \in \langle \nu(C) \rangle$. We often write $c(P)$, $cc(P)$ and $ic(P)$ instead of $c_{m,d}(P)$, $cc_{m,d}(P)$ and $ic_{m,d}(P)$, respectively, and call $r(P)$ the symmetric tensor rank of P, $b(P)$ its border rank and $z(P)$ its cactus rank (also called the scheme-rank) (we may drop the subscript m, because if P depends from $n < m$ variables, then $b_{n,d}(P) = b_{m,d}(P)$ (see [7, Proposition 2.1] for the tensor rank and the border tensor rank). If $m = 2$, then $cc(P) = c(P)$ for all P, because any plane curve is connected. Let $Gc(m, d)$ (resp. $Gcc(m, d)$, resp. $Gic(m, d)$) be the integer $c_{m,d}(P)$ (resp. $cc_{m,d}(P)$, resp. $ic_{m,d}(P)$) for a general $P \in \mathbb{P}^r$. Let $Mc(m, d)$ (resp. $Mcc(m, d)$, resp. $Mic(m, d)$) be the maximal integer $c_{m,d}(P)$ (resp. $cc_{m,d}(P)$, resp. $ic_{m,d}(P)$) for some $P \in \mathbb{P}^r$.

Obviously $c(P) = 1 \iff cc(P) = 1 \iff cc(P) = 1 \iff$ “ there is a line $L \subseteq \mathbb{P}^m$ such that $P \in \langle \nu_d(L) \rangle$ ”.

In this note we compute all integers $c(P)$, $cc(P)$ and $ic(P)$ when P has border rank ≤ 4 (Propositions 2, 3 , 4 and 5). Following [4] we also make the following observation.

Proposition 1. *We have $Mc(m, d) \le 2 \cdot Gc(m, d)$.*

Proof. There is a non-empty open subset $U \subset \mathbb{P}^r$ such that $c_{m,d}(O) = Gc(m, d)$ for all $O \in U$. Fix $P \in \mathbb{P}^r$ and take a general line $L \subset \mathbb{P}^r$ though P. Since $L \cap U \neq \emptyset$, there are $P_1, P_2 \in U \cap L$ such that $P_1 \neq P_2$. Take $C_i \subset \mathbb{P}^m$, $i = 1, 2$, □

Remark 1. Fix an integer $t \ge 1$. Instead of curves in the definition of $c_{m,d}(P)$ we may take finite unions of curves of degree $\le t$ (in the case $t = 1$ we use finite unions of lines, the so-called stick-figures). We call $lr(P)$ the minimal degree of a union $E \subset \mathbb{P}^m$ of lines with $P \in \langle \nu_d(E) \rangle$ (we write $lcr(P)$ if we add the condition that E is connected). Proposition 1 holds is for these invariants (if we don’t assume the connectedness).

Proposition 2. *All points of border rank ≤ 2 have $ic(P) = cc(P) = c(P) = 1$.*

Proof. The case $b = 1$ is obvious, because every point of $\mathbb{P}^m$ is contained in a line. Now assume $b = 2$. If $r(P) = 2$, then $ic(P) = c(P) = cc(P)$, because any two points of $\mathbb{P}m$ are contained in a line. If $r(P) \neq 2$, the proof of [3, Theorem 32] gives the existence of a line $L \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(L) \rangle$ and hence $ic(P) = c(P) = cc(P)$. □

Proposition 3. *Assume $m \geq 2$, $d \geq 4$, $b(P) = 3$ and that $P \notin \langle \nu_d(L) \rangle$ for any line $L \subset \mathbb{P}^2$. Then $c(P) = cc(P) = ic(P) = 2$.*

Proof. Since $P \notin \langle \nu_d(L) \rangle$ for any line, we have $c(P) > 1$. Since $d \geq b(P) - 1$ there is a degree 3 zero-dimensional scheme $Z \subset \mathbb{P}^3$ such that $P \in \langle \nu_d(Z) \rangle$ ([6, Lemma 2.6], [5, Proposition 2.5]). We have $Z \neq 2O$ for some $O \in \mathbb{P}^m$, because $b(P) > 2$ ([3, proof of Theorem 37]). By assumption Z is not contained in a line. Hence it is elementary to check that it is contained in a smooth conic C. Since $\langle \nu_d(C) \rangle \supseteq \langle \nu_d(Z) \rangle$, we get $ic(P) = 2$ and hence $c(P) = cc(P) = 2$. □

