International Journal of Pure and Applied Mathematics
Volume 96 No. 1 2014, 105-115
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijjpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v96i1.8
ijpam.eu

ON THE STRATIFICATION OF THE PROJECTIVE SPACE BY THE X-RANK FOR A CERTAIN CONFIGURATION X OF RATIONAL NORMAL CURVES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$, $m \ge 2$, $N := \binom{m+d}{m} - 1$, denote the Veronese embedding. Fix $O \in \mathbb{P}^m$ and let T_m be the union of m lines of $\mathbb{P}^m$ passing through O and with $\langle T_m \rangle = \mathbb{P}^m$, where $\langle\ \rangle$ denote the linear span (T_m is an angle or a coordinate frame). Set $T_{m,d} := \nu_d(T_m)$. In this note we study the stratification by $T_{m,d}$-rank of $\langle T_{m,d} \rangle$. We give a sharp upper bound for this rank, prove a concision result (in a precise quantitative way) and study some cases with high rank.

AMS Subject Classification: 14N05
Key Words: symmetric tensor rank, reducible curve, border rank, rational normal curve

1. Introduction

For all positive integers m, d let $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, denote the Veronese embedding induced by $|\mathcal{O}_{\mathbb{P}^m}(d)|$. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$.

Fix an integer $m \ge 2$, $O \in \mathbb{P}^m$ and m lines $L_1, \dots, L_m \subset \mathbb{P}^m$ such that $O \in L_i$ for all i and $\langle L_1 \cup \cdots \cup L_m \rangle = \mathbb{P}^m$, where $\langle\ \rangle$ denote the linear span. Set

Received: July 8, 2014

$T_m := L_1 \cup \cdots \cup L_m$ and $T_{m,d} := \nu_d(T_m)$. The curves T_m and $T_{m,d}$ do not depend upon the choice of O and the lines $L_1, \dots, L_m$, up to a projective equivalence. The curve T_m is reduced, connected, $p_a(T_m) = 0$ and its unique singular point, O, is a seminormal singularity with multiplicity m (T_2 is a reducible conic). We will say that T_m is an *angle* or a *coordinate frame*. The curve $\nu_d(T_m)$ has degree dm, each $\nu_d(L_i)$ is a degree d rational normal curve in its linear span $\langle \nu_d(L_i)$ and $\dim(\langle T_{m,d}\rangle) = md$. For each $P \in \langle T_{m,d}\rangle$, its $T_{m,d}$ rank $r_{T_{m,d}}(P)$ is the minimal cardinality of a set $S \subset T_m$ such that $P \in \langle \nu_d(S)\rangle$. For each $P \in \mathbb{P}^N$ the symmetric tensor rank $r_{m,d}(P)$ of P is the minimal cardinality of a subset $S \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(S)\rangle$. If $P \in \langle T_{m,d}\rangle$, then both $r_{m,d}(P)$ and $r_{T_{m,d}}(P)$ and obviously $r_{m,d}(P) \le r_{T_{m,d}}(P)$. In the (omitted) case $m = 1$ concision says that equality holds ([7]) In the case $m \ge 2$ easy examples shows that often strict inequality holds (Example 2), but in some interesting cases equality holds (Proposition 4).

Let $2O$ be the closed subscheme of $\mathbb{P}^m$ with $(\mathcal{I}_O)^2)$ as its ideal sheaf. We have $2O \subset T_m$ and $\langle \nu_d(2O)\rangle$ is an m-dimensional linear subspace of $\mathbb{P}^N$ which is the tangent space of the Veronese variety $X_{m,d}$ at O. For each closed subscheme $U \subseteq \mathbb{P}^m$ set $\{2O, U\} := 2O \cap U$.

Remark 1. Fix $P \in \langle T_{m,d}\rangle$ such that $P \notin \langle \nu_d(T)\rangle$ for any proper subcurve T of T_m. There are $P_i \in \langle \nu_d(L_i)\rangle \setminus \{\nu_d(O)\rangle$ such that $P \in \langle \{P_1, \dots, P_n\}\rangle$. The points $P_1, \dots, P_n$ are not unique, but if $Q_1, \dots, Q_n$ are another choice, then $Q_i \in \langle \{\nu_d(O), P_i\}\rangle$. Hence P_i is contained in the tangent line $\langle \nu_d(2O, L_i\}\rangle$ of $\nu_d(L_i)$ if and only if Q_i is contained in the tangent line $\langle \nu_d(2O, L_i\}\rangle$ of $\nu_d(L_i)$. Hence the number $\rho(P)$ of such points P_i depends only from P. Now take any $P \in \langle T_{m,d}\rangle \setminus \{\nu_d(O)\}$ and call T a minimal subcurve of T_m such that $P \in \langle \nu(T)\rangle$. If $T \ne T_m$, then define $\rho(P)$ using this curve $T \cong T_x$, $1 \le x < m$. In particular, if $\rho(P) = m$, then $P \notin \langle \nu_d(T)\rangle$ for any proper subcurve T of T_m.

Theorem 1. *Assume $d \ge 3$. For any $P \in \langle T_{m,d}\rangle$ we have $r_{T_{m,d}}(P) \le m(d-1) + 1 + \min\{n-1, \rho(P)-1\}$. If $d \ge 5$, then $r_{T_{m,d}}(P) = md$ if and only if $\rho(P) = m$.*

For points $P \in \langle \nu_d(2O)\rangle$, see Proposition 2.

