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THE MONOPHONIC GRAPHOIDAL COVERING NUMBER OF A GRAPH

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Abstract: A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A monophonic graphoidal cover of a graph G is a collection ψ_m of monophonic paths in G such that every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The minimum cardinality of a monophonic graphoidal cover of G is called the monophonic graphoidal covering number of G and is denoted by η_m. We determine bounds for it and characterize graphs which realize these bounds. Also, for any positive integer n with $q - p + 2 \leq n \leq q - 1$, there exists a tree T such that the monophonic graphoidal covering number is n.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary[6]. The concept of graphoidal cover was introduced by Acharya and Sampathkumar[2] and further studied in [1, 3 ,7 ,8].

A *graphoidal cover* of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions.

(i) Every path in ψ has at least two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in ψ.

(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the *graphoidal covering number* of G and is denoted by $\eta(G)$.

The collection ψ is called an *acyclic graphoidal cover* of G if no member of ψ is cycle; it is called a *geodesic graphoidal cover* if every member of ψ is a shortest path in G. The minimum cardinality of an acyclic (geodesic) graphoidal cover of G is called the *acyclic (geodesic) graphoidal covering number* of G and is denoted by $\eta_a(\eta_g)$. The acyclic graphoidal covering number and geodesic graphoidal covering number are studied in [4,5].

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ *of a vertex* v *in* G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius* is $rad_m(G) = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter* is $diam_m(G) = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [9,10].

The following theorems will be used in the sequal.

Theorem 1.1. [6] Every non-trivial connected graph has at least two vertices which are not cut vertices.

Theorem 1.2. [6] Let G be a connected graph with at least three vertices. The following statements are equivalent:

(i) G is a block

(ii) Every two vertices of G lie on a common cycle.

Theorem 1.3. [5] Let $K_{m,n} (1 \le m \le n)$ be a bipartite graph. Then

$$\eta_g(K_{m,n}) = \begin{cases} 1 & if\ m=1,\ n=1 \\ n-1 & if\ m=1,\ n\geq 2 \\ n & if\ m=2,\ n\geq 2 \\ m+n-1 & if\ m=3,\ n=3,4 \\ m+n & if\ m=3,\ n=5 \\ 2n-3 & if\ m=3,\ n\geq 6 \\ mn-m-n & if\ m,n\geq 4. \end{cases}$$

Throughout this paper G denotes a connected graph with at least two vertices.

2. Monophonic Graphoidal Cover

Definition 2.1. A *monophonic graphoidal cover* of a graph G is a collection ψ_m of monophonic paths in G such that every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The minimum cardinality of a monophonic graphoidal cover of G is called the *monophonic graphoidal covering number* of G and is denoted by $\eta_m(G)$.

Example 2.2. For the graph G given in Figure 2.1, $\psi_m = \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7), (v_3, v_{10}, v_1, v_8, v_7, v_9, v_5)\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = 2$.

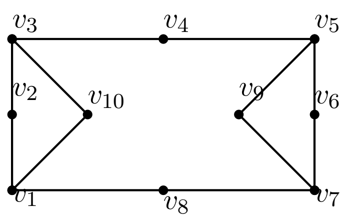

Figure 2.1: G

Theorem 2.3. *For any connected graph G, $\eta(G) \leq \eta_a(G) \leq \eta_m(G) \leq \eta_g(G)$.*

Proof. Since any acyclic graphoidal cover is a graphoidal cover and any monophonic graphoidal cover is an acyclic graphoidal cover, we have $\eta(G) \leq \eta_a(G) \leq \eta_m(G)$. Also, since every geodesic is a monophonic path, we have every geodesic graphoidal cover is a monophonic graphoidal cover and so $\eta_m(G) \leq \eta_g(G)$. Hence $\eta(G) \leq \eta_a(G) \leq \eta_m(G) \leq \eta_g(G)$. □

Remark 2.4. For the graph K_2, $\eta(K_2) = \eta_a(K_2) = 1$, for the cycle C_5, $\eta_a(C_5) = \eta_m(C_5) = 2$, for the cycle C_3, $\eta_m(C_3) = \eta_g(C_3) = 3$. Further, for a tree T, $\eta(T) = \eta_a(T) = \eta_m(T) = \eta_g(T) = n - 1$, where n is the number of end vertices of T. All the inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 2.2, $\eta(G) = 2, \eta_a(G) = 3, \eta_m(G) = 4$ and $\eta_g(G) = 5$. Thus we have $\eta(G) < \eta_a(G) < \eta_m(G) < \eta_g(G)$.

