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PRIMITIVE ELEMENTS AND PREIMAGE OF PRIMITIVE SETS OF FREE LIE ALGEBRAS

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Abstract: Let F and L be free Lie algebras of finite rank n and m respectively and ϕ be a homomorphism from F to L. We prove that the preimage of a primitive set of L contains a primitive set of F. As a consequence of this result we obtain that an element h of a subalgebra H of F is primitive in H if it is primitive in F.

Also we show that in a free Lie algebra of the form $F/\gamma_{m+1}(R)$ if the ideal $\langle g\rangle_{id}$ of this algebra contains a primitive element h then h and g are conjugate by means of an inner automorphism.

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1. Introduction

Let F be a free Lie algebra freely generated by a set $\{x_1, x_2, ..., x_n\}$ over a field K of characteristic zero. A set $\{y_1, ..., y_m\}$ of F with $m \leq n$ is called a primitive set in F if it can be extended to a free generating set of F. So a primitive set is a subset of a free generating of F. An element g of F is said primitive element if it can be included in a free generating set of F.

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Denote by $\langle a_1, ..., a_k\rangle$ the subalgebra generated by the elements $a_1, ..., a_k$ and by $\langle a_1, ..., a_k\rangle_{id}$ the ideal of F generated by these elements.

There are many interesting results about primitive elements and primitive systems of free Lie algebras, free associative algebras and free groups. We refer to $[1, 2, 3, 4, 5, 8, 9]$ for these results.

The starting point of this work is the following questions.

Question 1. Let H be a subalgebra of F. Assume that an element h of H is primitive in F. Is it true that h is also primitive in H.

The affirmative solution of this question for free groups is given by A. Clifford and R.Z. Goldstein in [4]. For free algebras of Schreier varieties of linear algebras, the affirmative answer follows from [7] . In this work we use elemantary transformations to give, for a homomorphism $\phi : F \longrightarrow L$ between free Lie algebras F and L and a primitive subset W in the image of ϕ a primitive set W in F so that :

ϕ is a bijective correspondence between V and W;

ϕ is injective on the subalgebra $\langle W\rangle$ of F;

and

ϕ is trivial on $\overline{W}$, where $W \cup \overline{W}$ is a primitive set of F.

The question 1 becomes more interesting when we take a subset $\{h_1, ..., h_k\}$ of H is primitive in F. Our main results answer this question in a different way which are independent from [6] and [7] for $1 \leq k \leq n$.

Question 2. What can we say about an element g of a relatively free Lie algebra L if the ideal $\langle g\rangle_{id}$ of L contains a primitive element h?

In the group case it is well known that the element g is conjugate to one of the elements $h^{\mp 1}$. For free metabelian Lie algebras, the similar result was recently obtained by I.V. Chirkov and M.A.Shevelin [3] . Does a similar result holds for the relatively free Lie algebra $F/\gamma_m(R)$, where R is an ideal of F and $\gamma_m(R)$ is the m th term of the lower central series of R? Firstly we consider the Lie algebra $F/\gamma_{m+1}(F')$, $m \geq 1$ and then the algebra $F/\gamma_{m+1}(R)$, where $F' = [F, F]$ and R is a verbal ideal of F contained in F'. We denote by F'' the derived subalgebra $[F', F']$.

Let $M = F/F''$. An inner automorphism $e^{adv}, v \in M'$, of M acts by $e^{adv}(w) = w + [w, v], w \in M$.

All Lie algebras we consider here assumed over a fixed field K of characteristic zero.

2. Preimage of Primitive Subsets

Let L be a free Lie algebra of rank m with a free generating set $Y = \{y_1, ..., y_m\}$.

Theorem 1. *Suppose that ϕ is a homomorphism from F to L. If $\{y_1, ..., y_s\} \subset \phi(F)$ then there is a free generating set $\{z_1, ..., z_n\}$ for F and some $r > s$ such that:*

1. *$\phi(z_i) = y_i$ for $1 \leq i \leq s$,*
2. *ϕ is injective on the subalgebra generated by $\{z_1, ..., z_r\}$,*
3. *$\phi(z_i) = 0$, for $r \leq i \leq n$.*

Proof. Consider the set $\{\phi(x_1), ..., \phi(x_n)\}$ in L. The elements of this set are written with respect to the generating set Y. Since $y_1, y_2, ..., y_s \in \phi(F)$ there is a sequence of elemantary transformations $\alpha_1, ..., \alpha_l$ taking the set $\{\phi(x_1), ..., \phi(x_n)\}$ to the set

$$\{y_1, y_2, ..., y_s, u_{s+1}, ..., u_r, 0, ..., 0\},$$

where $\{y_1, y_2, ..., y_s, u_{s+1}, ..., u_r,\}$ is a free generating set of $\phi(F)$.

