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POSTULATION OF GENERAL UNIONS OF DOUBLE LINES, LINES AND DOUBLE POINTS IN A PROJECTIVE SPACE

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: We study the Hilbert function of general unions $X \subset \mathbb{P}^r$ of double points and a " small " number of lines and double lines. Non asymptotic results are given only for $r = 3$.

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1. Introduction

For each $P \in \mathbb{P}^r$, $r \geq 3$, let $2P$ be the closed subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_P)^2$ as its ideal sheaf. We will say that $2P$ is a 2-point and that P is the support of $2P$. For any line $L \subset \mathbb{P}^r$, $r \geq 3$, let $2L$ be the closed subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_L)^2$ as its ideal sheaf. We will say that $2L$ is a 2-line and that P is the support of $2L$. Since the conormal bundle of L is isomorphic to $\mathcal{O}_L(-1)^{\oplus(r-1)}$, $2L$ a scheme with $\chi(\mathcal{O}_{2L}(t)) = rt + 1$ for all $t \in \mathbb{Z}$.

For all $(t, a, c) \in \mathbb{N}^3$ and all $r \geq 3$ let $L(t, a, c; r)$ denote the set of all schemes $X \subset \mathbb{P}^r$ which are the disjoint union of t lines, a 2-points and c 2-lines.

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Theorem 1. *Fix an integer $k \geq 5$ and $(t,a,c) \in \mathbb{N}^3$ such that $3t+c \leq \lfloor k^2/64 \rfloor$. Fix a general $X \in L(t,a,c;3)$.*

(i) If $t(k+1)+(3k+1)c+4a \leq \binom{k+3}{3}$, then $h^1(\mathcal{I}_X(k)) = 0$.

(ii) If $t(k+1)+(3k+1)c+4a \geq \binom{k+3}{3}$, then $h^0(\mathcal{I}_X(k)) = 0$

Remark 1. Assume $3t+c \leq \lfloor (k-1)^2/64 \rfloor$ and $t(k+1)+(3k+1)c+4a \leq \binom{k+3}{3}$. Fix a general $X \in L(t,a,c;3)$. By part (i) for the integer k and part (ii) for the integer $k-1$ we get that X has maximal rank. The best constant we could obtain when $k \gg 0$ with the proof in this paper is $3t+c \leq k^2/32$, but we have numerical problems for low k. For a fixed $c \leq k^2/32$ it is easy to get some t with $3t+c > k^2/32$ (but not by much) and for which the proof works; in step (ii) of the proof of Theorem 1 we need to modify step (i2) adding a step $k-4 \Longrightarrow k-2$ in which we add lines in Q.

In section 3 we describe all cases with $k \leq 5$, the cases with $k=6$ and $c=1$ and a few other cases. There are a few cases with $h^0(\mathcal{I}_X(k)) \cdot h^1(\mathcal{I}_X(k)) > 0$ for a general $X \in L(t,a,c;3)$, but each case may be explained by a geometrical reason for its bad Hilbert function in degree k.

In the case $r > 4$ we first take the case $c=0$ with an additional (unknown) zero-dimensional scheme Z (see [4], [6], [7] for the case $Z = \emptyset$ and/or $at = 0$).

Proposition 1. *For all integers $r \geq 4$ and $z \geq 0$ there is an integer $\beta_{r,z} > 0$ such that for every zero-dimensional scheme $Z \subset \mathbb{P}^r$ with $\deg(Z) \leq z$ and all $(t,a) \in \mathbb{N}^2$ with $a \geq \alpha_{r,x}$ a general union of Z and a general element of $L(t,a,0;r)$ has maximal rank.*

Proposition 2. *For all integers $r \geq 4$ and $z \geq 0$ there is an integer $\tau_{r,z} > 0$ with the following property: for every zero-dimensional scheme $Z \subset \mathbb{P}^r$ with $\deg(Z) \leq z$ and all $(t,a) \in \mathbb{N}^2$ with $t \geq \tau_{r,z}$ a general union of Z and a general element of $L(t,a,0;r)$ has maximal rank.*

Taking the union of Propositions 1 and 2 we get the following result.

Corollary 1. *For all integers $r \geq 4$ and $z \geq 0$ there is an integer $\delta_{r,z} > 0$ such that for every zero-dimensional scheme $Z \subset \mathbb{P}^r$ with $\deg(Z) \leq z$ and all $(t,a) \in \mathbb{N}^2$ with $a+t \geq \delta_{r,z}$ a general union of Z and a general element of $L(t,a,0;r)$ has maximal rank.*

Remark 2. Fix an integer $x > 0$ and assume $r \geq x+4$. Fix a degree x integer-valued polynomial ψ with $\deg(\psi) = x$ which is the Hilbert polynomial of some subscheme of $\mathbb{P}^r$. As in [5] we may use induction on r and x and get integers $\beta_{r,\psi}$, $\tau_{r,\psi}$ and $\delta_{r,\psi}$ which works as in Propositions 1, 2 and in Corollary 1.

In the case $z = 0$ and $c > 0$ see Proposition 3.

Conjecture 1. *Fix an integer $r \geq 3$. Is there an integer u_r with the following property: fix $(t,a,c) \in \mathbb{N}^3$ such that $t+a \geq u_r$; has a general element of $L(t,a,c;r)$ maximal rank ?*

We recall that [10] raised the question of the Hilbert function of unions of multiples of linear subspaces.

2. Preliminaries

For any finite subset $S \subset \mathbb{P}^r$ set $2S := \cup_{O \in S} 2O$. For any finite union $E \subset \mathbb{P}^r$ of disjoint lines set $2E := \cup_{L \subseteq E} 2L$. Fix a hyperplane $H \subset \mathbb{P}^r$. For any $P \in H$, any finite subset $S \subset H$, any line $L \subset H$ and any union $E \subset H$ of finitely may disjoint lines set $\{2P, H\} ::= 2P \cap H$, $\{2S, H\} := 2S \cap H$, $\{2L, H\} := 2L \cap H$ and $\{2E, H\} := 2E \cap H$. Write $L(t,a,c) := L(t,a,c;3)$ and call $X(t,a,c)$ a general element of $L(t,a,c)$.

For any closed subscheme $X \subset \mathbb{P}^r$ the residual scheme $\mathrm{Res}_H(X)$ of X with respect to H is the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_X : \mathcal{I}_H$ as its ideal sheaf.

