REFLEXIVITY IN ASSOCIATIVE TRIPLE SYSTEMS

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Abstract: In the present paper, we define the notions of regularity, strong regularity and reflexivity in associative triple systems and prove some theorems concerning these notions.

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1. Introduction and Preliminaries

A vector space $T$ over a field $F$ together with a trilinear map $(x, y, z) \rightarrow < x, y, z >$ is called a triple system. A triple system $T$ is said to be an associative triple system if $<< x, y, z >, u, v > = < x, < y, z, u >, v > = < x, y, < z, u, v >>$ for all $x, y, z, u, v \in T$. For example, if $A$ is an associative algebra, then the underlying vector space of $A$ together with the map $(x, y, z) \rightarrow xyz$ is an associative triple system which we denote by $T_A$. $T_A$ is called the triple system associated with the algebra $A$. Hence forward, $T$ will denote an associative triple system. We call an element $x$ in $T$ invertible if there exists an element $x_1 \in T$ such that $< x, x_1, t > = < x_1, x, t > = < t, x, x_1 > = < t, x_1, x > = t$ for all $t \in T$. $x_1$ is called an inverse of $x$. It can be easily seen that an invertible element of a triple system $T$ has a unique inverse. Triple system $T$
is said to be a division triple system if every non zero element of T is invertible. It can be easily shown that T is a division triple system if and only if the following condition is satisfied: For \( a, b, c \in T \) \( (a, b \neq 0) \), the equations \( < a, b, x > = c \), \( < a, x, b > = c \) and \( < x, a, b > = c \) have solutions. If \( A \) is a division algebra, then the triple system \( T_A \) associated with \( A \) is a division triple system.

The odd powers of an element \( a \in T \) are defined recursively as follows. \( a^1 = a \), \( a^{2(n+1)+1} = < a^{2n+1}, a, a > \). \( a \) is said to be nilpotent if \( a^{2n+1} = 0 \) for some positive integer \( n \). For \( x \in T \), we define a map \( U_x : T \to T \) as \( U_x(y) = < x, y, x > \). \( x \) is said to be a zero divisor if \( U_x \) is not injective.

An element \( a \in T \) is said to be regular if there exist an element \( x \in T \) such that \( a = U_a(x) = < a, x, a > \). \( x \) is referred to as a generalized inverse of \( a \). \( a \) is said to be unit regular if there exists an invertible element \( x \) such that \( a = U_a(x) = < a, x, a > \). \( a \) is said to be strongly regular if there exists an element \( x \in T \) such that \( a = < a, a, x > \). \( T \) is said to be strongly regular if every element of \( T \) is strongly regular. An element \( x \) is said to be a reflexive inverse of \( a \) if \( a = U_a(x) \) and \( x = U_x(a) \). If \( x \) is a reflexive inverse of \( a \), then it follows that \( a \) is regular with \( x \) as a generalized inverse.

2. Main Results

The relationship between regularity and strong regularity is given by the following.

**Theorem 2.1.** For each element \( a \) of a strongly regular triple system \( T \), there corresponds an element \( x \) such that \( < a, a, x > = < a, x, a > = < x, a, a > = a \). Thus every element of a strongly regular triple system \( T \) is regular.

**Proof.** We will first prove that a strongly regular triple system \( T \) cannot contain any nonzero nilpotent element. Suppose \( s \in T \) and \( s \) is nilpotent. Then \( s^{(2p+1)} = 0 \) for some positive integer \( p \). Since \( T \) is strongly regular, there exists \( t \in T \) such that \( s = < s, s, t > \). Now \( s = < s, s, t > = < s, < s, s, t >, t > = < s^3, t, t > \) (by associativity). We will now prove that \( s = < s^{(2n+1)}, t^{(2n-1)}, t > \) by induction on \( n \). The result is true when \( n = 1 \). Now assume that \( s = < s^{(2n+1)}, t^{(2n-1)}, t > \). Then

\[
\begin{align*}
  s &= < < s^{(2n-1)}, s, s >, t^{(2n-1)}, t > \\
  &= < < s^{(2n-1)}, s, < s, s, t > >, t^{(2n-1)}, t > \quad \text{(note that } s = < s, s, t >) \\
  &= < s^{(2n+1)}, s, t^{(2n+1)} > \quad \text{(by associativity)}
\end{align*}
\]
\[ s^{(2n+1)}, < s, s, t >, t^{(2n+1)} > \quad \text{since} \quad s = < s, s, t > \]
\[ = s^{(2n+3)}, t^{(2n+1)}, t > \quad \text{(by associativity)} \]

