GENERALIZED DIFFERENCES AND ABEL'S TYPE BINOMIAL THEOREMS

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Abstract: The first author proved Abel's type binomial theorems for the falling and/or rising factorial power in the previous paper [3]. The purpose of this paper is to give generalizations for these theorems by the use of the generalized forward and/or backward differences.

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1. Introduction

For an integer n , the falling factorial power $x^{\underline{n}}$ is defined as

$$x^{\underline{n}} = \begin{cases} x(x-1)(x-2)\cdots(x-n+1) & (n = 1, 2, 3, \dots) \\ 1 & (n = 0) \\ \frac{1}{(x+1)(x+2)\cdots(x-n)} & (n = -1, -2, -3, \dots). \end{cases}$$

Similarly the rising factorial power $x^{\overline{n}}$ is defined as

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$$x^{\overline{n}} = \begin{cases} x(x+1)(x+2)\cdots(x+n-1) & (n = 1, 2, 3, \dots) \\ 1 & (n = 0) \\ \frac{1}{(x-1)(x-2)\cdots(x-n)} & (n = -1, -2, -3, \dots). \end{cases}$$

The binomial theorems for the falling factorial power and the rising factorial power are known as [2, 3, 5]:

$$(x+y)^{\underline{n}} = \sum_{i=0}^n \binom{n}{i} x^i y^{\underline{n-i}}, \quad \text{and} \quad (x+y)^{\overline{n}} = \sum_{i=0}^n \binom{n}{i} x^i y^{\overline{n-i}}.$$

In the previous paper [3], the first author of this paper proved the following Abel's type binomial theorems for the falling and rising factorial power:

$$(x+y)^{\underline{n}} = \sum_{i=0}^n \binom{n}{i} x(x-1+iz)^{\underline{i-1}} (y-iz)^{\underline{n-i}}, \quad (1)$$

$$(x+y)^{\overline{n}} = \sum_{i=0}^n \binom{n}{i} x(x+1+iz)^{\overline{i-1}} (y-iz)^{\overline{n-i}}. \quad (2)$$

These formulas (1) and (2) are generalizations for the Abel's generalized formula [1] given by

$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x(x+iz)^{i-1} (y-iz)^{n-i}.$$

At the same time, those formulas (1) and (2) are generalizations of the binomial theorems for falling (rising) factorial power.

In this paper, we prove the following theorem which is a further generalization of (1) and (2).

Theorem 1. *Let n be a natural number and α a complex number, then the following equalities hold.*

$$(x+y)^{\underline{n},\alpha} = \sum_{i=0}^n \binom{n}{i} x(x-\alpha+iz)^{\underline{i-1},\alpha} (y-iz)^{\underline{n-i},\alpha}, \quad (3)$$

$$(x+y)^{\overline{n},\alpha} = \sum_{i=0}^n \binom{n}{i} x(x+\alpha+iz)^{\overline{i-1},\alpha} (y-iz)^{\overline{n-i},\alpha}. \quad (4)$$

In the above theorem we set a generalized falling factorial power $x^{\underline{n},\alpha}$ for any complex number α is defined as follows:

$$x^{\underline{n},\alpha} = \begin{cases} x(x-\alpha)(x-2\alpha)\cdots(x-(n-1)\alpha) & (n = 1, 2, 3, \dots) \\ 1 & (n = 0) \\ \frac{1}{(x+\alpha)(x+2\alpha)\cdots(x-n\alpha)} & (n = -1, -2, -3, \dots). \end{cases}$$

Similarly a generalized rising factorial power $x^{\overline{n},\alpha}$ is defined as follows:

$$x^{\overline{n},\alpha} = \begin{cases} x(x+\alpha)(x+2\alpha)\cdots(x+(n-1)\alpha) & (n = 1, 2, 3, \dots) \\ 1 & (n = 0) \\ \frac{1}{(x-\alpha)(x-2\alpha)\cdots(x+n\alpha)} & (n = -1, -2, -3, \dots). \end{cases}$$

For the convenience of the reader's help to understand Theorem 1, we give the following example.

