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ON WEAKLY ϕ-RICCI SYMMETRIC KENMOTSU MANIFOLDS

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Abstract: The object of the present paper is to introduce the notion of weakly ϕ-Ricci symmetric Kenmotsu manifolds and study characteristic properties of locally ϕ-Ricci symmetric and ϕ-recurrent spaces.

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Key Words: weakly ϕ-symmetric, weakly ϕ-Ricci symmetric, Kenmotsu manifold, Einstein manifold, η-Einstein manifold

1. Introduction

The notion of Kenmotsu manifolds was defined by K. Kenmotsu [8]. In [8], it is showed (1) that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and Kähler manifold N with warping function $f(t) = se^t$, where s is a non-zero constant; and (2) that a Kenmotsu manifold of constant ϕ - sectional curvature is a space of constant curvature - 1, and so it is locally hyperbolic surface. Kenmotsu manifolds have been studied by many authors. For example see [1, 4, 7, 10].

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In 1992, the notion of weakly symmetric and weakly Ricci symmetric manifolds were introduced by L. Tamássy and T.Q. Binh ([14] and [15]). These type manifolds were studied with different structures by several authors (see [3], [9] and [11]).

As a weaker notion of locally symmetric manifolds, T. Takahashi [13] introduced the notion of locally ϕ-Symmetric Sasakian manifolds. In [5], U.C. De studied ϕ-Symmetric Kenmotsu manifolds with several examples. Recently, U. C. De and A. Sarkar [6] introduced the notion of ϕ-Ricci Symmetric Sasakian manifolds and obtained some interesting results of this manifold. This notion of ϕ-Ricci Symmetry was studied by S.S. Shukla and M.K. Shukla [12] in the context of Kenmotsu manifolds.

The paper is organized as follows. Section 2 is concerned with preliminary results of Kenmotsu manifolds which are needed in the rest of Sections. In Section 3, we introduce the notion of weakly ϕ-Ricci symmetric Kenmotsu manifolds and studied the characteristic properties of locally ϕ-symmetric and ϕ-recurrent spaces. Here, it is proved that, if the weakly ϕ-Ricci symmetric Kenmotsu manifold is locally ϕ-Ricci symmetric then the sum of the associated 1-forms A, B and D is zero everywhere. Also it is shown that, if the weakly ϕ- Ricci symmetric Kenmotsu manifold is ϕ-Ricci recurrent then the associated 1-forms B and D are in the opposite directions.

2. Preliminaries

A smooth manifold (M^n, g) $(n = 2m + 1 > 3)$ is said to be an almost contact metric manifold [2] if it admits a (1, 1) tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following condition holds [8]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{4}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{5}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold, the following relations hold [8]:

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{7}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{8}$$

$$S(\xi,\xi) = -(n-1), \quad \text{i.e.,} \quad Q\xi = -(n-1)\xi, \tag{9}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{10}$$

for any vector field X, Y on M. Here R, S and Q are the Riemannian curvature tensor, the Ricci tensor of type (0,2) such that $g(QX,Y) = S(X,Y)$ and Ricci operator, respectively.

3. Weakly ϕ-Ricci Symmetric Kenmotsu Manifolds

In this section, we introduce the notion of weakly ϕ-Ricci symmetric Kenmotsu manifolds.

Definition 1. A Kenmotsu manifold M $(n > 2)$ is said to be weakly ϕ-Ricci symmetric if the non-zero Ricci curvature Q of type (1, 1) satisfies the condition

$$\phi^2(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + g(QX,Y)\rho \tag{11}$$

where the vector fields X and Y on M, ρ is a vector field such that $g(\rho,X) = D(X)$, A and B are associated vector fields (not simultaneously zero).