Let $X \subset \mathbb{P}^r$ be a closed subscheme. Fix a hypersurface $T \subset \mathbb{P}^r$ and set $y := \deg(T)$. The residual scheme $\mathrm{Res}_T(X)$ of X with respect to t is the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_X : \mathcal{I}_T$ as its ideal sheaf. For each $x \in \mathbb{Z}$ we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_T(X)}(x-y) \to \mathcal{I}_X(x) \to \mathcal{I}_{X \cap T, T}(x) \to 0 \tag{1}$$

which is often called the Castelnuovo's sequence or the Horace sequence.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. For any $P \in Q$, any finite subset $S \subset H$, any line $L \subset H$ and any union $E \subset H$ of finitely may disjoint lines set $\{2P, Q\} := 2P \cap Q$, $\{2S, Q\} := 2S \cap Q$, $\{2L, Q\} := 2L \cap Q$ and $\{2E, Q\} := 2E \cap Q$. Call $|\mathcal{O}_Q(1,0)|$ and $|\mathcal{O}_Q(0,1)|$ the two rulings of Q. Fix $D \in |\mathcal{O}_Q(e,f)|$ and a closed subscheme A of Q. The residual scheme $\mathrm{Res}_D(A)$ of A with respect to D is the closed subscheme of Q with $\mathcal{I}_A : \mathcal{I}_D$ as its ideal sheaf. For all $(a,b) \in \mathbb{Z}^2$ we have an exact sequence (called again a Castelnuovo's sequence):

$$0 \to \mathcal{I}_{\mathrm{Res}_D(A)}(a-e, b-f) \to \mathcal{I}_A(a,b) \to \mathcal{I}_{A \cap D, D}(a,b) \to 0 \tag{2}$$

Set $L(t,a,c) := L(t,a,c;3)$. For any $(t,a,b,c) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$ let $X(t,a,c)$ denote the general member of $L(t,a,c)$. We write $L(0,0,0) := \{\emptyset\}$. Therefore $X(0,0,0) = \emptyset$. For any $(t,a,c) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$ the minimal integer k such that $(k+1)t + (3k+1)c + 4a \leq \binom{k+3}{3}$ is called the *critical value* of k. We have $k = 1$ if and only if (t, a, b) is one of the following triples $(1, 0, 0)$, $(2, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. We claim that for any triple (t, a, c) with critical value 1 the scheme $X(t, a, c)$ has maximal rank. To check the Hilbert function in the case $(t, a, c; r) = (0, 0, 1; 3)$ it is sufficient to use the following well-known lemma.

Lemma 1. *We have* $h^1(\mathcal{I}_{X(0,0,1)}(x)) = 0$ *and* $h^0(\mathcal{I}_{X(0,0,1)}(x)) = \binom{x+3}{3} - 3x - 1$ *for all* $t > 0$.

We also use the following result ([4, Theorem 1]):

Lemma 2. *Fix* $(t, a) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ *and an integer* $k \geq 3$. *Then either* $h^0(\mathcal{I}_{X(t,a,0)}(k)) = 0$ *or* $h^1(\mathcal{I}_{X(t,a,0)}(k)) = 0$, *unless either* $(t, a, k) = (2, 3, 3)$ *or* $(t, a, k) = (0, 9, 4)$. *We have* $h^0(\mathcal{I}_{X(2,3,0)}(3)) = h^0(\mathcal{I}_{X(0,9,0)}(4)) = 1$.

We use the following well-known and elementary lemma.

Lemma 3. *Fix an integral surface* $T \subset \mathbb{P}^3$, *a closed subscheme* $X \subset \mathbb{P}^3$ *not containing* T, *integers* $t > 0$, $f > 0$ *and a general set* $S \subset T$ *such that* $\sharp(S) = f$. *Set* $e := \deg(T)$ *and* $u := h^0(T, \mathcal{I}_{X \cap T}(t))$.

(i) If $f \geq u$, *then* $h^0(\mathcal{I}_{X \cup S}(t)) = h^0(\mathcal{I}_{\mathrm{Res}_T(X)}(t - e))$.

(ii) If $f \leq u$, $h^1(\mathcal{I}_X(t)) = 0$ *and* $h^0(\mathcal{I}_{\mathrm{Res}_T(X)}(t - e)) \leq \max\{0, h^0(\mathcal{I}_X(t)) - f\}$, *then* $h^1(\mathcal{I}_{X \cup S}(t)) = 0$.

(iii) If $h^1(\mathcal{I}_X(t)) = 0$ *and* $h^0(\mathcal{I}_{\mathrm{Res}_T(X)}(t - e)) \leq \max\{0, h^0(\mathcal{I}_X(t)) - f\}$, *then either* $h^0(\mathcal{I}_{X \cup S}(t)) = 0$ *or* $h^1(\mathcal{I}_{X \cup S}(t)) = 0$.

Remark 3. Fix closed subschemes $Y \subset X \subset \mathbb{P}^r$ and $t \in \mathbb{N}$. If $h^0(\mathcal{I}_Y(t)) = 0$, then obviously $h^0(\mathcal{I}_X(t)) = 0$. Now assume that Y is the union of some of the connected components of Y. In this case the restriction map $H^0(\mathcal{O}_X(t)) \to H^0(\mathcal{O}_Y(t))$ is surjective. Hence $h^1(\mathcal{I}_Y(t)) \leq h^1(\mathcal{I}_X(t))$. If $X = A \sqcup Z$ and $Y = A \sqcup W$ with Z zero-dimensional, then the restriction map $H^0(\mathcal{O}_X(t)) \to H^0(\mathcal{O}_Y(t))$ is surjective. Hence $h^1(\mathcal{I}_Y(t)) \leq h^1(\mathcal{I}_X(t))$.

3. Low degrees

In this section we study the triples (t, a, c) whose associated integer k is low.

Lemma 4. *We have* $h^0(\mathcal{I}_{X(0,0,2)}(3)) = h^1(\mathcal{I}_{X(0,0,2)}(3)) = 0$.

Proof. Since $h^0(\mathcal{O}_{X(0,0,2)}(3)) = 2{\cdot}10 = \binom{6}{3}$, then $h^1(\mathcal{I}_{X(0,0,2)}(3)) = h^1(\mathcal{I}_{X(0,0,2)}(3))$. Let Q' be any smooth quadric surface containing the two lines of $X(0, 0, 2)_{\mathrm{red}}$, say as members of the ruling $|\mathcal{O}_{Q'}(1, 0)|$. Since $X(0, 0, 2) \cap Q' \in |\mathcal{O}_{Q'}(4, 0)|$, we have $h^0(Q', \mathcal{I}_{Q' \cap X(0,0,2)}(3)) = 0$. Hence Q' is a component of every cubic surface containing $X(0,0,2)$. Since there are infinitely many surfaces Q', we get $h^0(\mathcal{I}_{X(0,0,2)}(3)) = 0$. □

Lemma 5. *We have* $h^1(\mathcal{I}_{X(0,1,1)}(3)) = 0$ *and* $h^0(\mathcal{I}_{X(0,1,1)}(3)) = 6$.

Proof. Write $X(0,1,1) = 2P \sqcup 2L$ with L a line. Let H be the plane spanned by L and P. We have $\mathrm{Res}_H(2L \cup 2P) = L \cup \{P\}$ and hence

$$h^0(\mathcal{I}_{\mathrm{Res}_H(X(0,1,1)}(2)) = 6$$

and

$$h^1(\mathcal{I}_{\mathrm{Res}_H(X(0,1,1)}(2)) = 0.$$

Since $h^0(H, \mathcal{I}_{X(0,1,1)\cap H}(3)) = 0$, the Castelnuovo's sequence gives the lemma. □

Lemma 6. *We have* $h^1(\mathcal{I}_{X(1,1,1)}(3)) = 1$ *and* $h^0(\mathcal{I}_{X(1,1,1)}(3)) = 3$.