We also consider a scheme $Z \subset T_m$ with $Z \supset 2O$ and $\deg(Z) = \dim(2O)+1$, give an upper bound for a point $P \in \langle \nu_d(Z)\rangle \setminus \langle \nu_d(2O)\rangle$ (Proposition 3) and prove that if $m = 3$, then this upper bound $3d-3$ is the exact value both of $r_{T_{3,d}}(P)$ and $r_{3,d}(P)$ (Proposition 4). A kind of concision with very precise bounds holds for a proper subscurve $T \subsetneq T_m$: if $P \in \langle \nu_d(T)\rangle$ and $S \subset T_m$ is a finite subset such that $P \in \langle \nu_d(S)\rangle \setminus \langle \nu_d(S \cap T)\rangle$, then $\sharp(S) \ge r_{T_{m,d}}(P) + d$ (Proposition 1). Easy examples shows that sometimes this inequality is sharp

(Example 1).

2. The proofs

Linear algebra (or Mayer-Vietoris exact sequences obtained from decompositions like $T_m = T_{m-1} \cup L_m$) gives the following lemma.

Lemma 1. *Fix $I, J \subseteq \{1, \dots, m\}$ such that $I \neq \emptyset$ and $J \neq \emptyset$. If $I \cap J = \emptyset$, then $\langle \nu_d(\cup_{i \in I} L_i)\rangle \cap \langle \nu_d(\cup_{i \in J} L_i)\rangle = \nu_d(O)$. If $I \cap J \neq \emptyset$, then $\langle \nu_d(\cup_{i \in I} L_i)\rangle \cap \langle \nu_d(\cup_{i \in J} L_i)\rangle = \langle \nu_d(\cup_{i \in I \cap J} L_i)\rangle$.*

Lemma 2. *Fix any $S \subset T_m$ such that $\nu_d(O) \in \langle \nu_d(S)\rangle$ and $O \notin S$. Then $\sharp(S) \geq d+1$ and if $\sharp(S) \leq 2d$, then there is a line $L \subset T_m$ such that $\sharp(S \cap L) \geq d+1$.*

Proof. Taking if necessary a smaller S we may assume that $\nu_d(S)$ is linearly independent and in particular that no $d+2$ of the points of S are collinear. Since $O \notin S$, [4, Lemma 34] gives $\sharp(S \cup \{O\}) \geq d+2$ and that if $\sharp(S \cup \{O\}) \leq 2d+1$, then there is a line $L \subset \mathbb{P}^m$ such that $\sharp((S \cup \{O\}) \cap L) \geq d+2$. Since $\sharp(S \cap L) \geq 3$ and T_m is cut out by quadrics, $L = L_i$ for some i. □

The first part of the next result is related to the concision property for multivariate polynomials ([7, Exercise 3.2.2.2]).

Proposition 1. *Fix $P \in \langle T_{m,d}\rangle$ and assume the existence of a curve $T \subsetneq T_m$ such that $P \in \langle \nu_d(T)\rangle$. Then:*

1. *each subset of T_m evincing $r_{T_{m,d}}(P)$ is contained in T, i.e. $S \subset T$ for each $S \subset T_m$ such that $\sharp(S) = r_{T_{m,d}}(P)$ and $P \in \langle \nu_d(S)\rangle$;*

2. *take any finite $S \subset T_m$ such that $P \in \langle \nu_d(S)\rangle$ and $P \notin \langle \nu_d(S \cap T)\rangle$. Then $\sharp(S) - \sharp(S \cap T) \geq d+1$ and $\sharp(S) \geq r_{T_{m,d}}(P) + d$.*

Proof. It is sufficient to prove part (2). Take any $S \subset T_m$ such that $P \in \langle \nu_d(S)\rangle$ and $P \notin \langle \nu_d(S \cap T)\rangle$. Write $T_m = T \cup T'$ with T' the union of the lines of T_m not contained in T. We have $T \cap T' = \{O\}$. We have $P \in \langle \langle \nu_d(S \cap T')\rangle \cup \langle \nu_d(T \cap S)\rangle\rangle$. Since $\langle \nu_d(T')\rangle \cap \langle \nu_d(T)\rangle = \{\nu_d(O)\}$ and $P \notin \langle \nu_d(S \cap T)\rangle$, we get $O \notin S$, $P \in \langle \{O\} \cup (S \cap T)\rangle$ and $O \in \langle \nu_d(T \cap S)\rangle$. Hence it is sufficient to prove that $\sharp(T' \cap S) \geq d+1$. Apply Lemma 2. □

Example 1. The following example shows that part (2) of Proposition 1 is sharp. Take any $P \in \langle T_{m-1,d}\rangle$ and any $S \subset T_m$ such that $P \in \langle \nu_d(S)\rangle$

and $P \notin \langle \nu_d(S \cap T_{m-1})\rangle$. Since $P \in \langle\langle(\nu_d(S \cap L_m)\rangle \cup \langle \nu_d(S \cap T_{m-1})\rangle$, $P \notin \langle \nu_d(S \cap T_{m-1})\rangle$ and $\langle \nu_d(L_m)\rangle \cap \langle \nu_d(T_{m-1})\rangle = \{O\}$ (Lemma 1), we get $O \notin S$, $P \in \langle \nu_d(\{O\} \cup (S \cap T_{n-1})\rangle$ and $O \in \langle \nu_d(L_n \cap S)\rangle$. Since $O \notin L_n \cap S$, we get $\sharp(L_n \cap S) \geq d+1$. Fix any $A \subset L_n$ such that $O \notin A$ and $\sharp(B) = d+1$. For any $B \subset T_{n-1}$ such that $O \in B$, $P \in \langle \nu_d(B)\rangle$ and $P \notin \langle \nu_d(B')\rangle$ for any $B' \subsetneq B$. Set $E := A \cup (B \setminus \{O\})$. We have $P \in \langle \nu_d(E)\rangle$ and $\sharp(E) = d + \sharp(B)$. Part (2) of Proposition 1 and Lemma 1 give $P \notin \nu_d(E')\rangle$ for any $E' \subsetneq E$.