We have thus proved that \( s = s^{(2n+1)}, t^{(2n-1)}, t > \) for all positive integers \( n \).
Since \( s^{(2p+1)} = 0 \), we conclude that \( s = 0 \). We will now come to the proof of the theorem. Since \( T \) is strongly regular, so is \( a \). Hence there exists an element \( x \) such that \( a = < a, a, x > \). Using associativity of \( T \), we can easily prove that \( a - < a, x, a > = 0 \). Therefore \( a - < a, x, a > \) is a nilpotent element of the strongly regular triple system \( T \). By what we have proved above, we conclude that \( a - < a, x, a > = 0 \) showing that \( a = < a, x, a > \). Similarly, we can prove that \( a = < x, a, a > \).

\textbf{Theorem 2.2.} For a nonzero regular element \( a \) of a triple system \( T \), the following statements are equivalent:

(i) \( a \) has a unique generalized inverse.

(ii) \( a \) is not a zero divisor.

(iii) There exist an element \( x \in T \) such that \( < x, a, t > = < t, a, x > = t \) for all \( t \).

\textit{Proof.} (i) \( \Rightarrow \) (ii) Let \( x \) be the unique generalized inverse of \( a \). Then \( a = U_a(x) = < a, x, a > \). Let \( t \in T \) such that \( U_a(t) = 0 \). Now \( U_a(x + t) = < a, x + t, a > = < a, x, a > + < a, t, a > = a \) since \( < a, t, a > = 0 \). Hence \( x + t \) is also a generalized inverse of \( a \). By the uniqueness of \( x \), we conclude that \( x + t = x \) so that \( t = 0 \). \( U_a \) is thus injective proving that \( a \) is not a zero divisor.

(ii) \( \Rightarrow \) (iii)

Since \( a \) is regular, we can find an element \( x \in T \) such that \( a = U_a(x) = < a, x, a > \). Let \( t \in T \) be arbitrary. Then Since \( a \) is not a zero divisor, \( U_a \) is injective so that \( t - < x, a, t > = 0 \) proving that \( t = < x, a, t > \). Similarly, we can prove that \( t = < t, a, x > \) for all \( t \in T \).

(iii) \( \Rightarrow \) (i)

Since \( a \) is regular, \( a \) has a generalized inverse say \( t \) so that \( a = U_a(t) = < a, t, a > \). By (iii), there exists an element \( x \in T \) such that \( < x, a, t > = < t, a, x > = t \). Now \( < x, a, x > = < x, < a, t, a >, x > = < x, a, t >, a, x > = < t, a, x > = t \). This proves that \( t \) is unique.

\textbf{Theorem 2.3.} For a non zero element \( a \) of a strongly regular triple system \( T \), the following statements are equivalent.
(i) $a$ has a unique generalized inverse.

(ii) $a$ is not a zero divisor.

(iii) $a$ is invertible.

Proof. (i)$\Rightarrow$(ii) Since $T$ is strongly regular, it follows from theorem that $a$ is regular so that by theorem, $a$ is not a zero divisor.

(ii)$\Rightarrow$(iii) Since $T$ is strongly regular, it follows from the theorem that there exists an element $x$ such that $a = < a, a, x > = < a, x, a > = < x, a, a >$. Also by the theorem, $< x, a, t > = < t, a, x > = t$ for all $t \in T$. Now, $U_a(t-< a, x, t >) = < a, t-< a, x, t >, a > = < a, t, a > - < a, < a, x, t >, a > = < a, t, a > - < < a, a, x >, t, a > = < a, t, a > - < a, t, a > = 0$. Since $a$ is not a zero divisor, $U_a$ is injective so that $t = < a, x, t >$. Similarly using $a = < x, a, a >$, we can prove that $t = < t, x, a >$. $x$ is thus the inverse of $a$.