Proof. Since $h^0(\mathcal{O}_{X(1,1,1)}(3)) = 18$, we have

$$h^0(\mathcal{I}_{X(1,1,1)}(3)) = 2 + h^1(\mathcal{I}_{X(1,1,1)}(3)).$$

Write $X(1,1,1) = R \sqcup 2P \sqcup 2L$ with P a point and L, R lines. Let H be the plane spanned by R and P. We have $\mathrm{Res}_H(X(1,1,1)) = L \cup R \cup \{P\}$. Hence $h^0(\mathcal{I}_{\mathrm{Res}_H(X(1,1,1))}(2)) = 3$. Since $P \notin R$ and $L \cap R = 1$, $X(1,1,1) \cap H$ is the disjoint union of a multiple conic, the 2-point $\{2P, H\}$ of H and the point $R \cap H$. Hence $h^0(H, \mathcal{I}_{X(1,1,1)\cap H}(3)) = 0$. The Castelnuovo's sequence gives $h^0(\mathcal{I}_{X(1,1,1)}(3)) = 3$. □

Lemma 7. *We have* $h^0(\mathcal{I}_{X(0,2,1)}(3)) = 2$ *and* $h^1(\mathcal{I}_{X(0,2,1)}(3)) = 0$

Proof. Write $X(0,2,1) = 2L \sqcup 2O \sqcup 2P$ with L a line. Let H (resp. M) be the plane spanned by L and O (resp. L and P). Every cubic surface containing $2L \cup 2O \cup 2P$ contains $H \cup M$. Since $\mathrm{Res}_{H\cup M}(X(0,2,1)) = \{O, P\}$, we have $h^0(\mathcal{I}_{\mathrm{Res}_{H\cup M}(X(0,2,1))}(1)) = 2$ and $h^1(\mathcal{I}_{\mathrm{Res}_{H\cup M}(X(0,2,1))}(1)) = 0$. Apply the Castelnuovo's sequence. □

Lemma 8. *We have* $h^0(\mathcal{I}_{X(2,0,1)}(3)) = 2$ *and* $h^1(\mathcal{I}_{X(2,0,1)}(3)) = 0$

Proof. Let Q' be the only quadric containing the 3 disjoint lines $X(2,0,1)_{\mathrm{red}}$, say as a divisor of type $(3,0)$. Since $Q' \cap X(2,0,1)$ is a divisor of type $(4,0)$, we have $h^i(Q', \mathcal{I}_{X(2,0,1)\cap Q'}(3)) = 0$, $i = 0,1$. Since $\mathrm{Res}_{Q'}(X(2,0,1))$ is a line, we have $h^0(\mathcal{I}_{\mathrm{Res}_{Q'}(X(2,0,1))}(1)) = 2$ and $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}(X(2,0,1))}(1)) = 0$. Apply the Castelnuovo's sequence. □

In the same way we check the following lemma.

Lemma 9. *We have $h^0(\mathcal{I}_{X(t,a,c)}(3)) = 0$ if $t \ge t'$, $a \ge a'$, $c \ge c'$ and (t', a', c') is one of the following triples:* $(0,0,2)$, $(0,3,1)$, $(2,1,1)$, $(3,0,1)$, $(1,2,1)$.

Lemma 10. *We have $h^1(\mathcal{I}_{X(4,0,1)}(4)) = 0$ and $h^0(\mathcal{I}_{X(4,0,1)}(4)) = 2$.*

Proof. Since $h^0(\mathcal{O}_{X(4,0,1)}(4)) = 33 = \binom{7}{3} - 2$, we have $h^0(\mathcal{I}_{X(4,0,1)}(4)) = 2 + h^1(\mathcal{I}_{X(4,0,1)}(4))$. Take $L, M, R \in |\mathcal{O}_Q(1,0)|$ with $L \neq M$, $M \neq R$ and $L \neq R$. Take a general $Y \in L(2,0,0)$ and set $X := Y \cup 2L \cup M \cup R$. It is sufficient to prove that $h^1(\mathcal{I}_X(4)) = 0$. Since $Y \cap Q$ is a general union of 4 points, we have $h^1(Q, \mathcal{I}_{X\cap Q}(4)) = h^1(Q, \mathcal{I}_{Y\cap Q}(0,4)) = 0$. Since $\mathrm{Res}_Q(X) = Y \cup L$ and $h^1(\mathcal{I}_{Y\cup L}(2)) = 0$, Castelnuovo's sequence gives $h^1(\mathcal{I}_{X(4,0,1)}(4)) = 0$ □

Since any two points of $\mathbb{P}^3$ are contained in a line, Lemma 10 implies the following result.

Lemma 11. *We have $h^0(\mathcal{I}_{X(5,0,1)}(4)) = 0$.*

Lemma 12. *We have $h^0(\mathcal{I}_{X(4,1,1)}(4)) = 0$ and $h^1(\mathcal{I}_{X(4,1,1)}(x)) = 0$ for all $x \ge 5$.*

Proof. Take $L, M, R \in |\mathcal{O}_Q(1,0)|$ with $L \neq M$, $M \neq R$ and $L \neq R$ and a general $O \in Q$. Take a general $Y \in L(2,0,0)$ and set $X := Y \cup 2L \cup M \cup R \cup 2O$. It is sufficient to prove that $h^0(\mathcal{I}_X(4)) = 0$ and $h^1(\mathcal{I}_X(x)) = 0$ for all $x \ge 5$. Since $Y \cap Q$ is a general union of 4 points, we have $h^1(Q, \mathcal{I}_{X\cap Q}(4)) = 0$ and $h^1(Q, \mathcal{I}_{X\cap Q}(y)) = 0$ for all $y \ge 5$. Since $\mathrm{Res}_Q(X) = Y \cup L \cup \{O\}$, $Y \nsubseteq Q$ and O is general in Q, we have $h^0(\mathcal{I}_{\mathrm{Res}_Q(X)}(2)) = 0$ and $h^1(\mathcal{I}_{\mathrm{Res}_Q(X)}(x-2)) = 0$ for all $x \ge 5$. Apply the Castelnuovo's sequence. □

Lemma 13. *We have $h^1(\mathcal{I}_{X(3,1,1)}(4)) = h^0(\mathcal{I}_{X(3,2,1)}(4)) = 0$, while we have $h^1(\mathcal{I}_{X(3,2,1)}(x)) = 0$ for all $x \ge 5$.*

Proof. Fix $L, M, R \in |\mathcal{O}_Q(1,0)|$ with $L \neq M$, $M \neq R$ and $L \neq R$. Take a general $Y \in L(1,1,0)$ and set $X := 2L \cup M \cup R \in L(3,1,1)$. We have $\mathrm{Res}_Q(X) = Y$, $h^1(\mathcal{I}_Y(2)) = 0$ and $h^1(Q, \mathcal{I}_{X\cap Q}(4)) = 0$, proving the first equality. Fix a plane H, a line $L \subset H$, $O \in H \setminus L$, and a general $Y \in L(3,1,0)$. Set $X' := Y \cup 2L \cup 2O$. We have $\mathrm{Res}_H(X') = Y \cup \{O\} \cup L$. Since $Y \cup L$ is general in $L(4,1,0)$, we have $h^i(\mathcal{I}_{Y\cup L}(3)) = 0$, $i = 0, 1$ (Lemma 1) and hence $h^0(\mathcal{I}_{\mathrm{Res}_H(X')}(3)) = 0$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(X')}(x-1)) = 0$ for all $x \ge 5$. Use the Castelnuovo's sequence. □

Lemma 14. *We have* $h^0(\mathcal{I}_{X(2,3,1)}(4)) = h^1(\mathcal{I}_{X(2,3,1)}(4)) = 1$, *while we have* $h^1(\mathcal{I}_{X(2,3,1)}(x)) = 0$ *for all* $x \geq 5$.