Proposition 2. *Assume $d \geq 3$. For each $P \in \langle \nu_d(2O)\rangle \setminus \{\nu_d(O)\}$ let $E_P \subseteq \{1, \dots, m\}$ be a minimal subset such that $P \in \langle \cup_{i \in E_P}\{2O, L_i\}\rangle$. We have $\rho(P) = \sharp(E_P)$, $r_{T_{m,d}}(P) = d\rho(P)$ and $O \notin S$ for each finite subset $S \subset T_m$ with $\sharp(S) = d\rho(P)$ and $P \in \langle \nu_d(S)\rangle$.*

Proof. For any $I \subseteq \{1, \dots, n\}$, $I \neq \emptyset$, the linear space $\langle \nu_d(2O)\rangle \cap \langle \nu_d(\cup_{i\in I} L_i)\rangle$ is spanned by the lines $\langle \nu_d(\{2O, L_i\})\rangle$, $i \in I$ spanned by degree two zero-dimensional scheme $\nu_d(\{2O, L_i\})$. Therefore the set E_P is unique by Lemma 1 and hence the integer $\sharp(E_P)$ is well-defined. Obviously $\sharp(E_P) = \rho(P)$. If $\rho(P) = 1$, then Proposition 2 is true by a theorem of Sylvester ([4, Theorem 23], [8, Theorem 5.1]) and concision ([7, Exercise 3.2.2.2]). Hence we may assume $\rho(P) \geq 2$. By Proposition 1 it is sufficient to do the case $\rho(P) = m$. Since $\langle \nu_d(2O)\rangle$ is the linear span of the lines $\langle \nu_d(\{2O, L_i\})\rangle$, $i = 1, \dots, n$, and $\rho(P) = m$, there are points $P_i \in \langle \nu_d(\{2O, L_i\})\rangle \setminus \{\nu_d(O)\}$, $i = 1, \dots n$, such that $P \in \langle\{P_1, \dots, P_n\}\rangle$. For each P_i there is $S_i \subset L_i$ such that $P \in \langle S_i\rangle$ (e.g. by the quoted theorem of Sylvester or by [8, Proposition 4.1]). Hence $r_{T_{m,d}}(P) \leq md$. Hence it is sufficient to prove the opposite inequality and that no subset of T_m evincing $r_{T_{m,d}}(P)$ contains O. Set $Z_i := \{2O, L_i\}$. Since $d \geq 3$, the scheme Z_i is the only degree two zero-dimensional subscheme of $\mathbb{P}^m$ such that $P_i \in \langle \nu_d(Z_i)\rangle$. Fix a set $S' \subset T_m$ such that $\sharp(S') \leq md - 1$, $O \notin S'$ and set $S := \{O\} \cup S'$. Increasing if necessary S' we may assume $\sharp(S') = md - 1$.

Assume for the moment $\sharp(S' \cap L_i) \leq d$ for all $i = 1, \dots, n$. There is a unique index $j \in \{1, \dots, n\}$ such that $\sharp(S' \cap L_j) = m - 1$. Write $T := \cup_{i\neq j} T_i$, $S_2 := S' \cap L_j$ and $S_1 := S \cap T$. We have $O \in S_1$, $S_1 \cap S_2 = \emptyset$ and $\langle \nu_d(S_1)\rangle = \langle \nu_d(T)\rangle$. We have $\langle \nu_d(S_2)\rangle \cap \langle \nu_d(Z_j)\rangle = \emptyset$ because $h^1(L_j, \mathcal{I}_W(d)) = 0$ for each zero-dimensional scheme $W \subset L_j$ with $\deg(W) \leq d+1$. Hence $\langle \nu_d(Z_j \cup S_2)\rangle = \nu_d(L_j)$ and $\langle \nu_d(S_2)\rangle \cap \langle \nu_d(Z_j) = \emptyset$. Since $O \in S_1$ and the points P_j and $\nu_d(O)$ span the line L_j, we get that $P \in \langle \nu_d(S)\rangle$ if and only if $P_j \in \langle \nu_d(S_j)\rangle$ for all $j = 1, 2$. We saw that $\langle \nu_d(S_2)\rangle \cap \langle \nu_d(Z_2) = \emptyset$ and hence $P_2 \notin \langle \nu_d(S_2)\rangle$.

Now assume $\sharp(S' \cap L_i) \geq d + 1$ for some i. We still have at least one index j with $\sharp(S' \cap L_j) \leq m - 1$ and we work in the same way, increasing if necessary S (even with cardinality $> md$) to get $\sharp(S \cap L_i) \geq d$ for all $i \neq j$ and

$\langle\nu_d(T)\rangle = \langle\nu_d(T\cap S)\rangle$. □

Example 2. Assume $m \geq 2$. Let $L \subset \mathbb{P}^m$ be a line such that $O \in L$ and $L \neq L_i$ for each i. Fix any $P \in \langle\nu_d(\{2O, L\})\rangle \setminus \nu_d(O)$. We have $r_{m,d}(P) = d$ ([4, Theorem 32]). By Proposition 2 we have $r_{T_{m,d}}(P) = \rho(P)d$ and hence we may take (for suitable) as $r_{T_{m,d}}(P)$ any integer xd with $x \in \{2, \dots m\}$. For a general $P \in \langle\nu_d(2O)\rangle$ we have $r_{m,d}(P) = d$, $r_{T_{m,d}}(P) = md$ and hence $r_{T_{m,d}}(P) - r_{m,d}(P) = (m-1)d$.