We consider the composition of the elemantary transformations $\alpha_1, ..., \alpha_l$. Let Ψ be the automorphism of $\phi(F)$ defined as $\Psi = \alpha_1 ... \alpha_l$. By definition

$$\Psi\phi(x_i) = \begin{cases} y_i, 1 \leq i \leq s \\ u_i, s+1 \leq i \leq r \\ 0, r < i \leq n \end{cases}$$

Clearly $\Psi\phi(x_i)$ is a word in $\phi(x_1), ..., \phi(x_n)$ for each i, $1 \leq i \leq r$. Denote this word by w_i, i.e.,

$\Psi\phi(x_i) = w_i(\phi(x_1), ..., \phi(x_n))$

Now we perform the elemantary transformations $\alpha_1, ..., \alpha_l$ on the set $\{x_1, x_2, ..., x_n\}$ in F. Let $\bar{\alpha_1}, ..., \bar{\alpha_l}$ be the elemantary transformations of F obtained by performing $\alpha_1, ..., \alpha_l$ on the set $\{x_1, x_2, ..., x_n\}$ in F and let $\theta = \bar{\alpha_1}, ..., \bar{\alpha_l}$ Clearly

$$\theta(x_i) = w_i(x_1, x_2, ..., x_n), 1 \leq i \leq r.$$

For $1 \leq i \leq n$ set $z_i = \theta(x_i)$. Then

$$z_i = w_i(x_1, x_2, ..., x_n), 1 \leq i \leq n$$

Therefore

$$\phi(z_i) = w_i(\phi(x_1), ..., \phi(x_n)) = \Psi\phi(x_i),$$

where $1 \leq i \leq n$. This shows that there is a set $\{z_1, ..., z_n\}$ in F satisfying the conclusion of the theorem. □

Corollary 2. *Let ϕ be an epimorphism from F to L. Then there exists a free generating set $\{z_1, ..., z_n\}$ of F such that*

$\phi(z_i) = y_i$, where $1 \leq i \leq m$
and
$\phi(z_i) = 0$, where $m < i \leq n$

Proof. Since ϕ is an epimorphism from F to L then $\phi(F) = L$. Therefore the elements $y_1, ..., y_m$ are in the image of ϕ. Hence by Theorem 1 the result follows. □

Corollary 3. *Let ϕ be a monomorphism from F to L and let $Z = \{z_1, ..., z_s\}$ be subset of F. If $\{\phi(z_1), ..., \phi(z_s)\}$ is a primitive subset of L then Z is primitive in F.*

Proof. For $1 \leq j \leq s$ put $\phi(z_j) = y_j$. Since $\{y_1, ..., y_s\} \subset \phi(F)$, by Theorem 1 there exists a free generating $\{w_1, ..., w_n\}$ of F such that $\phi(w_i) = y_i$, where $i = 1, ..., s$. Hence $\{z_1, ..., z_s\}$ is primitive in F. □

Corollary 4. *Let H be a subalgebra of F. If $\{z_1, ..., z_s\}$ is a subset of H and $\{z_1, ..., z_s\}$ is primitive in F then it is a primitive set in H.*

Proof. Let $\{z_1, ..., z_s\} \subset H$ and let $\{z_1, ..., z_s\}$ be a primitive in F. Since H is subalgebra of F we can define the inclusion map $\phi : H \longrightarrow F$. Since $z_1, ..., z_s \in H$ then by definition of the inclusion map $\phi(z_i) = z_i$,where $i = 1, ..., s$. Clearly ϕ is a monomorphism. hence the primitivity of the set $\{z_i = \phi(z_i) \mid i = 1, ..., s\}$ and Corollary 3 implies that $\{z_1, ..., z_s\}$ is primitive in H. □

Taking $s = 1$ in Corollary 4 we get the following:

Corollary 5. *Let H be a subalgebra of F. If an element h of H is primitive in F then it is primitive in H.*

Proposition 6. *Let $g_1, g_2, ..., g_m \in F$ and $\{h_1, ..., h_m\}$ be a primitive system in F. If $h_1, ..., h_m \in \langle g_1, g_2, ..., g_m \rangle_{id}$*

then
$\langle h_1, ..., h_m \rangle_{id} = \langle g_1, g_2, ..., g_m \rangle_{id}$
where $n \geq m$.

Proof. Let $g_1, g_2, ..., g_m \in F, \{h_1, ..., h_m\}$ be a primitive system in F and let $h_1, ..., h_m \in \langle g_1, ..., g_m \rangle_{id}$. Clearly $\langle h_1, ..., h_m \rangle_{id} \subseteq \langle g_1, ..., g_m \rangle_{id}$

Since the set $\{h_1, ..., h_m\}$ is primitive in F without loss of generality we may assume that for an automorphism φ of F $\varphi(x_i) = h_i, 1 \leq i \leq m$.