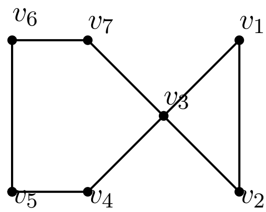

Figure 2.2: G

Since $q-p \leq \eta_a(G) \leq q$ and $\eta_a(G) \leq \eta_m(G) \leq q$, we have $q-p \leq \eta_m(G) \leq q$. Now, we proceed to characterize graphs G for which the bounds of $\eta_m(G)$ are attained.

For any monophonic graphoidal cover ψ_m of a graph G, let t_{ψ_m} denote the number of vertices of G which are not internal vertices of any path in ψ_m. Let $t_m = min\ t_{\psi_m}$, where the minimum is taken over all graphoidal covers of G.

Theorem 2.5. *For any graph G, $\eta_m(G) = q - p + t_m$.*

Proof. Let ψ_m be any monophonic graphoidal cover of G. Then $q = \sum_{P \in \psi_m} |E(P)| = |\psi_m| + \sum_{P \in \psi_m} t_m(P) = |\psi_m| + p - t_{\psi_m}$. Therefore $|\psi_m| = q-p+t_{\psi_m}$. Since $\eta_m(G)$ is the minimum cardinality of a monophonic graphoidal cover of G, we have $\eta_m(G) = q - p + t_m$. □

Corollary 2.6. *Let T be a tree with n pendant vertices. Then $\eta_m(T) = n - 1$.*

Corollary 2.7. *Let G be a graph having n simplicial vertices. Then $\eta_m(G) \geq q - p + n$. Furthermore, equality holds if and only if there exists a monophonic graphoidal cover ψ_m of G such that every non-simplicial vertex of G is an internal vertex of a unique monophonic path in ψ_m.*

The following proposition is the characterization result of the lower bound of $\eta_m(G)$ and it follows from Corollary 2.7.

Proposition 2.8. *For any connected graph G of order at least 3, $\eta_m(G) = q - p$ if and only if G has no simplicial vertices and there exists a monophonic graphoidal cover ψ_m such that every vertex of G is an internal vertex of a unique monophonic path in ψ_m.*

Theorem 2.9. *For any connected graph G, $\eta_m(G) = q$ if and only if G is complete.*

Proof. Let G be a complete graph. Since any two vertices of G are adjacent, the length of any monophonic path is one. Hence $E(G)$ is the unique monophonic graphoidal cover of G and so $\eta_m(G) = q$.

Conversely, suppose that $\eta_m(G) = q$. Claim that G is complete. If G is not complete, then there exists a monophonic path, say P, in G such that $|E(P)| > 1$. Then $\psi_m = \{E(G) - E(P)\} \cup \{P\}$ is a monophonic graphoidal cover of G and so $\eta_m(G) \leq q - 1$, which is a contradiction. □

Theorem 2.10. *For any connected graph G of order $p \geq 3$, $\eta_m(G) = q - 1$ if and only if $G = K_1 + \cup\, m_j K_j$, where $\sum m_j \geq 2$.*

Proof. Let $\eta_m(G) = q - 1$. Since $p \geq 3$, by Theorem 1.1 there exists a vertex x, which is not a cut vertex of G. If G has two or more cut vertices, then let P be a monophonic path containing at least two cut vertices. Then $|E(P)| \geq 3$. Clearly, $\psi_m = \{E(G) - E(P)\} \cup \{P\}$ is a monophonic graphoidal cover of G and so $\eta_m(G) \leq |\psi_m| = q - 2$, which is a contradiction. Thus the number of cut vertices k of G is at most one.

Case (i): If $k = 0$, then the graph G is a block. If $p = 3$, then $G = K_3$ and so by Theorem 2.9, $\eta_m(G) = q$, which is a contradiction to the assumption. If $p \geq 4$, we claim that G is complete. Suppose that G is not complete. Then there exists two vertices x and y in G such that $d(x, y) \geq 2$. By Theorem 1.2, x and y lie on a common cycle and hence x and y lie on a smallest cycle $C = x, x_1, x_2, ..., y, ..., x_n, x$ of length at least 4. Clearly, all the edges of C lie on either an $x - y$ monophonic path, say P_1, or an $y - x$ monophonic path, say P_2. Then $\psi_m = \{E(G) - E(C)\} \cup \{P_1, P_2\}$ is a monophonic graphoidal cover of G and so $\eta_m(G) \leq q - 2$, which is a contradiction. Hence G is complete and so by Theorem 2.9, $\eta_m(G) = q$, which is again a contradiction. Thus $k \neq 0$.