Example 1. We consider the case of $n = 4$ in Theorem 1. The equality (3) is calculated by

$$\begin{aligned} & (x+y)(x+y-\alpha)(x+y-2\alpha)(x+y-3\alpha) \\ &= y(y-\alpha)(y-2\alpha)(y-3\alpha) + 4x(y-z)(y-z-\alpha)(y-z-2\alpha) \\ & \quad + 6x(x+2z-\alpha)(y-2z)(y-2z-\alpha) + 4x(x+3z-\alpha)(x+3z-2\alpha)(y-3z) \\ & \quad + x(x+4z-\alpha)(x+4z-2\alpha)(x+4z-3\alpha). \end{aligned}$$

The equality (4) is also calculated by

$$\begin{aligned} & (x+y)(x+y+\alpha)(x+y+2\alpha)(x+y+3\alpha) \\ &= y(y+\alpha)(y+2\alpha)(y+3\alpha) + 4x(y-z)(y-z+\alpha)(y-z+2\alpha) \\ & \quad + 6x(x+2z+\alpha)(y-2z)(y-2z+\alpha) + 4x(x+3z+\alpha)(x+3z+2\alpha)(y-3z) \\ & \quad + x(x+4z+\alpha)(x+4z+2\alpha)(x+4z+3\alpha). \end{aligned}$$

2. Generalized Differences and their Properties

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[x]$ the polynomial algebra over $\mathbb{C}$. We consider $\mathbb{C}[x]$ a linear space over $\mathbb{C}$. For $a \in \mathbb{C}$, shift operator E_a is defined as follows: $E_a : \mathbb{C}[x] \ni f(x) \mapsto f(x+a) \in \mathbb{C}[x]$.

Definition 1. A linear operator $Q : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$ is called a delta operator, if the following conditions are satisfied:

- (1) if $f \in \mathbb{C}[x]$ is degree of n , then Qf is degree of $n-1$,
- (2) Q commutes with all shift operators, that is, $QE_a = E_aQ$ for all $a \in \mathbb{C}$.

We define a generalized forward difference $\Delta^{+,\alpha}$ and a generalized backward difference $\Delta^{-,\alpha}$ for any complex number α as follows:

$$\begin{aligned} \Delta^{+,\alpha} f(x) &= f(x+\alpha) - f(x) \quad \text{for } f \in \mathbb{C}[x], \\ \Delta^{-,\alpha} f(x) &= f(x) - f(x-\alpha) \quad \text{for } f \in \mathbb{C}[x]. \end{aligned}$$

In particular, if $\alpha = 1$ then the generalized forward (backward) difference is a standard forward (backward) difference. That is, $\Delta^{+,1}f(x) = f(x+1) - f(x)$ and $\Delta^{-,1}f(x) = f(x) - f(x-1)$. Generalized forward differences and generalized backward differences are delta operators.

Proposition 1. *We have the following equalities.*

(1) *For a generalized forward difference $\Delta^{+,\alpha}$ and $a \in \mathbb{C}$, we have*

$$\Delta^{+,\alpha}(x+a)^{\underline{n},\alpha} = n\alpha(x+a)^{\underline{n-1},\alpha}, \quad (n = 1, 2, \dots).$$

In particular, $\Delta^{+,\alpha}x^{\underline{n},\alpha} = n\alpha x^{\underline{n-1},\alpha}$, $(n = 1, 2, \dots)$.

(2) *For a generalized backward difference $\Delta^{-,\alpha}$ and $a \in \mathbb{C}$, we have*

$$\Delta^{-,\alpha}(x+a)^{\overline{n},\alpha} = n\alpha(x+a)^{\overline{n-1},\alpha}, \quad (n = 1, 2, \dots).$$

In particular, $\Delta^{-,\alpha}x^{\overline{n},\alpha} = n\alpha x^{\overline{n-1},\alpha}$, $(n = 1, 2, \dots)$.