Assume that the ideal $\langle g_1, ..., g_m \rangle_{id}$ strictly contains the ideal $\langle h_1, ..., h_m \rangle_{id}$ of F. Then there exists a non zero element $w = w(x_{m+1}, ..., x_n)$ of F such that $(w) \in \langle g_1, ..., g_m \rangle_{id}$ and $\varphi(w) \notin \langle h_1, ..., h_m \rangle_{id}$. Now consider the free generating set $\{\varphi(x_1) = h_1, ..., \varphi(x_m) = h_m, \varphi(x_{m+1}), ..., \varphi(x_n)\}$ of F. We can find a subset Y of this set which including n-m elements such that Freiheitssatz holds i.e.the intersection of the subalgebra generated by the set Y of F and the ideal $\langle g_1, ..., g_m \rangle_{id}$ is zero. Hence if

1. $Y = \{\varphi(x_{m+1}), ..., \varphi(x_n)\}$ then we get
   $\varphi(w) \in \langle Y \rangle \cap \langle g_1, ..., g_m \rangle_{id} \neq \{0\}$
   which is a contradiction,
2. If Y is a subset of $\{h_1, ..., h_m\}$ then
   $\langle Y \rangle \subseteq \langle h_1, ..., h_m \rangle_{id} \subset \langle g_1, ..., g_m \rangle_{id}$
   and hence $\langle Y \rangle \cap \langle g_1, ..., g_m \rangle_{id} \neq \{0\}$, which is a contradiction,
3. If Y contains either some of h_i or some of $\varphi(x_j), m + 1 \leq j \leq n$ then
   $\langle Y \rangle \cap \langle g_1, ..., g_m \rangle_{id} \neq \{0\}$
   which is a contradiction.

Hence these contradictions completes the proof. □

2.1. Free Metabelian Lie Algebras and Primitive Elements

Let M be a free metabelian Lie algebra of rank n. We identify M with F/F'' and we write $M = F/F''$.

The following theorem was proven in [1] by I.V.Chirkov and M. A. Shevelin.

Theorem 7. *Let z be a primitive element of the algebra M and let $y \in M$. If $z \in \langle y \rangle_{id}$ then the elements z and y are conjugate by means of an inner automorphism of M.*

In this section we prove a similar theorem for relatively free Lie algebras of the form $F/\gamma_{m+1}(R), n \geq 1$.

Denote by $\frac{\partial g}{\partial x_i}$ the Fox i th left derivative of an element g of F.

Consider the natural homomorphism $\theta : U(F) \longrightarrow U(F/R)$, where R is an ideal of F.

The image of derivative $\frac{\partial g}{\partial x_i}$ in $U(F/R)$ is called the value of $\frac{\partial g}{\partial x_i}$ in $U(F/R)$.

For an element of a free metabelian Lie algebra the following primitivity criterion holds.

The image of an element u of F in F/F'' is primitive if and only if the vector $\left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$ of the values of the dervatives in $U(F/F')$ is unimodular,i,e. There exist $a_1, ..., a_n \in U(F/F')$ such that $\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = 1$.

Although the following theorem is a consequence of Theorem 7, its proof can be applied in other cases in which the primitivity criterion holds.

Theorem 8. *Let g and h are elements of F and let $\overline{g}, \overline{h}$ be their images in the algebra $F/\gamma_{m+1}(F')$, respectively.*

Assume that $\overline{h}$ is an element of the ideal of $F/\gamma_{m+1}(F')$ generated by $\overline{g}$. If $\overline{h}$ is primitive then $\overline{g}$ is also primitive.

Proof. For $m \geq 1$ we have $\gamma_{m+1}(F') \subseteq F''$. It is clear that every automorphism of

$$F/\gamma_{m+1}(F')$$

induces an automorphism of $(F/\gamma_{m+1}(F'))/(F''/\gamma_{m+1}(F'))$.

Since

$$[F/\gamma_{m+1}(F')]/[F''/\gamma_{m+1}(F')] \simeq F/F''$$

it suffices to prove that the element $g + F''$ is primitive. Let $h + F''$ is an element of the ideal of M generated by $g + F''$. If $h + F''$ is primitive then by Theorem 7 the elements $h + F''$ and $g + F''$ are conjugate means of an inner automorphism of M. Put $\widehat{h} = h + F''$ and $\widehat{g} = g + F''$.

Hence there exists an element $\widehat{v}$ of M such that $\widehat{g} = \widehat{h} + \left[\widehat{h}, \widehat{v}\right] = e^{ad\widehat{v}}\left(\widehat{h}\right)$, where $e^{ad\widehat{v}} = I + ad\widehat{v}$ and $ad\widehat{v}(\widehat{h}) = \left[\widehat{h}, \widehat{v}\right]$. As is known if elements $g_1, ..., g_r$ generate a free Lie algebra H modulo a nilpotent ideal then these elements generate the algebra H. Hence $\overline{g}$ is primitive in $F/\gamma_{m+1}(F')$.Since $\widehat{h}$ is primitive then $\widehat{g}$ is also primitive. □

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