Proof of Theorem 1. First assume $\rho(P) = 0$. Fix any $P_1, \dots, P_n$ as in Remark 1. We will take suitable $S_i \subset L_i$, set $S := S_1 \cup \cdots \cup S_n$, prove that $P \in \langle\nu_d(S)\rangle$ and give an upper bound for $\sharp(S)$. If P_i is not contained in the tangential surface of $\langle\nu_d(L_i)\rangle$, then it has rank $\leq d-1$ with respect to the degree d rational normal curve $\nu_d(L_i)$ by Sylvester's theorem. In this case we take as S_i any subset of L_i evincing this rank. Hence in this case $\sharp(S_i) \leq d-1$. Now assume that P_i is contained in the tangential surface of $\langle\nu_d(L_i)\rangle$. In this case P_i has rank d with respect to $\nu_d(L_i)$, but there is $S_i \subset L_i$ such that $\sharp(S_i) = d$, $O \in S_i$ and $P_i \in \langle\nu_d(S_i)\rangle$ ([1, part (b3) of Proposition 1]). Set $S := S_1 \cup \cdots \cup S_n$. We have $\sharp(S) \leq m(d-1)+1$ and equality holds only if some P_i is in the tangential surface of $\nu_d(L_i)$. Since $P \in \langle\{P_1, \dots, P_n\}\rangle$, we have $P \in \langle\nu_d(S)\rangle$. Now assume $\rho(P) > 0$. If $\rho(P) = m$, then use Proposition 2. Assume $0 < \rho(P) < m$ and set $E := \{i \in \{1, \dots, n\} : P_i \in \langle\{2O, L_i\}\rangle\}$. For each $i \in \{1, \dots, n\} \setminus E$ take S_i as above. For each $i \in E$ fix $S_i \subset L_i$ evincing the rank of P_i with respect to $\nu_d(L_i)$. In this way we get $r_{T_{m,d}} \leq m(d-1)+1+\rho(P)$. To conclude the proof we only need to check the case $\rho(P) = m-1$. Assume $d \geq 5$. Without losing generality we may assume $E = \{1, \dots, n-1\}$. If P_n has rank $\leq d-1$ with respect to $\nu_d(L_n)$, then the proof just given shows that $r_{T_{m,d}}(P) < md$. Now assume that P_i has rank d. Therefore there is $Q \in L_n$ such that $P_n \in \langle\nu_d(\{2Q, L_n\})\rangle$, where $\{2Q, L_n\}$ denote the degree two effective divisor of L_n with Q as its support. Since $n \notin E$, we have $Q \neq O$. Fix $Q_n \in \langle\{P_n, \nu_d(O)\rangle \setminus \{P_n, \nu_d(O)\}$. We have $Q_n \in \langle\{\{2Q, L_n\}, \nu_d(O)\}\rangle$ and hence Q_n has border rank ≤ 3 with respect to $\nu_d(L_n)$. Assume the existence of a connected degree 2 zero-dimensional scheme $W \subset L_n$ such that $Q_n \in \langle\nu_d(W)\rangle$. Call O' the support of W. Since $d \geq 5$, we have $\nu_d(O) \notin \langle\{2Q, L_n\}\rangle$. Hence $O' \neq Q$. Since $n \notin E$, we have $O' \neq O$. Hence $W \cap \{Q, O\} = \emptyset$. Since $d \geq 4$ and $\deg(W \cup \{O\} \cup \{2Q, L_n\}) \leq 5$, we have $h^1(L_n, \mathcal{I}_{W\cup\{O\}\cup\{2Q,L_n\}}(d)) = 0$. Since $W \cap \{Q, O\} = \emptyset$, we get $\dim(\langle\nu_d(W \cup \{O\} \cup \{2Q, L_n\})\rangle) = 4$. Hence $\langle\nu_d(W)\rangle \cap \langle\nu_d(\{O\} \cup \{2Q, L_n\})\rangle = \emptyset$, contradicting the fact that $Q_n \in \langle\nu_d(W)\rangle \cap \langle\nu_d\{O\} \cup \{2Q, L_n\})\rangle$. Hence Q_n has border rank ≥ 3 and so rank $\leq d-1$ ([4, Theorem 23],[8, Theorem 4.1]). Take $A_n \subset L_n$ such that $\sharp(A_n) \leq d-1$ and $Q_n \in \langle\nu_d(A_n)\rangle$. Take $Q_i \in \langle\nu_d(O), P_i\rangle$,

$i = 1, \dots, n-1$, such that $P \in \langle \{Q_1, \dots, Q_n\}$. Take $A_i \subset L_i$, $1 \le i \le n-1$, such that $\sharp(A_i) \le d$ and $Q_i \in \langle \nu_d(A_i) \rangle$ and set $A := A_1 \cup \dots \cup A_n$. We have $\sharp(A) < md$ and $P \in \langle \nu_d(A) \rangle$. □

Remark 2. Take $\rho(P) = m-1$ and assume $d \ge 5$, $E = \{1, \dots, n-1\}$, $P \in \langle \{P_1, \dots, P_n\} \rangle$ with $P_i \in \langle \nu_d(L_i) \rangle$ and call b (resp. r') the border rank (resp. the rank) of P_n with respect to P_n. We have $b \le \lceil (d+1)/2 \rceil$ and either $r' = b$ or $r' = d+2-b'$. We saw in the proof of Theorem 1 that $r_{T_{m,d}}(P) \le (m-1)d + r'$. Now assume $r' = d$. We claim that $r_{T_{m,d}}(P) \le md - 2$. If $m = 2$, then the claim is [3, Lemma 5.4]. If $m > 2$, then using the result for $m = 2$ (say for $L_{n-1} \cup L_n$) and taking suitable $A_i \subset L_i$, $i = 1, \dots, m-2$, we get the inequality $r_{T_{m,d}}(P) \le md - 2$.

Example 3. Fix $O \in \mathbb{P}^m$, $m \ge 2$. Here we describe all zero-dimensional schemes $Z \subset \mathbb{P}^m$ such that $\deg(Z) = m+2$, $Z \supset 2O$ and either Z is Gorenstein or Z is in linearly general position.