Case (ii): If $k = 1$, let x be the cut vertex of G. If $p = 3$, then $G = P_3 = K_1 + \cup\, m_j K_1$ where $\sum m_j = 2$. If $p \geq 4$, we claim that $G = K_1 + \cup\, m_j K_j$, $\sum m_j \geq 2$. It is enough to prove that every block of G is complete. Suppose that there exists a block B, which is not complete. Let u and v be two vertices in B such that $d(u, v) \geq 2$. Then as in Case (i), $\eta_m(G) \leq q - 2$, which is a contradiction. Thus every block of G is complete so that $G = K_1 + \cup\, m_j K_j$, where K_1 is the vertex x and $\sum m_j \geq 2$. □

Theorem 2.11. *For any cycle C_p $(p \geq 4)$, $\eta_m(C_p) = 2$.*

Proof. Let $C_p : v_1, v_2, v_3, ..., v_p, v_1$ be a cycle of order p. Then $\psi_m = \{(v_1, v_2, v_3),\ (v_3, v_4, ..., v_p, v_1)\}$ is a minimum monophonic graphoidal cover of C_p and hence $\eta_m(C_p) = 2$. □

Since every monophonic path in $K_{m,n}$ is a geodesic, we have the following result by Theorem 1.3.

Theorem 2.12. *Let $K_{m,n}(1 \leq m \leq n)$ be a bipartite graph. Then*

$$\eta_m(K_{m,n}) = \begin{cases} 1 & if\ m = 1,\ n = 1 \\ n-1 & if\ m = 1,\ n \geq 2 \\ n & if\ m = 2,\ n \geq 2 \\ m+n-1 & if\ m = 3,\ n = 3, 4 \\ m+n & if\ m = 3,\ n = 5 \\ 2n-3 & if\ m = 3,\ n \geq 6 \\ mn-m-n & if\ m, n \geq 4. \end{cases}$$

Theorem 2.13. *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G having length greater than 3 and let k be the number of vertices of degree greater than 2 on C. Then*

$$\eta_m(G) = \begin{cases} 2 & if\, k = 0 \\ n & if\ \text{there exists two non} - \text{adjacent vertices} \\ & \quad \text{of degree} > 2\, \text{on}\ C\, \text{(or) all vertices in}\ C\, \text{are of} \\ & \quad \text{degree} > 2 \\ n+1 & \quad \text{otherwise.} \end{cases}$$

Proof. Let $C : v_0, v_1, v_2, ..., v_l, v_0$ be the unique cycle in G having length greater than 3.

Case (i) : $k = 0$. Then $G = C$ and by Theorem 2.11, $\eta_m(G) = 2$.

Case (ii): $k = 1$. Let v_0(say) be the unique vertex of degree greater than 2 on C. Let $G' = G - \{v_1\}$. Then G' is a tree with $n + 1$ pendant vertices and hence by Corollary 2.6, $\eta_m(G') = n$. Let ψ'_m be a minimum monophonic graphoidal cover of G'. Clearly any path in ψ'_m is a monophonic path in G, we have $\psi_m = \psi'_m \cup \{(v_0, v_1, v_2)\}$ is a monophonic graphoidal cover of G. Hence $\eta_m(G) \leq n + 1$.

Also, at least one vertex on C and all the n pendant vertices are exterior vertices of any minimum monophonic graphoidal cover of G, we have $t_m \geq n+1$. Then by Theorem 2.5, $\eta_m(G) = q - p + t_m \geq n + 1$. Hence $\eta_m(G) = n + 1$.

Case (iii): $k = 2$ and the vertices of degree greater than 2 on C are adjacent in G.

Let v_0, v_1 be vertices of degree greater than 2 on C. Let $P = (v_1, v_2, v_3)$ be a $v_1 - v_3$ monophonic path in G. Let G' be the subgraph obtained by deleting v_2 from G. Clearly G' is a tree with $n+1$ pendant vertices and hence by Corollary 2.6, $\eta_m(G') = n$. If ψ'_m is a minimum monophonic graphoidal cover of G', then $\psi'_m \cup \{P\}$ is a monophonic graphoidal cover of G and hence $\eta_m(G) \leq n+1$. Also, at least one vertex on C and all the n pendant vertices are exterior vertices of any minimum monophonic graphoidal cover of G, we have $t_m \geq n + 1$. Then by Theorem 2.5, $\eta_m(G) \geq n + 1$. Hence $\eta_m(G) = n + 1$.

Case (iv): $k \geq 2$ and there exists two non-adjacent vertices of degree greater than 2 on C.

Let u, v be vertices of degree greater than 2 on C such that all vertices in a $(u - v)$- section of C other than u, v have degree 2. Let P denote this (u, v) - section and let G' be the subgraph obtained by deleting all the internal vertices of P. Clearly G' is a tree with n pendant vertices and hence by Corollary 2.6, $\eta_m(G') = n - 1$. If ψ'_m is a minimum monophonic graphoidal cover of G', then $\psi'_m \cup \{P\}$ is a monophonic graphoidal cover of G and hence $\eta_m(G) \leq n$. Also, since G has n pendant vertices, $t_m \geq n$ so that $\eta_m(G) = n$.