Proposition 4. *Assume $m = 2$, $d \geq 7$ and take P with border rank 4 and with $P \notin \langle \nu_d(L) \rangle$ for any line $L \subset \mathbb{P}^2$. Then $cc(P) = c(P) = 2$. We have $ic(P) = 1$ if $r(P) = d - 2$ and for some (but not all) points P with $r(P) = 4$. We have $ic(P) \leq 3$ and $ic(P) = 3$ only if either there is a line $L \subset \mathbb{P}^2$ with $\deg(Z \cap L) = 3$ or Z is the the complete intersection of two conics through some $O \in \mathbb{P}^2$ and $Z \supset 2O$.*

Proof. The assumption " $P \notin \langle \nu_d(L) \rangle$ for any line $L \subset \mathbb{P}^2$ " gives $c(P) > 1$. Since $m = 2$, we have $cc(P) = c(P)$. Since $d \geq b(P) - 1$, there is a zero-dimensional scheme $Z \subset \mathbb{P}^2$ with $\deg(Z) = 4$ such that $P \in \langle \nu(Z) \rangle$ ([6, Lemma 2.6], [5, Proposition 2.5]). Every zero-dimensional scheme of $\mathbb{P}^2$ is smoothable. Since $d \geq 2b(P) - 1$, the scheme Z evincing $b(P)$ is unique (proof as in [6, Theorem 1.8] which consider the case of reduced schemes, using [3, Lemma 34], which is stated for arbitrary zero-dimensional schemes).

(i) Assume $ic(P) = 2$, i.e. assume the existence of a smooth conic C such that $P \in \langle \nu_d(C) \rangle$, and assume $r(P) \geq d+3$. By [2, case (e) of Theorem 1.2] we have $r(P) = 2d-2$. The curve $\nu_d(C)$ is a rational normal curve of degree $2d$ in its linear span. Call r' and b' the rank and the border rank of P with respect to $\nu_d(C)$. We have $r' \geq r$ and $b' \geq b$. A theorem of Sylvester gives that $b' \leq d+1$ and that either $r' = b'$ or $r' + b' = 2d + 2$ ([9, Theorem 4.1], [3, Theorem 23]). Since $r > d + 1$ and $b = 4$, we get $b' = 4$ and $r' = 2d - 2$. Let $Z' \subset C$ the degree 4 scheme evincing b'. Since $d \geq 7$, the uniqueness of Z gives $Z' = Z$. Hence $Z \subset C$. Since C is a smooth conic, we get that Z is curvilinear and that $\deg(Z \cap L) \leq 2$ for each line L.

(ii) Assume $r(P) = 4$, i.e. assume that Z is formed by 4 non-collinear points. If 3 of these points are contained in a line, then $c(P) = cc(P) = 2$, and $ic(P) \leq 3$, because Z is contained in a smooth plane cubic. Step (i) gives $ic(P) > 2$ and hence $ic(P) = 3$. If no 3 of the points of Z are contained in a line, then Z is contained in a smooth conic and hence $ic(P) = 2$.

(iii) We have $h^0(\mathcal{I}_Z(2)) \geq 2$. Since we work in characteristic $\neq 2$, no pencil of conics is formed only by double lines. Hence $c(P) \leq 2$. Since $c(P) > 1$, in all cases we have $c(P) = 2$. By step (i) we have $ic(2) > 2$ only if $r(P) \leq d+2$. By step (ii) we may assume $r(P) > 4$. Z is a complete intersection of two conics if and only if there is no line $L \subset \mathbb{P}^2$ with $\deg(Z \cap L) = 3$. If Z is a complete intersection of two conics, then it is contained in a smooth conic if and only if it is curvilinear. If Z is a complete intersection of two conics, then Z is not curvilinear if and only if it contains $2O$ for some $O \in \mathbb{P}^2$.

(iv) Assume that Z is a complete intersection of two conics and that $Z \supset 2O$ for some $O \in \mathbb{P}^2$. Since $h^1(\mathcal{I}_Z(2)) = 0$, the homogeneous ideal of Z is generated in degree ≤ 3. Since Z is contained in an irreducible curve of large degree (e.g. for an arbitrary scheme with $Z_{red} = \{O\}$ take any irreducible curve of degree $\deg(Z)+2$ with multiplicity $\deg(Z)+1$ at O), we get that Z is contained in an irreducible cubic. Hence $ic(P) = 3$.