(iii)$\Rightarrow$(i) Since by Theorem, $a$ is regular, (iii)$\Rightarrow$(i) follows from Theorem 2.2.

Theorem 2.4. $T$ is a division triple system if and only if every non zero element of $T$ is strongly regular with a unique generalized inverse.

Proof. If $T$ is a division triple system, then every nonzero element $a \in T$ has an inverse say $a^{(-1)}$. Hence $< a, a, a^{(-1)} > = a$ which proves that $a$ is strongly regular. Thus $T$ is a strongly regular triple system in which $a$ is invertible so that by Theorem 2.3, $a$ has a unique generalized inverse. Conversely, if every non zero element of $T$ is strongly regular with a unique generalized inverse, then $T$ is a strongly regular triple system so that by Theorem 2.3, every non zero element of $T$ is invertible. Hence $T$ is a division triple system.

Lemma 2.5. If $a$ is a regular element of $T$ with a generalized inverse $x$, then $< x, a, x >$ is a reflexive inverse of $a$.

Proof. $x$ is a generalized inverse of $a$ means $a = U_a(x) = < a, x, a >$. If $y = < x, a, x >$, then $U_a(y) = < a, y, a > = < a, < x, a, x >, a > = < < a, a, x >, x, a > = a$. Again $U_y(a) = < y, a, y > = < < x, a, x >, a >, < x, a, x > > = < x, a, < x, a, x > > = < x, < a, x, a >, x > = < x, a, x > = y$. This shows that $y$ is a reflexive inverse of $a$. We call an element $x \in T$ a strong reflexive inverse of $a$ if $< a, a, x > = < a, x, a > = < x, a, a > = a$ and $< x, x, a >= < x, a, x > = < a, x, x > = x$. 

Theorem 2.6. If \(a\) is an element of \(T\) with a unique reflexive inverse \(x\), then \(<a, a, x> = <x, a, a>\) and \(<a, x, x> = <x, x, a>\).

Proof. Since \(x\) is a reflexive inverse of \(a\), we have \(U_a(x) = a\) and \(U_x(a) = x\). For \(y \in T\), we have \(U_a(x + y - <x, a, y>) = <a, x + y - <x, a, y>, a>\) = \(<a, x, a> + <a, y, a> - <a, <x, a, y>, a> = U_a(x) + U_a(y) - <a, x, a>, y, a > = U_a(x) + U_a(y) - U_a(y)\) since \(<a, x, a> = U_a(x) = a\) = \(U_a(x) = a\). Also \(U_a(x + y - <y, a, x>) = <a, x + y - <y, a, x>, a> = <a, x, a> + <a, y, a> - <a, <y, a, x>, a> = U_a(x) + U_a(y) - <a, y, <a, x, a>>, a > = U_a(x) + U_a(y) - \(<x, a, y, x >, a > = U_a(x) + U_a(y) = U_a(x) = a\). Thus \(x + y - <x, a, y >)\) and \(x + y - <y, a, x >)\) are both reflexive inverses of \(a\). From the Lemma, it follows that \(<x + y - <x, a, y >, a + y - <x, a, y >)\) and \(<x + y - <y, a, x >, a + y - <y, a, x >)\) are both reflexive inverses of \(a\). Since \(x\) is the unique reflexive inverse of \(a\), we have \(x = <x + y - <x, a, y >, a + x - <x, a, y >)\). Using associativity of \(T\), \(<x, a, x > = x\) and \(<a, x, a > = a\) and simplifying, we obtain \(x = x + <y, a, x > - <x, a, y >, a > so that \(<y, a, x > = <x, a, y >, a > \) for all \(y \in T\). Similarly, using the fact that \(<x + y - <y, a, x >, a + x - <y, a, x >)\) is a reflexive inverse of \(a\), we can prove that \(<x, a, y > = <x, a, <y, a, x >)\) for all \(y \in T\). Thus for all \(y \in T\),

\[
<x, a, y >= <y, a, x>
\]