Proof. (1) By simple calculations, firstly we can prove the equality $\Delta^{+,\alpha}x^{\underline{n},\alpha} = n\alpha x^{\underline{n-1},\alpha}$. Since $Q \equiv \Delta^{+,\alpha}$ commutes with any shift operator E_a , that is, $QE_a = E_aQ$, we then have

$$\begin{aligned} Q(x+a)^{\underline{n},\alpha} &= QE_ax^{\underline{n},\alpha} = E_aQx^{\underline{n},\alpha} \\ &= E_a(n\alpha x^{\underline{n-1},\alpha}) = n\alpha E_ax^{\underline{n-1},\alpha} = n\alpha(x+a)^{\underline{n-1},\alpha}. \end{aligned}$$

Similarly we can prove (2). □

Proposition 2. (Product rule of generalized differences)

(1) *The equality $\Delta^{+,\alpha}\{f(x)g(x)\} = (\Delta^{+,\alpha}f(x))g(x) + f(x+\alpha)(\Delta^{+,\alpha}g(x))$ holds.*

In particular, $\Delta^{+,\alpha}\{xf(x)\} = \alpha f(x) + (x+\alpha)(\Delta^{+,\alpha}f(x))$.

(2) *The equality $\Delta^{-,\alpha}\{f(x)g(x)\} = (\Delta^{-,\alpha}f(x))g(x) + f(x-\alpha)(\Delta^{-,\alpha}g(x))$ holds.*

In particular, $\Delta^{-,\alpha}\{xf(x)\} = \alpha f(x) + (x-\alpha)(\Delta^{-,\alpha}f(x))$.

Proof. (1) By simple calculations, we get

$$\begin{aligned} \Delta^{+,\alpha}\{f(x)g(x)\} &= f(x+\alpha)g(x+\alpha) - f(x)g(x) \\ &= (f(x+\alpha) - f(x))g(x) - f(x+\alpha)g(x) + f(x+\alpha)g(x+\alpha) \\ &= (f(x+\alpha) - f(x))g(x) + f(x+\alpha)(g(x+\alpha) - g(x)) \\ &= (\Delta^{+,\alpha}f(x))g(x) + f(x+\alpha)(\Delta^{+,\alpha}g(x)). \end{aligned}$$

In the above equality, let $f(x) = x, g(x) = f(x)$. Since $\Delta^{+, \alpha} x = \alpha$, we have $\Delta^{+, \alpha} \{x f(x)\} = (\Delta^{+, \alpha} x) f(x) + (x + \alpha)(\Delta^{+, \alpha} f(x)) = \alpha f(x) + (x + \alpha)(\Delta^{+, \alpha} f(x))$. Similarly we can prove (2). □

3. Proof of Theorem 1

In this section, we prove Theorem 1. For this purpose, we prepare the following lemmas. Let polynomials $A_k(x, z)$, $(k = 0, 1, 2, \dots)$ in x, z be as follows:

$A_k(x, z) = x(x - \alpha + kz)^{\underline{k-1}, \alpha}$, with the convention $A_0(0, z) \equiv 1$. That is,

$$\begin{aligned} A_0(x, z) &= 1, \quad A_1(x, z) = x, \\ A_2(x, z) &= x(x - \alpha + 2z)^{\underline{1}, \alpha} = x(x - \alpha + 2z), \\ A_3(x, z) &= x(x - \alpha + 3z)^{\underline{2}, \alpha} = x(x - \alpha + 3z)(x - 2\alpha + 3z), \\ &\dots\dots \end{aligned}$$

Lemma 1. *Let polynomials $A_k(x, z) = x(x - \alpha + kz)^{\underline{k-1}, \alpha}$ in x, z , $(k = 0, 1, 2, \dots)$. If $f(x) \in C[x]$ is a polynomial in x of degree n , then*

$$\begin{aligned} f(x) &= \sum_{k=0}^n \frac{1}{\alpha^k} \frac{1}{k!} (\Delta_x^{+, \alpha})^k f(-kz) A_k(x, z) \\ &= \sum_{k=0}^n \frac{1}{\alpha^k} \frac{1}{k!} (\Delta_x^{+, \alpha})^k f(-kz) x(x - \alpha + kz)^{\underline{k-1}, \alpha}, \end{aligned} \tag{5}$$

where $\Delta_x^{+, \alpha}$ is a generalized forward difference with respect to x , given by $\Delta_x^{+, \alpha} f(x) \equiv f(x + \alpha) - f(x)$.

Proof. We prove this lemma along to the following four steps.

(i). First we should remark that the polynomial $A_k(x, z)$ is degree of k with respect to x and $A_k(0, z) = 0$, $(k = 1, 2, \dots)$.