Case (v): $k \geq 3$ and all the vertices of C are of degree greater than 2.

Let $H = G - \{v_1v_2, v_2v_3\}$. Let H' and H'' be the components of H with H' contain the vertices

v_1,v_3 and H'' contains the vertex v_2. Let r be the number of pendant vertices in H' and let s be the number of pendant vertices in H''. Since any pendant vertex of H' or H'' is a pendent vertex of G, we have $n = r + s$. Let $G' = H'$ and $G'' = H'' \cup \{v_1v_2, v_2v_3\}$. Then G' contains r pendant vertices and G'' contains $s + 2$ pendant vertices. Clearly G' and G'' are trees and hence by Corollary 2.6, $\eta_m(G') = r - 1$ and $\eta_m(G'') = s + 1$. Let ψ'_m be a minimum monophonic graphoidal cover of G' and let ${\psi_m}''$ be a minimum monophonic graphoidal cover of G''. Then $\psi'_m \cup \psi''_m$ is a monophonic graphoidal cover of G and hence $\eta_m(G) \leq r - 1 + s + 1 = n$. Also, since G has n pendant vertices, $t_m \geq n$ so that $\eta_m(G) = n$. □

We have seen that if G is a connected graph of order $p \geq 3$, then $q - p \leq \eta_m(G) \leq q$. Also we have $\eta_m(G) = q - p$ if and only if G has no simplicial vertices and there exists a monophonic graphoidal cover ψ_m such that every vertex of G is an internal vertex of a unique monophonic path in ψ_m and $\eta_m(G) = q$ if and only if G is complete. Also, it is proved that $\eta_m(G) = q-1$ if and only if $G = K_1 + \cup\, m_j K_j$, where $\sum m_j \geq 2$. In the following theorem, we give an improved bounds for the monophonic graphoidal covering number of a graph interms of its size and monophonic diameter.

Theorem 2.14. *For any connected graph G of order $p \geq 2$, $\lceil q/d_m \rceil \leq \eta_m(G) \leq q - d_m + 1$, where d_m is the monophonic diameter of G.*

Proof. Let ψ_m be a minimum monophonic graphoidal cover of G. Since every edge of G is in exactly one monophonic path in ψ_m, we have $q = \sum_{P\in\psi_m} |E(P)|$. Since $|E(P)| \leq d_m$ for each P in ψ_m, we have $q \leq \eta_m(G).d_m$. Hence $\eta_m(G) \geq \lceil q/d_m \rceil$. Let Q be a monophonic diametral path of G. It is clear that $\{(E(G) - E(Q)) \cup Q\}$ is a monophonic graphoidal cover of G. Hence $\eta_m(G) \leq |E(G) - E(Q)| + 1 = q - d_m + 1$. □

Now we give a realization result for the monophonic graphoidal covering number with some suitable conditions.

Theorem 2.15. *For any positive integer n with $q-p+2 \leq n \leq q-1$, there exists a tree T such that the monophonic graphoidal covering number is n. .*

Proof. Let $P : v_1, v_2, v_3, ..., v_{q-n+2}$ be a path of order $q-n+2$. Let T be a tree obtained from P by adding $n-1$ new vertices $u_1, u_2, ...u_{n-1}$ and joining each vertex $u_i (1 \leq i \leq n-1)$ to the vertex v_{q-n+1}. The tree T is given in Figure 2.3 and it has $n+1$ pendant vertices. Then by Corollary 2.6, $\eta_m(T) = n$. □

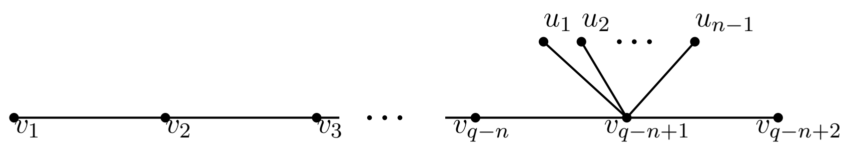

Figure 2.3: T

Remark 2.16. In a tree T, $q = p-1$ and so $q-p$, $q-p+1$ are non-positive numbers. Hence there does not exist a tree T whose monophonic graphoidal covering number is either $q-p$ or $q-p+1$. Also, by Theorem 2.9, $\eta_m(G) = q$ if and only if G is complete. Thus there does not exist a tree with the monophonic graphoidal covering number is $q-p$ or $q-p+1$, or $q-1$.

Problem 2.17. For any positive integer n with $q - p \leq n \leq q$, does there exist a connected graph G such that G is not a tree and the monophonic graphoidal covering number is n?.

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