(2.1)

Taking \(y = a\) in 2.1, we obtain \(<x, a, a > = <a, a, x >\). Again, taking \(y = <x, a, x >\) in 2.1, we obtain \(<x, a, <x, a, x >) = <x, x, a >, a > \) i.e \(<x, a, x >, x, a > = <x, x, a >, a >\). Since \(<x, a, x > = x\), we have

\[
<x, a, x > = <x, x, a >, a > = <x, x, a >, x >
\]

(2.2)

Taking \(y = <a, a, x >\) in 2.1, we obtain \(<x, a, <a, a, x >, x > = <a, a, x >, x > = <a, a, x >, x >\). i.e \(<a, a, x >, x > = <a, a, x >\) since \(<x, a, x > = x\). Since we have already proved that \(<a, a, x > = <x, a, a >\), this implies that

\[
<x, <a, a >, x > = <a, x, x >
\]

(2.3)

From 2.2 and 2.3 we obtain \(<x, a, x > = <a, x, x >\). This completes the proof of the theorem.

\[
\square
\]

Theorem 2.7. Let \(a\) be an element of \(T\) with a reflexive inverse \(x\) such that \(<x, a, x > = x\). Further, let \(x\) be the unique generalized inverse of \(a\). Then \(a\) is invertible.
Proof. Since $x$ is a reflexive inverse of $a$, we have $a = U_a(x) = < a, x, a >$ and $x = U_x(a) = < x, a, x >$. Since $x$ is the unique generalized inverse of $a$, $x$ is also the unique reflexive inverse of $a$. Hence it follows from Theorem 2.6 that $< a, a, x > = x, a, a >$ and $< a, x, x > = < x, x, a >$. This together with the hypothesis gives $< a, x, x > = < x, x, a > = < x, a, x > = x$. Now for any $t \in T$, we have $< a, x, t > = < a, x, a >, t > = < a, < x, a, x >, t > = < a, < a, x, a >, t >$.

Similarly, we can prove that $< t, a, x > = < t, x, a >$ for all $t \in T$. Now by Theorem 2.2, $< x, a, t > = < t, x, a >$ for all $t \in T$ so that $< a, x, t > = < x, a, t > = < t, a, x > = < t, x, a > = t$ for all $t \in T$ proving that $a$ is invertible with inverse $x$.

Theorem 2.8. Let $a$ be an element of $T$ possessing a reflexive inverse. Furthermore, let $< a, x, t > = < x, a, t >$ and $< t, a, x > = < t, x, a >$ for all $t \in T$ and for all generalized inverses $x$ of $a$. Then $a$ has a unique reflexive inverse.

Proof. Let $y$ and $z$ both be reflexive inverses of $a$. Then $a = < a, y, a > = < a, z, a >$, $y = < y, a, y >$ and $z = < z, a, z >$. Now $< a, y, t > = < y, a, t >$ (by hypothesis) $= < a, y, z >, t > = < y, a, z >, a, t > = < a, y, z >, a, t >$ (by hypothesis since $y$ is a generalized inverse) $= < a, y, z, t >$ (again by hypothesis since $z$ is a generalized inverse) $= < a, y, z, a, t >$. We have thus proved that $< a, y, t > = < a, z, t > = < z, a, t >$ for all $t \in T$. Similarly, we can prove that $< t, a, y > = < t, y, a > = < t, a, z > = < t, z, a >$ for all $t \in T$. Now $y = < y, a, y > = < z, a, y >$ (since by above, $< y, a, t > = < z, a, t >$ for all $t \in T$) $= < z, a, z >$ (since $< t, a, y > = < t, a, z >$ for all $t \in T$) $= z$. This proves that reflexive inverse of $a$ is unique.

References