(a) We first check that these two conditions are equivalent for zero-dimensional schemes $Z \subset \mathbb{P}^m$ such that $\deg(Z) = m+2$, $Z \supset 2O$. Write $\mathbb{P}^m = \mathbb{P}(W)$ with W an $(m+1)$-dimensional vector space. First assume that Z is in linearly general position. Since $Z \supset 2O$ and $m \ge 2$, Z is not curvilinear. By [5, Theorem 1.3] Z is Gorenstein. Now assume that Z is not in linearly general position, i.e. there is a hyperplane $H \subset \mathbb{P}^m$ such that $\deg(H \cap Z) \ge m+1$. Since $2O \nsubseteq H$, we get $\deg(H \cap Z) = m+1$ and $\mathrm{Res}_H(Z) = \{O\}$. Hence for each hyperplane $N \subset \mathbb{P}^n$ with $O \in N$ we have $Z \subset H \cup N$. Fix a system of homogeneous coordinates $x_0, \dots, x_m$ of $\mathbb{P}^m$ such that $O = (1 : 0 : \dots : 0)$, $H = \{x_1 = 0\}$ and set $z_i := x_i/x_0$, $i = 1, \dots, n$. We get that $z_1 z_{j|Z} \equiv 0$ for each $j = 1, \dots, m$, i.e. that z_1 represents an element of the socle of $\mathcal{O}_{Z,O}$ which is not in μ^2, where μ is the maximal ideal of $\mathcal{O}_{Z,O}$. Since $Z \supset 2O$ and $\deg(Z) = \deg(2O) + 1$, the $\mathbb{C}$-vector space μ^2/μ^3 has dimension 1, $\mu^3 = 0$ and any element of μ^2 is contained in the socle of $\mathcal{O}_{Z,O}$. Hence $\mathcal{O}_{Z,O}$ has socle degree ≥ 2, i.e. it is not Gorenstein.

(b) Now we take Z in linearly general position and hence Gorenstein. For each hyperplane $H \subset \mathbb{P}^m$ we have $\deg(Z \cap H) = m$ and hence $Z \cap H = \{2O, H\}$ and $\deg(\mathrm{Res}_H(Z)) = 2$. Therefore there is a unique line L spanned by the scheme $\mathrm{Res}_H(Z)$. We get $Z \subset H \cup N$ for each hyperplane $N \supset L$. From the residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(Z)}}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z \cap H, H}(2) \to 0 \qquad (1)$$

we get $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \ge 2$. Hence $h^0(\mathcal{I}_Z(2)) = \binom{m+2}{2} - m - 2$. For $m = 2$ we get that Z is the complete intersection of two singular conics through

O (and the converse holds).

Proposition 3. *Assume $d \geq 3$. Take $Z \subset \mathbb{P}^m$ as in Example 3 with $Z_{red} = \{O\}$ and assume $T_m \supset Z$. Take $P \in \langle \nu_d(Z) \rangle$ such that $P \notin \langle \nu_d(Z') \rangle$ for any $Z' \subsetneq Z$. Fix a finite set $B \subset T_n$. Then there is $S \subset T_n \setminus (\{O\} \cup B)$ such that $\sharp(S \cap L_i) = d-1$ for all i and $P \in \langle \nu_d(S) \rangle$.*

Proof. The case $m = 2$ is true ([3, Lemma 5.4]) and hence we may assume $m > 2$ and use induction on m.

We first check that $\rho(P) = 0$. Take $(P_1, \dots, P_m)$ with $P_i \in \langle \nu_d(L_i) \rangle$ and $P \in \langle \{P_1, \dots, P_m\} \rangle$. Assume that $P_i \in \langle \nu_d(\{2O, L_i\}) \rangle$ for some i. Fix a homogeneous system of coordinates $x_0, \dots, x_m$ such that $O = (1 : 0 : \cdots : 0)$ and $L_j = \{x_j = 0\}$. Let $H \subset \mathbb{P}^m$ be the hyperplane $\{x_i = 0\}$. Since Z is in linearly general position, the scheme $Z \cap H$ is the 2-point $\{2O, H\}$ of H and it has degree m. Hence the residual scheme $\mathrm{Res}_H(Z)$ has degree two and it is supported by O. We claim that $\mathrm{Res}_H(Z)$ is not contained in H. Indeed, if it were contained in H, then we would get that x_i^2 is in the homogeneous ideal of Z. Since all reducible quadrics $\{x_i x_j\}$, $j = 1, \dots, m$, $j \neq i$, contain T_m, we would get $x_j x_{i|Z} \equiv 0$ for each $j = 1, \dots, n$. Hence the local ring $\mathcal{O}_{Z,O}$ would have socle degree ≥ 2 (its socle also contains a generator as a $\mathbb{C}$-vector space of the ideal sheaf of $2O$ in Z), a contradiction. Fix any finite set $A \subset \cup_{j \neq i} L_j$ such that $\cup_{j \neq i} \{P_j\} \subset \langle \nu_d(A) \rangle$. We get $P \in \langle \nu_d(\{2O, L_i\} \cup H \rangle$ and hence $P \in \langle \nu_d(2H) \rangle$. Since the quadric Q contains $2O$, $\deg(Z) = \deg(2O) + 1$ and $P \notin \langle \nu_d(2O) \rangle$, we have $\langle \nu_d(Z) \rangle = \langle \{P\} \cup \nu_d(2O) \rangle$. Hence $\langle \nu_d(Z) \rangle \subset \langle \nu_d(2H) \rangle$. Let g be a degree $d-2$ polynomial not vanishing at O. The polynomial $g x_i^2$ vanishes on $2H$, but it does not vanish on Z. Hence $\nu_d(Z) \nsubseteq \langle \nu_d(2H) \rangle$, a contradiction.