Proof. Since $h^0(\mathcal{I}_{X(2,3,1)}(4)) = \binom{7}{3}$, we have

$$h^0(\mathcal{I}_{X(2,3,1)}(4)) = h^1(\mathcal{I}_{X(2,3,1)}(4)).$$

Write $X(2,3,1) = 2L \sqcup M \sqcup R \sqcup 2S$ with $\sharp(S) = 3$. Let Q_1 (resp. Q_2) be the only quadric containing $L \cup M \cup S$ (resp. $L \cup R \cup S$). Since $X(2,3,1) \subset Q_1 \cup Q_2$, we have $h^0(\mathcal{I}_{X(2,3,1)}(4)) \geq 1$. Fix a plane H, a line $R \subset H$, general $P_1, P_2 \in H$ and a general $Y \in L(1,1,1)$. Set $X := Y \cup R \cup 2P_1 \cup 2P_2$. We have $h^i(H, \mathcal{I}_{X \cap H}(4)) = 0$, $i = 0, 1$, and $\mathrm{Res}_H(X) = Y \cup \{P_1, P_2\}$. We have $h^1(\mathcal{I}_Y(3)) = 1$ and $h^0(\mathcal{I}_Y(3)) = 3$ (Lemma 6). Since $h^0(\mathcal{I}_Y(3)) < 4$, we have $h^0(\mathcal{I}_Y(2)) = 0$. Lemma 3 gives $h^0(\mathcal{I}_{Y \cup \{P_1,P_2\}}(3)) = 1$. Apply the Castelnuovo's sequence. Since $h^1(\mathcal{I}_{Y \cup \{P_1,P_2\}}(4)) = 0$, we also get $h^1(\mathcal{I}_X(x)) = 0$ for all $x \geq 5$. □

Lemma 15. *We have* $h^0(\mathcal{I}_{X(0,5,1)}(4)) = 3$ *and* $h^1(\mathcal{I}_{X(0,5,1)}(4)) = 1$. *We have* $h^1(\mathcal{I}_{X(1,5,1)}(x)) = 0$ *for all* $x \geq 5$.

Proof. Since $h^0(\mathcal{O}_{X(0,5,1)}(4)) = 33$, $h^0(\mathcal{I}_{X(0,5,1)}(4)) = 2 + h^1(\mathcal{I}_{X(0,5,1)}(4))$. Set $D := X(0,5,1)_{\mathrm{red}}$. We have $h^0(\mathcal{I}_D(2)) = 2$. Fix any $Q_1, Q_2 \in |\mathcal{I}_D(2)|$. Since $X(3,0,1) \subset Q_1 \cup Q_2$, we get $h^0(\mathcal{I}_{X(0,5,1)}(4)) \geq 3$. Fix any smooth $Q \in |\mathcal{I}_D(2)|$. Since $h^0(Q, \mathcal{I}_Z(2,4)) = h^1(Q, \mathcal{I}_Z(2,4)) = 1$ for a general union $Z \subset Q$ of five 2-points of Q (this is one of the exceptional cases in [14, Table I], $\mathrm{Res}_Q(X(0,5,1)) = D$ and $h^1(\mathcal{I}_D(2)) = 0$, we get the first two assertions. Since $h^1(Q, \mathcal{I}_Z(y-2, y)) = 0$ for all $y \geq 5$, we get $h^1(\mathcal{I}_{X(1,5,1)}(x)) = 0$ for all $x \geq 5$. □

Lemma 16. *We have* $h^0(\mathcal{I}_{X(1,5,1)}(4)) = 0$ *and* $h^1(\mathcal{I}_{X(1,5,1)}(x)) = 0$ *for all* $x \geq 5$.

Proof. Write $X(1,5,1) = R \cup 2L \cup 2S$ with $S = \{P_1, P_2, P_3, P_4, P_5\}$ general points of $\mathbb{P}^3$. Fix a smooth quadric surface $Q \in |\mathcal{I}_{S \cup L}(2)|$. Since $h^0(Q, \mathcal{I}_Z(2,4)) = 1$ and $h^1(Q, \mathcal{I}_Z(3,5)) = 0$ for a general union $Z \subset Q$ of 5 2-points of Q and $R \cap Q$ is a general union of 2 points of Q, we get $h^0(Q, \mathcal{I}_{X(1,5,1)}(4)) = 0$ and $h^1(Q, \mathcal{I}_{X(1,5,1)}(x)) = 0$ for all $x \geq 5$. Since $\mathrm{Res}_Q(X(1,5,1)) = L \cup R \cup S$ is contained in no quadric surface, a Castelnuovo's sequence gives the lemma. □

Lemma 17. *We have* $h^0(\mathcal{I}_{X(0,6,1)}(4)) = 1$ *and* $h^1(\mathcal{I}_{X(0,6,1)}(4)) = 3$. *We have* $h^1(\mathcal{I}_{X(0,6,1)}(x)) = 0$ *for all* $x \geq 5$.

Proof. Set $L \cup S := X(0,6,1)_{\mathrm{red}}$ with L the line. Let Q be the only quadric containing $L \cup S$. Q is smooth. We have $\mathrm{Res}_Q(2S \cup 2L) = S \cup L$. We have $h^0(Q, \mathcal{I}_{X(0,6,1)\cap Q}(4)) = h^0(Q, \mathcal{I}_{(2S\cap Q)}(2,4)) = 0$. Hence the Castelnuovo's sequence gives $h^0(\mathcal{I}_{X(0,6,1)}(4)) = h^0(\mathcal{I}_{L\cup S}(2)) = 1$. Since $h^0(\mathcal{I}_{X(0,6,1)}(4)) = 37$, we have $h^1(\mathcal{I}_{X(0,6,1)}(4)) = 2 + h^0(\mathcal{I}_{X(0,6,1)}(4)) = 0$. Since $h^1(Q, \mathcal{I}_{(2S\cap Q)}(3,5)) = 0$, the Castelnuovo's sequence gives $h^1(\mathcal{I}_{X(0,6,1)}(x)) = 0$ for all $x \geq 5$. □

Lemma 18. *We have* $h^0(\mathcal{I}_{X(1,1,2)}(4)) = h^1(\mathcal{I}_{X(1,1,2)}(4)) = 1$*, while we have* $h^1(\mathcal{I}_{X(1,1,2)}(x)) = 0$ *for all* $x \geq 5$*.*

Proof. Since $h^0(\mathcal{O}_{X(1,1,2)}(4)) = 35$, we have $h^0(\mathcal{I}_{X(1,1,2)}(4)) = h^1(\mathcal{I}_{X(1,1,2)}(4))$. Write $X(1,1,2) = Z \sqcup 2L \sqcup 2R \sqcup M$ with L, M, N general lines. Let Q be the only cubic containing $L \cup R \cup M$, say as a divisor of type $(3,0)$. Since Z is general, we have $Z_{\mathrm{red}} \notin Q$, i.e. $\mathrm{Res}_Q(X(1,1,2)) = L \cup R \cup Z$. Since $h^0(Q, \mathcal{I}_{X(1,1,2)\cap Q}(4)) = 0$, we have $h^0(\mathcal{I}_{X(1,1,2)}(4)) = h^0(\mathcal{I}_{Z\cup L\cup R}(2))$. Set $\{P\} := Z_{\mathrm{red}}$. The only quadric surface containing $Z \cup L \cup R$ is the union of the two planes through P containing one of the lines L, R. Since $h^1(Q, \mathcal{I}_{Q\cap X(1,1,2)}(t)) = 0$ and $h^1(\mathcal{I}_{Z\cup L\cup R}(x-2)) = 0$ for all $x \geq 5$, we get the last assertion. □