(ii). Let $\Delta_x^{+, \alpha}$ be a generalized forward difference with respect to x . We prove the following equality.

$$\Delta_x^{+, \alpha} A_k(x, z) = \begin{cases} k\alpha A_{k-1}(x + z, z) & (k = 1, 2, \dots), \\ 0 & (k = 0). \end{cases}$$

From Proposition 1 and 2 (product rule of generalized differences),

$$\begin{aligned}
 \Delta_x^{+, \alpha} A_k(x, z) &= \Delta_x^{+, \alpha} \{x(x - \alpha + kz)^{\overline{k-1, \alpha}}\} \\
 &= \alpha(x - \alpha + kz)^{\overline{k-1, \alpha}} + (x + \alpha) \Delta_x^{+, \alpha} \{(x - \alpha + kz)^{\overline{k-1, \alpha}}\} \\
 &= \alpha(x - \alpha + kz)^{\overline{k-1, \alpha}} + (x + \alpha) \alpha(k-1)(x - \alpha + kz)^{\overline{k-2, \alpha}} \\
 &= \alpha(x - \alpha + kz)^{\overline{k-2, \alpha}} \{(x - \alpha + kz - (k-2)\alpha) + (k-1)(x + \alpha)\} \\
 &= k\alpha(x + z)(x - \alpha + kz)^{\overline{k-2, \alpha}}.
 \end{aligned}$$

Hence $\Delta_x^{+, \alpha} A_k(x, z) = k\alpha(x + z)(x - \alpha + kz)^{\overline{k-2, \alpha}}$.

Since $A_{k-1}(x, z) = x(x - \alpha + (k-1)z)^{\overline{k-2, \alpha}}$,

$$\begin{aligned}
 A_{k-1}(x + z, z) &= (x + z)(x + z - \alpha + (k-1)z)^{\overline{k-2, \alpha}} \\
 &= (x + z)(x - \alpha + kz)^{\overline{k-2, \alpha}}
 \end{aligned}$$

Therefore $\Delta_x^{+, \alpha} A_k(x, z) = k\alpha(x + z)(x - \alpha + kz)^{\overline{k-2, \alpha}} = k\alpha A_{k-1}(x + z, z)$.

(iii). Using the above (ii) repeatedly, we get for $m = 1, 2, \dots, k$

$$(\Delta_x^+)^m A_k(x, z) = k(k-1) \cdots (k-m+1) \alpha^m A_{k-m}(x + mz, z). \quad (6)$$

In fact, we assume the above equality holds for some m . Then we have

$$\begin{aligned}
 (\Delta_x^{+, \alpha})^{m+1} A_k(x, z) &= \Delta_x^{+, \alpha} \{(\Delta_x^{+, \alpha})^m A_k(x, z)\} \\
 &= \Delta_x^{+, \alpha} \{k(k-1) \cdots (k-(m-1)) \alpha^m A_{k-m}(x + mz, z)\} \\
 &= k(k-1) \cdots (k-(m-1)) \alpha^m \Delta_x^{+, \alpha} A_{k-m}(x + mz, z) \\
 &= k(k-1) \cdots (k-m) \alpha^{m+1} A_{k-(m+1)}(x + (m+1)z, z).
 \end{aligned}$$

Thus we find that the equality (6) holds by induction on m .

(iv). Let $f(x)$ be a polynomial $f(x) \in \mathbb{C}[x]$ of degree n . Since $A_k(x, z)$ is a polynomial of degree k with respect to x and $f(x)$ is a polynomial of degree n , we can set

$$f(x) = \sum_{k=0}^n a_k(z) A_k(x, z) \quad (7)$$

Substituting $x = 0$ in the equality (7), from the fact that $A_0(0, z) = 1$, $A_k(0, z) = 0$, ($k = 1, 2, \dots$), we get $a_0(z) = f(0)$. Operating $\Delta_x^{+, \alpha}$ to the both sides of the equality (7),

$$\begin{aligned}
 \Delta_x^{+, \alpha} f(x) &= \Delta_x^{+, \alpha} \left\{ \sum_{k=0}^n a_k(z) A_k(x, z) \right\} \\
 &= \sum_{k=0}^n a_k(z) \Delta_x^{+, \alpha} A_k(x, z) = \sum_{k=1}^n a_k(z) k\alpha A_{k-1}(x + z, z)
 \end{aligned}$$