Since $\rho(P) = 0$, to prove Proposition 3 using the proof of Theorem 1 it is sufficient to prove that no P_i has rank d with respect to the rational normal curve $\nu_d(L_i)$. Assume that this is the case for at least one index, i. Since $\rho(P) = 0$, there is a point $Q \in L_i \setminus \{O\}$ such that $P_i \in \langle \nu_d(\{2Q, L_i\}) \rangle$. Since $d \geq 4$, we have $h^1(L_i, \mathcal{I}_W(4)) = 0$ for each zero-dimensional scheme $W \subset L_i$ with $\deg(W) \leq 5$. Hence the only point of the plane $\langle \nu_d(\{2Q, L_i\}) \rangle$ contained in the linear span of a degree 2 divisors of $\nu_d(L_i)$ are the points of the lines $\langle \{\nu_d(Q), \nu_d(O)\} \rangle$. Hence each point $P_i' \in \langle \{P_i, \nu_d(O)\} \rangle \setminus \{P_i\}$ has rank $\leq d-3$ with respect to $\nu_d(L_i)$. However, fixing P_i' we have to change the other points P_j, $j \neq i$. We need to check that we may do the same trick simultaneously for all bad indices i without creating new bad indices. Fix an index $j \neq i$ with P_j of rank two and call $Q_j \subset L_j$ the only point of L_j such that $P_j \in \langle \nu_d(\{2Q_j, L_j\}) \rangle$. We saw that $r_{\nu_d(L_j}(P_j') < d$ for any $P_j' \in \langle \{P_j, \nu_d(O)\} \rangle \setminus \{P_j\}$ and hence we may simultaneously handle all bad indices. Let F be the set of all $j \in \{1, \dots, m\}$ such that P_j has rank in the interval $2 \le x \le d-3$. Fix any $j \in F$. Since the tangential variety $\tau(\nu_d(L_j))$ of $\nu_d(L_j)$ is closed, we see that moving by a small amount each bad P_i the corresponding point P'_j does not go to $\tau(\nu_d(L_j))$ and hence P'_j has rank $\le d-3$. Now assume the existence of some $j \in \{1, \dots, m\}$, say $j = n$, with P_n of rank 1, i.e. with $P_m \in L_m$. Fix $P' \in \langle T_{m-1,d} \rangle$ such that $P \in \langle \{P', P_n\} \rangle$; apply Proposition 3 to T_{m-1} and then add 1 to get $r_{T_{m,d}}(P) \le (m-1)(d-1)+1$. Therefore for a general choice of $(P'_1, \dots, P'_m)$ with $P'_i \in \langle \nu_d(L_i) \rangle$ and $P \in \langle \{P'_1, \dots, P'_m\} \rangle$ each P'_i has rank $\le d-3$ with respect to $\nu_d(L_i)$. □

The additional condition imposed by the set B in the statement of Proposition 3 is related to the definition of open rank ([6]).

In the case $m = 3$ we can prove the inequality opposite to Proposition 3 and hence that $r_{T_{3,d}}(P) = r_{m,d}(P) = 3d-3$.

Proposition 4. *Take Z as in Example 3 and assume $m = 3$ and that $Z \subset T_3$. Fix an integer $d \ge 7$ and $P \in \langle \nu_d(Z) \rangle$ such that $P \notin \langle \nu_d(Z') \rangle$ for all $Z' \subsetneq Z$. Then $r_{3,d}(P) \ge 3d-3$ and there is no set $S \subset \mathbb{P}^3$ with $\sharp(S) = 3d-3$ and $O \in S$.*

Proof. In order to obtain a contradiction we assume the existence of a set $B \subset \mathbb{P}^3$ such that $\sharp(B \cup \{O\}) \le 3d-3$ and $P \in \langle \nu_d(B) \rangle$. Taking B minimal we may also assume $P \notin \langle \nu_d(B') \rangle$ for all $B' \subsetneq B$. Set $W_0 := \deg(Z \cup S)$. Notice that for each line $L \subset \mathbb{P}^3$ either $O \notin L$ and $\deg(L \cap Z) = 0$ or $O \in L$ and $\deg(Z \cap L) = 2$. Hence either $\deg(L \cap W_0) = \sharp(L \cap B)$ (case $O \notin L$) or $\deg(W_0) = 2 + \sharp(B \cap (L \setminus \{O\}))$ (case $O \in L$). By assumption $w_0 := \deg(W_0) \le 3d+1$. Let $H_1 \subset \mathbb{P}^3$ be a plane such that $e_1 := \deg(W_0 \cap H_1)$ is maximal. Set $W_1 := \mathrm{Res}_{H_1}(W_0)$. Fix an integer $i \ge 2$ and assume to have defined the integers e_j, the hyperplanes H_j and the schemes W_j for all $j = 1, \dots, i-1$. Let H_i be any plane such that $e_i := \deg(H_i \cap H_{i-1})$ is maximal. Set $W_i := \mathrm{Res}_{H_i}(W_{i-1})$. We have $e_i \ge e_{i+1}$ for all i. For each integer $i > 0$ we have an exact sequence

$$0 \to \mathcal{I}_{W_i}(d-i) \to \mathcal{I}_{W_{i-1}}(d+1-i) \to \mathcal{I}_{W_{i-1} \cap H_i, H_i}(d+1-i) \to 0 \qquad (2)$$

Since $h^1(\mathcal{I}_{W_0}(d)) > 0$ ([2, Lemma 1]),the exact sequence (2) show the existence of an integer $i > 0$ such $h^1(H_i, \mathcal{I}_{W_{i-1} \cap H_i, H_i}(d+1-i)) > 0$. We call g the minimal such an integer. Since any degree 3 zero-dimensional subscheme of $\mathbb{P}^3$ is contained in a plane, if $e_i \le 2$, then $W_i = \emptyset$ and $e_j = 0$ for all $j > i$. Since $w_0 \le 3d+1$, we get $e_{d+1} = 0$ and $e_d \le 1$. Since $h^1(\mathcal{O}_{\mathbb{P}^3}(d)) = h^1(\mathcal{I}_Q) = 0$ for any $Q \in \mathbb{P}^3$, we get $g \le d$. By [4, Lemma 34] either $e_g \ge 2(d+1-g)+2$ or there is a line $L \subset H_g$ such that $\deg(L \cap W_{i-1}) \ge d+3-g$. Assume for the moment $g \geq 2$ and $e_g \leq 2(d+1-g)+1$. Since $e_g > 0$, W_{i-2} spans $\mathbb{P}^3$. Hence $e_{g-1} \geq d+4-g$ in all cases. Hence if $g \geq 2$ we get $3d+1 \geq g(d+4-g)-1$. Hence $1 \leq g \leq 2$.