Lemma 19. *We have* $h^0(\mathcal{I}_{X(2,0,2)}(4)) = 1$ *and* $h^1(\mathcal{I}_{X(2,0,2)}(4)) = 2$*. We have* $h^1(\mathcal{I}_{X(2,0,2)}(x)) = 0$ *for all* $x \geq 5$*.*

Proof. Write $X(2,0,2) = L \sqcup R \sqcup 2M \sqcup 2N$. Let Q_1 (resp. Q_2) be the only quadric containing $L \cup M \cup N$ (resp. $N \cup M \cup N$), say as a curve of type $(3,0)$ (each Q_i is smooth). Since each curve $Q_i \cap X(2,0,2)$ has type $(5,0)$, no curve of type $(4,4)$ contains it. Hence $Q_1 \cup Q_2$ is the only quartic surface containing $X(2,0,2)$. Since $h^0(\mathcal{O}_{X(2,0,2)}(4)) = 36$, we have $h^1(\mathcal{I}_{X(2,0,2)}(4)) = h^0(\mathcal{I}_{X(2,0,2)}(4)) + 1$. Since $h^1(Q_1, \mathcal{I}_{X(2,0,2)\cap Q_1}(5)) = 0$, we get $h^1(\mathcal{I}_{X(2,0,2)}(x)) = 0$ for all $x \geq 5$. □

Lemma 20. *We have* $h^0(\mathcal{I}_{X(0,2,2)}(4)) = 3$ *and* $h^1(\mathcal{I}_{X(0,2,2)}(4)) = 2$*.*

Proof. Write $X(0,2,2) = 2O \sqcup 2P \sqcup 2M \sqcup 2N$ with M, N lines and O, P points. Since M, N, O, and P are general, we have $h^0(\mathcal{I}_{\{O,P\}\cup M\cup N}(2)) = 2$. Fix any $Q_1, Q_2 \in |\mathcal{I}_{\{O,P\}\cup M\cup N}(2)|$. Since $Q_1 \cup Q_2 \in |\mathcal{I}_{X(0,2,2)}(4)|$, we get $h^0(\mathcal{I}_{X(0,2,2)}(4)) \geq 3$. To check the opposite inequality fix a smooth $Q' \in |\mathcal{I}_{\{O,P\}\cup M\cup N}(2)|$, a general $P_1 \in Q'$ and general points $P_2, P_3 \in \mathbb{P}^3$. We have $h^0(Q', \mathcal{I}_{(X(0,2,2)\cap Q')\cup\{P_1\}}(4)) = 0$, because $h^0(Q', \mathcal{I}_{\{2O,Q'\}\cup\{2P,Q'\}}(0,4)) = 1$. Since $\{P_2, P_3\}$ are general in $\mathbb{P}^3$, we have $h^0(\mathcal{I}_{\{P_2,P_3,O,P\}\cup M\cup N}(2)) = 0$. Since $\mathrm{Res}_{Q'}(\{P_1, P_2, P_3\} \cup X(0,2,2)) = \{P_2, P_3, O, P\} \cup M \cup N$, the Castelnuovo's sequence gives $h^0 0(\mathcal{I}_{X(0,2,2)}(4)) \leq 3$.

Since $h^0(\mathcal{O}_{X(0,2,2)}(4)) = 34$, we have $h^1(\mathcal{I}_{X(0,2,2)}(4)) = h^0(\mathcal{I}_{X(0,2,2)}(4)) - 1$. □

Lemma 21. *We have* $h^0(\mathcal{I}_{X(3,0,2)}(4)) = h^0(\mathcal{I}_{X(2,1,2)}(4)) = h^0(\mathcal{I}_{X(1,2,2)}(4)) = 0$.

Proof. Since a line contains two points and a 2-point contains a point, Lemma 19 implies $h^0(\mathcal{I}_{X(3,0,2)}(4)) = h^0(\mathcal{I}_{X(2,1,0)}(4)) = 0$. Write $X(1,2,2) = L \sqcup 2N \sqcup 2M \sqcup 2P \sqcup 2O$ and call H the plane spanned by N and P. The scheme $X(1,2,2) \cap H$ is the union of an unreduced conic, two general 2-points of H (one of them being $M \cap H$) and a general point (i.e. $L \cap H$). Hence $h^0(H, \mathcal{I}_{X(1,2,2)\cap H}(4)) = 0$. The scheme $\mathrm{Res}_H(X(1,2,2))$ is a general union of a general element of $L(1,1,1)$ and a general point, O, of H. Lemmas 3 and 6 and a Castelnuovo's sequence gives $h^0(\mathcal{I}_{X(1,2,2)}(4)) = 0$; alternatively, use Lemma 18. □

Lemma 22. *We have* $h^0(\mathcal{I}_{X(0,3,2)}(4)) = 1$ *and* $h^1(\mathcal{I}_{X(0,3,2)}(4)) = 4$.

Proof. Write $X(0,3,2) = 2P_1 \sqcup 2P_2 \sqcup 2P_3 \sqcup L \sqcup 2M$ with L, M lines and P_1, P_2, P_3 general points. Let Q' be the only quadric surface containing $\{P_1, P_1, P_3\} \cup L \cup M$. Since $2Q'$ is a quartic surface containing $X(0,3,2)$, we have $h^0(\mathcal{I}_{X(0,3,2)}(4)) \geq 1$. The quadric Q' is smooth. Call $\mathcal{O}_{Q'}(1,0)$ the ruling of Q' containing L. The singular points of any $D \in |\mathcal{O}_{Q'}(0,e)|$, $e > 0$, are the union of the multiple components of D. Since $X(0,3,2) \cap Q'$ is the union of a divisor of type $(4,0)$ and3 general 2-points of Q' , we have $h^0(Q', \mathcal{I}_{X(0,3,2)\cap Q'}(4)) = 0$. Since Q' is the only quadric surface containing $\{P_1, P_1, P_3\} \cup L \cup M = \mathrm{Res}_{Q'}(X(0,3,2))$, the Castelnuovo's sequence gives $h^0(\mathcal{I}_{X(0,3,2)}(4)) \leq 1$. Since $h^0(\mathcal{O}_{X(0,3,2)}(4)) = 38$, we have $h^1(\mathcal{I}_{X(0,3,2)}(4)) = h^0(\mathcal{I}_{X(0,3,2)}(4)) + 3$. □

Lemma 23. *We have* $h^0(\mathcal{I}_X(4)) = 1$, $h^1(\mathcal{I}_X(4)) = 5$ *and* $h^1(\mathcal{I}_X(x)) = 0$ *for every* $x \geq 5$ *and every* $X \in L(0,0,3)$.