Let us consider a weakly ϕ-Ricci symmetric Kenmotsu manifold $(M,g)(n > 2)$. Since the manifold is weakly ϕ-Ricci symmetric, we have (11) which, by virtue of (1) yields

$$-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi \;=\; A(X)QY + B(Y)QX + S(Y,X)\rho,$$

from which it follows that,

$$\begin{aligned} & -g(\nabla_X Q(Y),Z) + S(\nabla_X Y,Z) + \eta((\nabla_X Q)(Y))\eta(Z) \\ = \; & A(X)S(Y,Z) + B(Y)S(X,Z) + S(Y,X)D(Z). \end{aligned} \tag{12}$$

Putting $Y = \xi$ in (12) and so using (4), (8) and (9) we have

$$[B(\xi) - 1]S(X,Z) = (n-1)[g(X,Z) + A(X)\eta(Z) + D(Z)\eta(X)]. \tag{13}$$

Setting $X = Z = \xi$ in (13) we obtain

$$A(\xi) + B(\xi) + D(\xi) = 0. \tag{14}$$

Claim. *$A + B + D = 0$ holds for all vector fields on M.*
Next, setting $Z = \xi$ in (13) we get,

$$A(X) = -[B(\xi) + D(\xi)]\eta(X). \tag{15}$$

In view of (13), the relation (15) reduces to

$$A(X) = A(\xi)\eta(X). \tag{16}$$

In a similar manner we can obtain

$$B(X) = B(\xi)\eta(X) \tag{17}$$

and

$$D(X) = D(\xi)\eta(X). \tag{18}$$

Adding (16), (17), (18) and then using (14) we obtain

$$A(X) + B(X) + D(X) = 0, \tag{19}$$

for all X. Hence from (19) we can state the following:

Theorem 2. *In a weakly ϕ-Ricci symmetric Kenmotsu manifold M ($n > 2$) the sum of associated 1-forms A, B and D is zero everywhere.*

In view of (16) and (18), the relation (13) yields

$$[B(\xi) - 1]S(X, Z) = (n - 1)[g(X, Z) + A(\xi) + D(\xi)\eta(X)\eta(Z)]. \tag{20}$$

Using (14) in (20) we obtain

$$S(X, Z) = \gamma g(X, Z) + \delta\eta(X)\eta(Z), \tag{21}$$

where

$$\gamma = \frac{n - 1}{B(\xi) - 1}$$

and

$$\delta = \frac{(1 - n)B(\xi)}{B(\xi) - 1}$$

are smooth functions such that $B(\xi) \neq 1$. Hence we state the following:

Theorem 3. *A weakly ϕ-Ricci symmetric Kenmotsu manifold $M(n > 2)$ is an η-Einstein manifold provided that $B(\xi) \neq 1$.*

Definition 4. A weakly ϕ-Ricci symmetric Kenmotsu manifold M $(n > 2)$ is said to be locally ϕ-symmetric if

$$\phi^2(\nabla Q) = 0. \tag{22}$$

Definition 5. A weakly ϕ-Ricci symmetric Kenmotsu manifold M $(n > 2)$ is said to be ϕ-Ricci recurrent if it satisfies the condition

$$\phi^2(\nabla_X Q)(Y) = A(X)Q(Y). \tag{23}$$

Next, we proceed with the above definitions.

Suppose a weakly ϕ-Ricci symmetric Kenmotsu manifold $M(n > 2)$ is locally ϕ-Ricci symmetric. Then from (11) and Definition 4, we have

$$A(X)Q(Y) + B(Y)Q(X) + g(QX, Y)\rho. \tag{24}$$

Taking inner product with ξ in (24) we obtain

$$A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(X, Y)\rho. \tag{25}$$

Setting $X = Y = \xi$ in (25) and then taking inner product with ξ, we find (14).