(a) Assume $g = 2$. Since $e_2 \leq (3d+1)/2 \geq 2(d-1)+1$, there is a line $L \subset H_2$ such that $\deg(L \cap W_1) \geq d+1$. Let $N_1 \subset \mathbb{P}^3$ be a plane containing L and such that $f_1 := \deg(W_0 \cap L)$ is maximal among the planes containing L. Since Z spans $\mathbb{P}^3$, we have $f_1 \geq d+2$. Set $Z_0 := W_0$ and $Z_1 := \mathrm{Res}_{N_1}(Z_0)$. Let N_2 be a plane such that $f_2 := \deg(N_2 \cap Z_1)$. Set $Z_1 := \mathrm{Res}_{N_2}(Z_1)$. Define inductively the plane N_i, $i \geq 3$, the scheme $Z_i := \mathrm{Res}_{N_i}(Z_{i-1})$ and the integer $f_i := \deg(N_i \cap Z_{i-1})$ with f_i maximal among all planes. We have $f_i \geq f_{i+1}$ for all $i \geq 3$ and if $f_i \leq 2$, then $f_{i+1} = 0$ and $Z_i = \emptyset$. We have the residual exact sequence similar to (2) with N_i instead of H_i, Z_{i-1} instead of W_{i-1} and Z_i instead of W_i. Hence there is an integer $i > 0$ such that $h^1(N_i, \mathcal{I}_{N_i \cap Z_{i-1}, H_i}(d+1-i)) > 0$ and we call g' the first such an integer. Since $\sum_{i \geq 2} f_i \leq 2d-1$ and $f_{i+1} = 0$ if $f_i \leq 2$, we have $g' < d$. Hence either $f_{g'} \geq 2(d+1-g')+2$ or there is a line $R \subset N_{g'}$ such that $\deg(R \cap Z_{i-1}) \geq d+3-g'$. Assume for the moment $g' \geq 3$. We get $f_{g'-1} \geq d+4-g'$. Hence $2d-1 \geq \sum_{i \geq 2} f_i \geq (g'-1)(d+4-g')-1$. We get $1 \leq g' \leq 2$.

(a1) Assume $g' = 2$. Since $f_1 \geq d+2$, we have $f_2 \leq 2(d-1)+1$. By [4, Lemma 34] there is a line $R \subset N_2$ such that $\deg(R \cap Z_1) \geq d+1$.

(a1.1) Assume $L \cap R = \emptyset$. Since Z is connected and spans $\mathbb{P}^3$, there is a smooth quadric $Q \subset \mathbb{P}^3$ containing $L \cup R$ and with $\deg(Q \cap W_0) \geq 1 + \deg(W_0 \cap R) + \deg(W_0 \cap L) \geq 2d+3$. Since $\mathrm{Res}_Q(W_0)$ has degree $\leq d-2$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2)) = 0$. Since Z is connected, [3, Lemma 5.1] gives $W_0 \subset Q$. Since Q is a smooth quadric, we have $Q \not\supseteq Z$, a contradiction.

(a1.2) Assume $L \cup R \neq \emptyset$. Since $\deg(R \cap (B \setminus B \cap L)) \geq d+1-\deg(R \cap Z) = d-1 > 0$, we have $L \neq R$. Hence $L \cap R$ is a point, O' and $\Pi := \langle L \cup R \rangle$ is a plane. First assume $O' \neq O$. We get $W_0 \cap (L \cup R) \geq 2d+2$, because $B \cap W_1 \subseteq B \setminus B \cap L$. Hence $\deg(\mathrm{Res}_\Pi(W_0)) \leq d-1$. Therefore $h^1(\mathcal{I}_{\mathrm{Res}_\Pi(W_0)}(d-1)) = 0$. Since Z is connected, [3, Lemma 5.1] gives $Z \cup B \subset \Pi$, contradicting the inclusion $Z \supset 2O$. Now assume $O' = O$. We may also assume $h^1(\mathcal{I}_{\mathrm{Res}_\Pi(W_0)}(d-1)) > 0$. In this case we have $\deg(\mathrm{Res}_\Pi(W_0)) \geq 3d+1-(2d-1)$. By [4, Lemma 34] there is a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap \mathrm{Res}_\Pi(W_0)) \geq d+1$. Looking at $\sharp(B \setminus B \cap R)$ and $\sharp(B \setminus B \cap L)$ we see that $D \neq R$ and $D \neq L$. Since $e_1 \leq 3d+1-d-1$, we have $D \nsubseteq \Pi$. First assume $O \in D \cap L \cap R$. We have $h^0(\mathcal{I}_{D \cup L \cup R}(2)) = 3$ and a general $Q \in |\mathcal{I}_{D \cup L \cup R}(2)|$ is an irreducible quadric cone. Since $\deg_Q(\mathrm{Res}_Q(W_0)) \leq d-1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2)) = 0$. By [3, Lemma 5.1] we get $W_0 \subset Q$. Since $\deg(L \cap Z) = \deg(D \cap Z) = \deg(R \cap Z) = 2$, we have $\sharp(B \cap (D \setminus \{O\})) \geq d-1$, $\sharp(B \cap (R \setminus \{O\})) \geq d-1$ and $\sharp(B \cap (L \setminus \{O\}) \geq d-1$ and hence $\sharp(B \cup \{O\}) \geq 3d-2$, a contradiction. If either $D \cap L = \emptyset$ or $D \cap R = \emptyset$, then we conclude as in step (a1.1). If D intersects exactly one of the line R and L. In this case $R \cup L \cup D$ is contained in a smooth quadric and we get a contradiction as in step (a1.1).