Proof. Set $D := X_{\mathrm{red}}$ and call Q' the only quadric containing D. Q' is smooth and we call $(1,0)$ the ruling of Q' such that $X \cap Q' \in |\mathcal{O}_{Q'}(6,0)|$. Since any two D's are projectively equivalent, all $X \in L(0,0,3)$ are projectively equivalent. We have $h^0(Q', \mathcal{I}_{X\cap Q'}(4)) = 0$, $h^1(Q', \mathcal{I}_{X\cap Q'}(t)) = 0$ for all $t \geq 5$, and $\mathrm{Res}_{Q'}(X(0,0,3)) = D$. The Castelnuovo's sequence gives $h^0(\mathcal{I}_{X(0,0,3)}(4)) = h^0(\mathcal{I}_D(2)) = 1$. Since $h^0(\mathcal{O}_X(4)) = 3 \cdot 13 = \binom{7}{3} + 4$, we have $h^1(\mathcal{I}_{X(0,0,3)}(4)) = h^0(\mathcal{I}_{X(0,0,3)}(4)) + 4$. Since $h^1(\mathcal{I}_D(x-2)) = 0$ for all $x \geq 5$, we get $h^1(\mathcal{I}_X(x)) = 0$ for all $x \geq 5$. □

Remark 4. By Lemmas 23, 21, and 22 we have $h^0(\mathcal{I}_{X(t,a,c)}(4)) = 0$ if $t \geq t'$, $c \geq c'$, $a \geq a'$ and (t', a', c') is one of the following triples: $(1, 0, 3)$, $(0, 1, 3)$, $(0, 0, 4)$, $(3, 0, 2)$, $(2, 1, 2)$, $(1, 2, 2)$.

Lemma 24. *We have* $h^0(\mathcal{I}_{X(1,0,3)}(5)) = 4$ *and* $h^1(\mathcal{I}_{X(1,0,3)}(5)) = 2$. *We have* $h^1(\mathcal{I}_{X(1,0,3)}(x)) = 0$ *for all* $x \geq 6$.

Proof. Write $X(1, 0, 3) = L \sqcup A$ with L a line and let Q' be the only quadric containing $D := A_{\mathrm{red}}$, say as an element of $|\mathcal{O}_{Q'}(3, 0)|$. Since $A \cap Q' \in |\mathcal{O}_{Q'}(6, 0)|$, we have $h^0(Q', \mathcal{I}_{X(1,0,3)\cap Q'}(5)) = 0$. The Castelnuovo's sequence (1) gives $h^0(\mathcal{I}_{X(1,0,3)}(5)) = h^0(\mathcal{I}_{L\cup D}(3)) = 4$ ([13]). Since $h^0(\mathcal{O}_{X(1,0,3)}(5)) = 54 = \binom{8}{3} - 2$, we get $h^1(\mathcal{I}_{X(1,0,3)}(5)) = 2$. Since $X(1, 0, 3) \cap Q'$ is the union of $A \cap Q'$ and two general points of Q', we have $h^1(Q', \mathcal{I}_{X(1,0,3)\cap Q'}(x)) = 0$ for all $x \geq 6$. The Castelnuovo's sequence (1) gives $h^1(\mathcal{I}_{X(1,0,3)}(x)) = h^1(\mathcal{I}_{D\cup A}(x)) = 0$ for all $x \geq 6$. □

Lemma 25. *We have* $h^0(\mathcal{I}_{X(2,0,3)}(5)) = 0$ *and* $h^1(\mathcal{I}_{X(2,0,3)}(x)) = 0$ *for all* $x \geq 6$.

Proof. Write $X(2, 0, 3) = R \sqcup A$ with $R \in L(2, 0, 0)$ and $A \in L(0, 0, 3)$. Let Q' be the only quadric surface containing $D := A_{\mathrm{red}}$, say as an element of $|\mathcal{O}_{Q'}(3, 0)|$. Since $X(2, 0, 3) \cap Q'$ is a general union of an element of $|\mathcal{O}_{Q'}(6, 0)|$ and 4 general points of Q', the Castelnuovo's sequence gives $h^0(\mathcal{I}_{X(2,0,3)}(5)) = h^0(\mathcal{I}_{R\cup D}(3)) = 0$ and $h^1(\mathcal{I}_{X(2,0,3)}(x)) = h^0(\mathcal{I}_{R\cup D}(x - 2)) = 0$ for all $x \geq 6$ ([13]). □

Lemma 26. *We have* $h^0(\mathcal{I}_{X(1,1,3)}(5)) = 0$ *and* $h^1(\mathcal{I}_{X(1,1,3)}(x)) = 0$ *for all* $x \geq 6$.

Proof. Write $X(1, 1, 3) = L \sqcup Z \sqcup A$ with $A \in L(0, 0, 3)$ and $Z \in L(0, 1, 0)$. Let Q' be the only quadric containing $D := A_{\mathrm{red}}$, say as a divisor of type $(3, 0)$. Since $X(1, 1, 3)$ is general, then $Z \cap Q' = \emptyset$ and $L \cap Q'$ is a general union of 2 points of Q'. Hence $h^0(\mathcal{I}_{X(1,1,3)}(5)) = h^0(\mathcal{I}_{L\cup D\cup Z}(3)) = 0$ ([4]) and $h^1(\mathcal{I}_{X(1,1,3)}(x)) = h^0(\mathcal{I}_{L\cup D\cup Z}(x - 2)) = 0$ for all $x \geq 6$. □

Lemma 27. *We have* $h^i(\mathcal{I}_{X(0,2,3)}(5)) = 0$, $i = 0, 1$.

Proof. Write $X(0, 2, 3) = Z \sqcup A$ with $A \in L(0, 0, 3)$ and $Z \in L(0, 2, 0)$. Let Q' be the only quadric containing $D := A_{\mathrm{red}}$, say as a divisor of type $(3, 0)$. Since Z is general, then $Z \cap Q' = \emptyset$. Hence $h^i(\mathcal{I}_{X(0,2,3)}(5)) = h^i(\mathcal{I}_{D\cup Z}(3)) = 0$, $i = 0, 1$ (Lemma 1). □

Lemma 28. *We have $h^1(\mathcal{I}_{X(t,a,1)}(5)) = 0$ for all $(a,t) \in \mathbb{N}^2$ such that $6t + 4a + 16 \le 56$.*

Proof. Increasing if necessary a we reduce to the case $37 \le 6t + 4a \le 40$. Set $e := \lfloor (10-t)/3 \rfloor$ and $f := 10 - t - 3e$. The quadruples (t,a,e,f) are the following ones: $(6,1,1,1)$, $(5,2,1,2)$, $(4,4,2,0)$, $(3,5,2,1)$, $(2,7,2,2)$, $(1,8,3,0)$, $(0,10,3,1)$. By the semicontinuity theorem it is sufficient to prove the existence of $X \in L(t,a,1)$ such that $h^1(\mathcal{I}_X(5)) = 0$.