Substituting $x = -z$ in the above equality,

$$\Delta_x^{+, \alpha} f(-z) = \sum_{k=1}^n a_k(z) k \alpha A_{k-1}(0, z) = \alpha a_1(z).$$

Similarly operating $(\Delta_x^{+, \alpha})^m$ to the both sides of the equality (7), we obtain with the repeated computations,

$$(\Delta_x^{+, \alpha})^m f(x) = \sum_{k=m}^n a_k(z) k(k-1) \cdots (k-(m-1)) \alpha^m A_{k-m}(x+mz, z).$$

Substituting $x = -mz$ in the above equality,

$$\begin{aligned} (\Delta_x^{+, \alpha})^m f(-mz) &= \sum_{k=m}^n a_k(z) k(k-1) \cdots (k-(m-1)) \alpha^m A_{k-m}(0, z) \\ &= m(m-1) \cdots 2 \cdot 1 \cdot \alpha^m a_m(z). \end{aligned}$$

Therefore we have $a_m(z) = \frac{1}{\alpha^m} \frac{1}{m!} (\Delta_x^{+, \alpha})^m f(-mz)$ for $m = 1, 2, \dots, n$.

□

We also have the following lemma which can be proven by the similar way to the proof of Lemma 1 with the generalized backward difference $\Delta_x^{-, \alpha}$.

Lemma 2. *Let polynomials $B_k(x, z) = x(x + \alpha + kz)^{\overline{k-1, \alpha}}$ in x, z , ($k = 0, 1, 2, \dots$). If $g(x) \in \mathbb{C}[x]$ is a polynomial in x of degree n , then*

$$\begin{aligned} g(x) &= \sum_{k=0}^n \frac{1}{\alpha^k} \frac{1}{k!} (\Delta_x^{-, \alpha})^k g(-kz) B_k(x, z) \\ &= \sum_{k=0}^n \frac{1}{\alpha^k} \frac{1}{k!} (\Delta_x^{-, \alpha})^k g(-kz) x(x + \alpha + kz)^{\overline{k-1, \alpha}}, \end{aligned}$$

where $\Delta_x^{-, \alpha}$ is a generalized backward difference with respect to x , given by $\Delta_x^{-, \alpha} g(x) \equiv g(x) - g(x - \alpha)$.

Proof of Theorem 1. We prove the equality (3). Let $f(x)$ be a polynomial in x of degree n $f(x) = (x+y)^{\underline{n, \alpha}} \in \mathbb{C}[x]$. Then we get $\Delta_x^{+, \alpha} f(x) = n\alpha(x+y)^{\underline{n-1, \alpha}}$ and $(\Delta_x^{+, \alpha})^2 f(x) = n(n-1)\alpha^2(x+y)^{\underline{n-2, \alpha}}$. More generally we have

$$(\Delta_x^{+, \alpha})^k f(x) = n(n-1) \cdots (n-k+1) \alpha^k (x+y)^{\underline{n-k, \alpha}}, \quad (k = 1, 2, \dots, n).$$

From the equality (5) in Lemma 1, we have

$$\begin{aligned}
 (x+y)^{n,\alpha} &= \sum_{k=0}^n \frac{1}{\alpha^k} \frac{1}{k!} (\Delta_x^{+,\alpha})^k f(-kz) x(x-\alpha+kz)^{\overline{k-1},\alpha} \\
 &= \sum_{k=0}^n \frac{1}{\alpha^k} \frac{1}{k!} n(n-1)\cdots(n-k+1) \alpha^k (-kz+y)^{\overline{n-k},\alpha} x(x-\alpha+kz)^{\overline{k-1},\alpha} \\
 &= \sum_{k=0}^n \binom{n}{k} x(x-\alpha+kz)^{\overline{k-1},\alpha} (y-kz)^{\overline{n-k},\alpha}
 \end{aligned}$$

Therefore the equality (3) has been proved. The equality (4) can be also proven by the similar way using Lemma 2. □

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