(a2) Assume $g' = 1$. Since $h^1(N_1, \mathcal{I}_{W_0 \cap N_1}(d)) > 0$ and $f_1 \le e_1 \le 2d$, there is a line $J \subset N_1$ such that $\deg(J \cap W_0) \ge d + 2$. Since $L \subset N_1$ and $\deg(L \cap W_1) \ge d + 1$, we get $J = L$. We work as in step (b).

(b) Assume $g = 1$. Since $h^1(H_1, \mathcal{I}_{W_0 \cap H_1}(d)) > 0$, we have $e_1 \ge d + 2$ and hence $\deg(W_1) \le 2d - 1$. First assume $h^1(\mathcal{I}_{W_1}(d-1)) = 0$. Since Z is connected, [3, Lemma 5.1] gives $Z \subset H_1$, a contradiction. Now assume $h^1(\mathcal{I}_{W_1}(d-1)) > 0$. Since $\deg(W_1) \le 2(d-1)$, [4, Lemma 34] gives the existens of a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap W_1) \ge d + 1$. Fix any plane $M \supset L$. We have $\mathrm{Res}_{H_1 \cup M}(W_0) = \mathrm{Res}_M(W_1)$. Hence $\deg(\mathrm{Res}_M(W_1)) \le d - 2$. Since $\deg(\mathrm{Res}_M(W_1)) \le 3d + 1 - (d+2) - (d+1) = d - 2$, we have $h^1(\mathcal{I}_{\mathrm{Res}_M(W_1)}(d-2)) = 0$. By [2, Lemma 5.1] we get $W_0 \subset H_1 \cup M$ and in particular $B \subset H_1 \cup L$. Set $Z_0 := W_0$. Let N_1 be a plane containing L and such that $f_1 := \deg(N_1 \cap Z_0)$ is maximal among the planes containing L. Since Z spans $\mathbb{P}^3$, we have $f_1 \ge d + 2$. Set $Z_0 := W_0$ and $Z_1 := \mathrm{Res}_{N_1}(Z_0)$. Let N_2 be a plane such that $f_2 := \deg(N_2 \cap Z_1)$. Set $Z_1 := \mathrm{Res}_{N_2}(Z_1)$. Define inductively the plane N_i, $i \ge 3$, the schemes $Z_i := \mathrm{Res}_{N_i}(Z_{i-1})$ and the integer $f_i := \deg(N_i \cap Z_{i-1})$ with f_i maximal among all planes. We have $f_i \ge f_{i+1}$ for all $i \ge 3$ and if $f_i \le 2$, then $f_{i+1} = 0$ and $Z_i = \emptyset$. We have the residual exact sequence similar to (2) with N_i instead of H_i, Z_{i-1} instead of W_{i-1} and Z_I instead of W_i. Hence there is an integer $i > 0$ such that $h^1(N_i, \mathcal{I}_{N_i \cap Z_{i-1}, H_i}(d+1-i)) > 0$ and we call g' the first such an integer. As in step (a) we get $g' \in \{1, 2\}$.

(b1) Assume $g' = 2$. As in step (a1) we get a line $R \subset N_2$ such that $\deg(R \cap Z_1) \ge d + 1$. Steps (a1.1) and (a1.2) work verbatim.

(b2) Assume $g' = 1$. Since $h^1(N_1, \mathcal{I}_{W_0 \cap N_1}(d)) > 0$ and $f_1 \le e_1 \le 2d$, there is a line $J \subset N_1$ such that $\deg(J \cap W_0) \ge d + 2$. Since Z spans $\mathbb{P}^3$, then $f_1 \ge 1 + \deg(J \cap W_0) = d + 3$. Since $L \subset N_1$ and $\deg(L \cap W_1) \ge d + 1$, we get $J = L$. If $h^1(\mathcal{I}_{Z_1}(d-1)) = 0$, then [3, Lemma 5.1] gives $Z \subset N_1$, a contradiction. Now assume $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$. Since $f_1 \ge d + 3$, we have $\deg(Z_1) \le 2d - 2$. By [4, Lemma 34] there is a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_1) \ge d + 1$. Since $B \cap W_1 \subseteq (B \setminus (\{O\} \cup L)$ we get $R \ne L$. We work as in steps (a1.1) and (a1.2) (here the inequalities are stronger by 1, because $\deg(L \cap Z_0) \ge d + 2$). $\square$

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Subsets of the variety $X \subset \mathbb{P}^n$ evincing the X-rank of a point of $\mathbb{P}^n$, Houston J. Math. (to appear).

[2] E. Ballico and A. Bernardi, Decomposition of homogeneous polynomials with low rank, Math. Z. 271 (2012), 1141–1149; DOI 10.1007/s00209-011-0907-6.

[3] E. Ballico andA. Bernardi, Stratification of the fourth secant variety of Veronese variety via the symmetric rank, Adv. Pure Appl. Math. **4** , no. 2, 215–250 (2013) DOI: 10.1515/apam-2013-0015

[4] A. Bernardi, A. Gimigliano, M. Idà, Computing symmetric rank for symmetric tensors, J. Symbolic. Comput. **46** (2011), no. 1, 34–53.

[5] D. Eisenbud and J. Harris, Finite projective schemes in linearly general position, J. Algebraic Geom. 1 (1992), no. 1, 15–30.

[6] J. Jelisiejew, An upper bound for the Waring rank of a form, arXiv:1305.6957.

[7] J. M. Landsberg, Tensors: Geometry and Applications, Graduate Studies in Mathematics, Vol. 128, Amer. Math. Soc. Providence, 2012.

[8] J. M. Landsberg and Z. Teitler, On the ranks and border ranks of symmetric tensors, Found. Comput. Math. 10 (2010), no. 3, 339–366.