Putting $Y = Z = \xi$ in (25), we get

$$A(X)S(\xi, \xi) = -B(\xi) + D(\xi)S(X, \xi).$$

Similarly, we have

$$B(Y)S(\xi, \xi) = -A(\xi) + D(\xi)S(Y, \xi),$$
$$D(V)S(\xi, \xi) = -A(\xi) + B(\xi)S(V, \xi).$$

Adding above equations by taking $X = Y = Z$ and using (14), we get

$$A(X) + B(X) + D(X) = 0.$$

for any vector field X on M so that $A + B + D = 0$. Hence we state

Theorem 6. *If a weakly ϕ-Ricci symmetric Kenmotsu manifold M $(n > 2)$ is locally ϕ-Ricci symmetric, then the sum of the associated $1 - forms$ A, B and D is zero everywhere.*

On the other hand, if a weakly ϕ-Ricci symmetric Kenmotsu manifold M is ϕ-Ricci recurrent, then from (11) and Definition 5, we find

$$B(Y)S(X,Z) + S(X,Y)D(Z) = 0. \tag{26}$$

for any vector fields X, Y and Z on M.

Next putting $X = Y = Z = \xi$ in (26) and then using (9), we obtain

$$B(\xi) + D(\xi) = 0. \tag{27}$$

Further proceeding as in the proof of Theorem 6, using the fact that $B(\xi) + D(\xi) = 0$, obviously, one can get $B(X) + D(X) = 0$. for any vector field X on M, so that $B + D = 0$ everywhere on M. Hence we state

Theorem 7. *If a weakly ϕ-Ricci symmetric Kenmotsu manifold M $(n > 2)$ is ϕ-Ricci recurrent, then the 1-forms B and D are in the opposite directions.*

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References

[1] C.S. Bagewadi, D.G. Prakasha and Venkatesha *Projective curvature tensor on a Kenmotsu manifold with respect to semi-symmetric metric connection*, Stud. Cercet. Stiint. Ser. Mat. Univ. Bacau., **17** (2007), 21–32.

[2] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. **509**, Springer-Verlag, 1976.

[3] U.C. De, A.A. Shaikh and S. Biswas, *On weakly symmetric contact metric manifolds*, Tensor (N.S.), **64 (2)** (2003), 170–175.

[4] U.C. De and G. Pathak, *On 3-dimensional Kenmotsu manifolds*, Indian J. Pure. Appl. Math., **35** (2004), 159–165.

[5] U.C. De, *On ϕ-symmetric Kenmotsu manifolds*, Int. Electronic J. Geom., **1(1)** (2008), 33–38.

[6] U.C. De and A. Sarkar , *On ϕ-Ricci symmetric Sasakian manifolds*, Proceedings of the Jangjeon Math. Soc., **11(1)** (2008), 47–52.

[7] S.P. Hong, C. Özgür and M.M. Tripathi, *On some special classes of Kenmotsu manifolds*, Kuwait J. Sci. Eng., **33/2** (2006), 19–32.

[8] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., **24** (1972), 93–103.

[9] C. Özgür, *On weakly symmetric Kenmotsu manifolds*, Diff. Geom.-Dyn. Syst., **8** (2006), 204–209.

[10] D.G. Prakasha, A.T. Vanli, C.S. Bagewadi and D.A. Patil, *Some classes of Kenmotsu manifolds with respect to semi-symmetric metic connection*, Acta Math. Sinica. English Series, **29 (7)** (2013), 1311–1322.

[11] A.A. Shaikh and T.Q. Binh, *On weakly symmetric $(LCS)_n$ manifolds*, J. Adv. Math. Studies., **2 (2)** (2009), 103–118.

[12] S.S. Shukla and M.K. Shukla, *On ϕ-Ricci symmetric Kenmotsu manifolds*, Novi Sad J. Math., **39(2)** (2009), 89–95.

[13] T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tohoku Math. J., **29**(1977), 91–113.

[14] L. Tamássy and T.Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc. J. Bolyai, **56** (1992), 663–670.

[15] L. Tamássy and T.Q. Binh, *On weak symmetrics of Einstein and Sasakian manifolds*, Tensor N. S., **53** (1993), 140–148.