(i) Assume $t \le 4$. Let $H \subset \mathbb{P}^3$ be a plane. We fix $L \in L(0,0,1)$ contained in H and take a general $S \cup S' \subset H$ with $\sharp(S) = e$, $\sharp(S') = f$. Set $A := 2L$. Fix a general $W \in L(t, a-e-f, 0)$. We have $h^i(H, \mathcal{I}_E(3)) = 0$, $i = 0,1$, for a general union $E \subset H$ of $f + t$ points and e 2-points of H. By the differential Horace lemma for double points ([1], [11, Lemma 5], [2] in characteristic $\ne 2$) to prove that a general union of $W \cup 2L \cup 2S$ and f 2-points (and hence to prove the lemma in these cases) it is sufficient to prove that $h^1(\mathcal{I}_{W\cup L\cup S\cup\{2S',H\}}(4)) = 0$.

Claim 1: $h^1(\mathcal{I}_{W\cup L\cup\{2S',H\}}(4)) = 0$.

Proof of Claim 1: If $t \in \{4,1\}$, then $S' = \emptyset$ and hence Claim 1 is true by [4]. Now assume $t \in \{0,3\}$. In this case S' is a point. Since any union of a line and a point is contained in a plane, $W \cup L \cup 2S'$ may be considered as a general element of $L(t+1, a-e+1, 0)$. In these two cases we have $h^1(\mathcal{I}_{W\cup L\cup 2S'}(4)) = 0$ by [4], Hence $h^1(\mathcal{I}_{W\cup L\cup\{2S',H\}}(4)) = 0$ (Lemma 3).

Now assume $(t,a,e,f) = (2,7,2,2)$. Since $W \cap H$ is a general subset with cardinality 2, we have $h^1(H, \mathcal{I}_{H\cap(W\cup L\cup S\cup\{2S',H\})}(4)) = 0$. Hence by the Castelnuovo's sequence it is sufficient to use that $h^1(\mathcal{I}_W(3)) = 0$ in this case, because $W \in L(3,2,0)$ (Lemma 1).

Claim 1 implies $h^0(\mathcal{I}_{W\cup L\cup\{2S',H\}}(4)) = e + 40 - 4a - 6t$. Since S is general in H and $\mathrm{Res}_H(W \cup L \cup \{2S',H\}) = W$, to get $h^1(\mathcal{I}_{W\cup L\cup S\cup\{2S',H\}}(4)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_W(3)) \le 40 - 6t - 4a$. We have $h^0(\mathcal{O}_W(3)) = 4t + 4(a-e-f)$; we use Lemma 1, because $W \notin L(2,3,0)$.

(ii) Now assume $t \in \{5,6\}$. First assume $(t,a) = (6,1)$. Let $Z \subset \mathbb{P}^3$ be a general 2-point. Take $L \in |\mathcal{O}_Q(1,0)|$ and $F \in |\mathcal{O}_Q(3,0)|$ such that F is reduced and $L \cap F = \emptyset$. Set $A := 2L$, so that $A \cap Q \in |\mathcal{O}_Q(2,0)|$ and $\mathrm{Res}_Q(A) = L$. Set $W := G \cup F \cup A$. Since G is general, $Q \cap G$ is a general union of 6 points. Hence $h^i(Q, \mathcal{I}_{W\cap Q}(5)) = 0$, $i = 0,1$. Hence by the Castelnuovo's sequence it is sufficient to use that $h^i(\mathcal{I}_{Z\cup L\cup G}(3)) = 0$, $i = 0,1$ by [4]. Now assume $(t,a) = (5,2)$. Take L, $A = 2L$ and F as in the previous case. Fix a general $G' \in L(2,0,0)$ and a general 2-point Z' of $\mathbb{P}^3$. Set $Z := Z' \cup 2O$. By the semicontinuity theorem it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup G'\cup A\cup F}(5)) = 0$. Since $(G' \cup Z) \cap Q$ is a general union of 4 points and one 2-point of Q, we have $h^i(Q, \mathcal{I}_{Q\cap(Z\cup G'\cup A\cup F)}(5)) = 0$. Hence by the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{G'\cup Z'\cup\{O\}\cup L}(3)) = 0$. We have $h^1(\mathcal{I}_{G'\cup L\cup Z'}(3)) = 0$ ([4]) and hence $h^0(\mathcal{I}_{G'\cup L\cup Z'}(3)) = 3$. Since $h^0(\mathcal{I}_{G'\cup Z}(1)) = 0$ and O is general in Q, we get $h^0(\mathcal{I}_{G'\cup L\cup Z'\cup\{O\}}(3)) = 2$. and hence $h^1(\mathcal{I}_{G'\cup L\cup Z'\cup\{O\}}(3)) = 0$.

(iii) Assume $(t, a) = (2, 7)$. We take $R, L, M \in |\mathcal{O}_Q(1, 0)|$ with $L \neq R$, $L \neq M$ and $R \neq M$. We take a general $S \subset Q$ with $\sharp(S) = 4$ and a general $Z \in L(0, 3, 0)$. Set $X := 2L \cup R \cup M \cup 2S \cup Z$. It is sufficient to prove that $h^1(\mathcal{I}_X(5)) = 0$. Since $X\cap Q$ is the union of an element of $|\mathcal{O}_Q(4, 0)|$ and 4 general 2-points of Q, we have $h^i(Q, \mathcal{I}_{X\cap Q}(5)) = 0$, $i = 0, 1$ ([14, Proposition 5.2 and Theorem 7.2]). Hence by the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup S\cup L}(3)) = 0$. We have $h^1(\mathcal{I}_{Z\cup L}(3)) = 0$ and $h^0(\mathcal{I}_{Z\cup L}(4)) = 4$ (Lemma 1). Since S is general in Q and $h^0(\mathcal{I}_Z(1)) = 0$, Lemma 3 gives $h^i(\mathcal{I}_{\mathrm{Res}_Q(X)}(3)) = 0$, $i = 0, 1$. □

Lemma 29. *We have* $h^1(\mathcal{I}_{X(t,a,2)}(5)) = 0$ *for all* $(a, t) \in \mathbb{N}^2$ *such that* $6t + 4a + 32 \leq 56$.

Proof. Set $\beta := 24 - 6t - 4a$. Increasing if necessary a we reduce to the case $0 \leq 6t+4a \leq 3$. Hence it is sufficient to check the following triples (t, a, β): $(4, 0, 0)$, $(3, 1, 2)$, $(2, 3, 0)$, $(1, 4, 2)$, $(0, 6, 0)$.

(i) First assume $3 \leq t \leq 4$. Fix $L, R, M \in |\mathcal{O}_Q(1, 0)|$ such that $R \neq L$, $R \neq M$ and $L \neq M$. Fix a general $W \in L(t-1, a, 0)$ and set $X := W\cup 2L\cup 2R\cup M$. Since $W\cap Q$ is a general union of $2t-2$ points and $(2L\cup 2R\cup M) \in |\mathcal{O}_Q(5, 0)|$, we have $h^1(Q, \mathcal{I}_{X\cap Q}(5)) = 0$. By the Castelnuovo's sequence it is sufficient to use that $h^1(\mathcal{I}_{W\cup L}(3)) = 0$ by Lemma 1.

