$\mathcal{AP}$
ijpam.eu

POISSON APPROXIMATION FOR RANDOM SUMS
OF POISSON RANDOM VARIABLES

K. Teerapabolarn

Department of Mathematics
Faculty of Science
Burapha University
Chonburi, 20131, THAILAND

Abstract: In this paper, we use the Stein-Chen method to determine a bound for the total variation distance between the distribution of random sums of independent Poisson random variables and an appropriate Poisson distribution. Two examples have been given to illustrate the result obtained.

1. Introduction

Let $X_1, X_2, ...$ be a sequence of independent Poisson random variables, each with probability $P(X_i = k) = \frac{e^{-\mu_i}\mu_i^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$, where $\mu_i > 0$. It is well-known that the sum $S_n = \sum_{i=1}^{n} X_i$ has a Poisson distribution with mean $\mu_n = \sum_{i=1}^{n} \mu_i$, that is, $S_n = \mathcal{P}_{\mu_n}$ is a Poisson random variable with mean λ_n. Let N be a non-negative integer-valued random variable and independent of the X_i's. Consider the random sums of N independent Poisson random variables, $S_N = \sum_{i=1}^{N} X_i$, it is observed that S_N has a Poisson distribution with mean μ_N when N is given. In this paper, we are interested to approximate the distribution of S_N

---

by a Poisson distribution with mean $\lambda = E(\mu_N)$, in terms of the total variation distance between the distributions of S_N and $\mathcal{P}_\lambda$, $d_{TV}(S_N, \mathcal{P}_\lambda)$, together with its upper bound, where $d_{TV}(S_N, \mathcal{P}_\lambda) = \sup_{A \subseteq \mathbb{N} \cup \{0\}} |P(S_N \in A) - P(\mathcal{P}_\lambda \in A)|$.
The method for determining the desired result of this study is the well-known Stein-Chen method, which is described and used to determine the result in Sections 2 and 3, respectively. In Section 4, some examples are provided to illustrate applications of the result. Conclusion of this study is presented in the last section.

2. Method

Stein [3] introduced a powerful and general method for bounding the error in the normal approximation. This method was developed and applied in the setting of Poisson approximation by Chen [2]. Stein's equation for Poisson distribution with mean $\lambda > 0$ is, for given h, of the form

$$h(x) - \wp_\lambda(h) = \lambda f(x+1) - x f(x), \tag{2.1}$$

where $\wp_\lambda(h) = e^{-\lambda} \sum_{l=0}^{\infty} h(l) \frac{\lambda^l}{l!}$ and f and h are bounded real valued functions defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let function $h_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

For $C_x = \{0, ..., x\}$, the solution f_A of (2.1) is of the form

$$f_A(x) = \begin{cases} (x-1)! \lambda^{-x} e^\lambda [\wp_\lambda(h_{A \cap C_{x-1}}) - \wp_\lambda(h_A)\wp_\lambda(h_{C_{x-1}})] & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.2}$$

and Barbour et al. [1] showed that

$$\sup_A |f_A(x)| \leq \min\left\{1, 0.74\lambda^{-1/2}\right\}. \tag{2.3}$$

3. Result

The following theorem presents a bound for $d_{TV}(S_N, \mathcal{P}_\lambda)$, which is the desired result.

Theorem 3.1. *For $\lambda = E(\mu_N)$, then we have*

$$d_{TV}(S_N, \mathcal{P}_\lambda) \leq \min\left\{1, 0.74\lambda^{-1/2}\right\} E|\mu_N - \lambda|. \qquad (3.1)$$

Proof. For $A \subseteq \mathbb{N} \cup \{0\}$, let $f = f_A$ be defined as in (2.2). Thus, applying (2.1), we can have

$$
\begin{aligned}
d_{TV}(S_N, \mathcal{P}_\lambda) &= |E\{\lambda f(S_N + 1) - S_N f(S_N)\}| \\
&= |E\{\lambda E[f(S_N + 1)|N] - E[S_N f(S_N)|N]\}| \\
&= |E\{\lambda E[f(S_N + 1)|N]\} - \mu_N E[f(S_N + 1)|N]\}| \\
&\leq E|(\lambda - \mu_N)E[f(S_N + 1)|N]| \\
&\leq \sup_A |f(x)| E|\mu_N - \lambda| \\
&\leq \min\left\{1, 0.74\lambda^{-1/2}\right\} E|\mu_N - \lambda| \quad \text{(by (2.3))},
\end{aligned}
$$

which completes the proof. $\qquad \square$

If X_i's are identically distributed, then the following corollary is an immediately consequence of the Theorem 3.1.

Corollary 3.1. *If $\mu_1 = \mu_2 = \cdots = \mu$, then we have the following:*

$$d_{TV}(S_N, \mathcal{P}_\lambda) \leq \min\left\{1, 0.74[E(N)\mu]^{-1/2}\right\} E|N - E(N)|\mu. \qquad (3.2)$$

4. Examples

This section gives two examples to illustrate the result in the case of X_i's to be identically distributed.

Example 3.1. For n ($n \in \mathbb{N}$) is fixed, let N be a random variable with probability function

$$
P(N = k) = \begin{cases} \frac{1}{2} & , k = n, \\ \frac{1}{2} & , k = 2n, \\ 0 & , \text{otherwise.} \end{cases}
$$

Therefore, $E(N) = \frac{3n}{2}$ and $E|N - E(N)| = \frac{n}{2}$. Let $\mu_1 = \mu_2 = \cdots = \mu$, then $\lambda = \frac{3n\mu}{2}$ and we have

$$d_{TV}(S_N, \mathcal{P}_{\frac{3n\mu}{2}}) \leq \min\left\{\frac{n\mu}{2}, 0.3\,(n\mu)^{1/2}\right\}.$$

Example 3.2. Let N be a random variable with probability function

$$P(N = n) = \frac{1}{2^n},\ \ n = 1, 2, ...,$$

then we have $E(N) = 2$ and $E|N - E(N)| = 1$. If $\mu_1 = \mu_2 = \cdots = \mu$, then $\lambda = 2\mu$ and we can obtain

$$d_{TV}(S_N, \mathcal{P}_{2\mu}) \leq \min\left\{\mu, 0.52\mu^{1/2}\right\}.$$

5. Conclusion

In this study, the Stein-Chen method was used to determine a bound for the total variation distance between the distribution of the random sums of independent Poisson random variables and a Poisson distribution. With this bound, it can be seen that the result gives a good approximation when all μ_i are small.

References

[1] A.D. Barbour, L. Holst, S. Janson, *Poisson approximation*, Oxford Studies in Probability 2, Clarendon Press, Oxford, 1992.

[2] L.H.Y. Chen, Poisson approximation for dependent trials, *Ann. Probab.*, **3** (1975), 534–545.

[3] C.M. Stein, A bound for the error in normal approximation to the distribution of a sum of dependent random variables, *Proc. Sixth Berkeley Sympos. Math. Statist. Probab.*, **3** (1972), 583–602.