(v) Assume that Z is not a complete intersection of two conics. Since Z is not contained in a line, there is a line $L \subset \mathbb{P}^2$ such that $\deg(L \cap Z) = 3$. As in step (i) we get $ic(P) > 2$. The residual scheme $\mathrm{Res}_L(Z)$ of Z with respect to L has a degree one and hence it is a point, O. We have $h^1(L, \mathcal{I}_{Z \cap L}(2)) = 0$. Since $h^1(\mathcal{I}_O(1)) = 0$, the residual exact sequence (also called the Castelnuovo's sequence) gives $h^1(\mathcal{I}_Z(2)) = 0$. As in step (iv) we get $ic(P) = 3$. □

Proposition 5. *Assume $d \geq 7$, $m \geq 3$ and fix P with $b(P) = 4$ and $P \notin \langle \nu(H) \rangle$ for any plane H. Let Z be the only zero-dimensional scheme evincing the border rank of P. We have $2 \leq c(P) \leq ic(P) = cc(P) = 3$. We have $c(P) = 2$ if and only if either Z has at least 3 connected components or Z has two connected components of degree 2.*

Proof. Since $d \geq 7$, there is a unique zero-dimensional scheme Z with $\deg(Z) = 4$ and evincing the border rank of P ([6, Lemma 2.6], [5, Proposition 2.5]). The possible Z's are classified in [2, Subection 5.3] and they are all curvilinear. Since $P \notin \langle \nu(H) \rangle$ for any plane, we have $c(P) \geq 2$, $cc(P) \geq 3$, $ic(P) \geq 3$, $c(P) = 2$ if and only if $P \in \langle \nu_d(E) \rangle$ with $E \subset \mathbb{P}^3$ the union of two disjoint lines, and $ic(P) = 3$ if and only if $P \in \langle \nu_d(C) \rangle$ with $C \subset \mathbb{P}^3$ a rational normal curve. Since $P \notin \langle \nu(H) \rangle$ for any H a plane, Z spans $\mathbb{P}^3$ and (having degree 4) it is in linearly general position.It is easy to check that every degree 4 curvilinear scheme in linearly general position is contained in a rational normal curve (when $Z = Z' \sqcup \{O\}$ with $\deg(Z') = 3$ one may use [1]; in the general case one can use [8, Theorem 3.2]).

Since $\langle Z \rangle = \mathbb{P}^3$, Z is contained in a disjoint union of 2 lines if and only if either Z has at least 3 connected components or it has two connected components of degree two. In these cases we have $c(P) = 2$.

Now assume $Z = Z' \sqcup \{O\}$ with Z' connected of degree 3 and O a point with $O \notin \langle Z' \rangle$. In this case we have $r(P) = 2d$ ([2, Proposition 5.22]). Assume the existence of two disjoint lines L, R such that $P \in \langle \nu_d(L \cup R) \rangle$. Take $O \in \langle \nu_d(L) \rangle$ and $Q \in \langle \nu_d(R) \rangle$ such that $P \in \langle \{O, Q\} \rangle$. Take $A \subset L$ (resp. $B \subset R$) be a set evincing the rak of O (resp. Q) with respect to the rational normal curve $\nu_d(L)$ (resp. $\nu_d(R)$). Since every degree d bivariate polynomial has rank ≤ 2, we have $\sharp(A) \le d$ and $\sharp(B) \le d$. Since $P \in \langle \nu_d(A \cup B) \rangle$ and $r(P) = 2d$, we have $\sharp(A) = \sharp(B) = d$. By a theorem of Sylvester ([3, Theorem 23], [9, Theorem 4.1]) there are connected zero-dimensional schemes $A' \subset L$, $B' \subset R$ such that $\deg(A') = \deg(B') = 2$, $O \in \langle \nu_d(A') \rangle$ and $Q \in \langle \nu_d(B') \rangle$. Since $P \in \langle \nu_d(A \cup B) \rangle$, the uniqueness of Z gives $Z' \sqcup \{O\} = A \cup B$, a contradiction.

Now assume that Z is connected. We have $r(P) = 3d - 2$ ([2, Proposition 5.19]). Since every bivariate polynomial of degree d has rank at most d and $r(P) > 2$, $P \notin \langle \nu_d(E) \rangle$ for any union of two distinct lines. □

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