(ii) Assume $(t, a) = (2, 3)$. Fix a plane H, a general line R, a line $L \subset H$, a general $Y \in L(2, 0, 0)$, a general $O \in H$ and a general $B \subset H$ such that $\sharp(B) = 2$. Set $X' := 2L \sqcup 2R \sqcup 2O \sqcup Y$. We have $h^i(H, \mathcal{I}_{(X'\cap H)\cup S'}(6)) = 0$, $i = 0, 1$, and $\mathrm{Res}_H(X') = 2R \cup L \cup Y \cup \{O\}$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5], [2] in characteristic $\neq 2$) to prove that a general union X of X' and two general 2-points of $\mathbb{P}^3$ satisfies $h^i(\mathcal{I}_X(5)) = 0$, $i = 0, 1$, (and hence to prove the lemma for the pair $(t, a) = (2, 3)$) it is sufficient to prove that $h^i(\mathcal{I}_{\{2B,H\}\cup 2R\cup L\cup Y\cup\{O\}}(4)) = 0$, $i = 0, 1$. Let $M \subset \mathbb{P}^3$ be a general plane containing R. Set $D := H \cap M$. Write $B = \{P_1, P_2\}$ with P_1, P_2 general points of H. We specialize O to a general point O_1 of D. We specialize $\{2P_1, H\} \cup \{2P_2, H\}$ to $\{2P_1, H\} \cup \{2O', H\}$ with O' general point of D. Set $Y'' := \{2P_1, H\} \cup \{2O', H\} \cup 2R \cup L \cup Y \cup \{O_1\} = Y_1 \sqcup 2R$. We have $\{2O', H\} \cap M = 2O' \cap D$, i.e. $\{2O', H\} \cap M$ is the degree two scheme with O' as its reduction and contained in D. We get $h^i(M, \mathcal{I}_{Y_1\cap M}(2)) = 0$, $i = 0, 2$, because $(Y \cup L) \cap M$ is a general union of $\{2O', H\} \cap M$, the general point O_1

of D and 3 general points of M. Since $h^i(\mathcal{I}_{\mathrm{Res}_M(Y')}(3)) = 0$, $i = 0,1$, we are done.

(iii) Take $(t,a,\beta) = (1,4,2)$. Take $L, R, M \in |\mathcal{O}_Q(1,0)|$ with $L \neq R$, $L \neq M$ and $R \neq M$. Fix a general $S \subset Q$ such that $\sharp(S) = 3$ and a general $Y \in L(1,1,0)$. Set $X := 2L \cup 2R \cup 2S \cup Y$. It is sufficient to prove that $h^1(\mathcal{I}_X(5)) = 0$. Since $X \cap Q$ is a union of a divisor of type $(4,0)$, 3 2-points of Q and two points, we have $h^1(Q, \mathcal{I}_{X\cap Q}(5)) = 0$. Since $\mathrm{Res}_Q(X) = L \cup R \cup Y \cup S$, by the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{L\cup R\cup Y\cup S}(3)) = 0$. We have $h^1(\mathcal{I}_{L\cup R\cup Y}(3)) = 0$ by Lemma 1 and hence $h^0(\mathcal{I}_{L\cup R\cup Y}(3)) = 4$. Since S is general in Q and $\sharp(S) \leq 4$, it is sufficient to observe that $h^0(\mathcal{I}_Y(1)) = 0$ (Lemma 3).

(iv) Take $(t,a,\beta) = (0,6,0)$. Take $L, R \in |\mathcal{O}_Q(1,0)|$ with $L \neq R$. Fix a general $S \subset Q$ with $\sharp(S) = 4$ and a general $Z \in L(0,2,0)$. Set $X = Z \cup 2L \cup 2R \cup 2S$. It is sufficient to prove that $h^i(\mathcal{I}_X(5)) = 0$. Since $X \cap Q$ is a the union of a divisor of type $(4,0)$ and 3 general 2-points of Q, we have $h^i(Q, \mathcal{I}_{X\cap Q}(5)) = 0$, $i = 0,1$ ([14, Proposition 5.2 and Theorem 7.2]). Hence it is sufficient to prove that $h^i(\mathcal{I}_{Z\cup L\cup R\cup S}(3)) = 0$, $i = 0,1$. We first check that $h^1(\mathcal{I}_{Z\cup L\cup R}(3)) = 0$. Since any two disjoint lines of $\mathbb{P}^3$ are contained in a smooth quadric, $Z \cup L \cup R$ may be considered as a general element of $L(2,2,0)$. Hence $h^1(\mathcal{I}_{Z\cup L\cup R}(3)) = 0$ and $h^0(\mathcal{I}_{Z\cup L\cup R}(3)) = 4$. Since S is general in Q, $\mathrm{Res}_Q(Z \cup L \cup R) = Z$ and $\sharp(S) = 4$, to prove that $h^i(\mathcal{I}_{Z\cup L\cup R\cup S}(3)) = 0$, $i = 0,1$, it is sufficient to use the obvious observation that $h^0(\mathcal{I}_Z(1)) = 0$ (Lemma 3). □

Lemma 30. *We have $h^0(\mathcal{I}_{X(t,a,1)}(5)) = 0$ for all $(a,t) \in \mathbb{N}^2$ such that $6t + 4a + 16 \geq 56$.*

Proof. Set $\beta' := 6t + 4a - 40$. Increasing if necessary a we reduce to the case $0 \leq \beta' \leq 3$. Notice that β' is even. Since all cases with $\beta' = 0$ are covered by Lemma 29, it is sufficient to check all cases with $\beta' = 2$, i.e. the following pairs (t,a): $(7,0)$, $(5,3)$, $(3,6)$, and $(1,9)$.

(i) Assume $(t,a) = (7,0)$. Take a disjoint union $E \in |\mathcal{O}_Q(3,0)|$ of 3 lines and $L \in |\mathcal{O}_Q(1,0)|$ with $L \cap E = \emptyset$. Take a general $Y \in L(4,0,0)$ and set $X := Y \cup E \cup 2L$. By the semicontinuity theorem it is sufficient to prove that $h^0(\mathcal{I}_X(5)) = 0$. Since $X \cap Q$ is the disjoint union of a divisor of type $(5,0)$ and 6 general points, we have $h^0(Q, \mathcal{I}_{Q\cap X}(5)) = 0$. Since $\mathrm{Res}_Q(X) = Y \cup L$ may be considered as a general union of 5 lines, we have $h^0(\mathcal{I}_{Y\cup L}(3)) = 0$ ([13]). Apply the Castelnuovo's sequence.

(ii) Assume $(t,a) = (5,3)$. Fix a general $Y \in L(3,1,0)$, 3 distinct lines $L, M, R \in |\mathcal{O}_Q(1,0)|$ and a general $S \subset Q$ such that $\sharp(S) = 2$. Set $X := Y \cup 2S \cup 2L \cup M \cup R$. By semicontinuity it is sufficient to prove that $h^0(\mathcal{I}_X(5)) = 0$.

We have $h^1(Q,\mathcal{I}_{\{2S,Q\}}(1,5)) = 0$ ([14, Proposition 5.2 and Theorem 7.2]). Since $Y \cap Q$ is a general union of 6 points, we get $h^i(Q,\mathcal{I}_{X\cap Q}(5)) = 0$, $i = 0,1$. We have $\mathrm{Res}_Q(X) = Y \cup L \cup S$. Since $Y \cup L$, may be considered as a general element of $L(4,1,0)$, we have $h^i(\mathcal{I}_{Y\cup L}(3)) = 0$, $i = 0,1$ (Lemma 1). Hence $h^0(\mathcal{I}_{Y\cup L\cup S}(3)) = 0$. Use the Castelnuovo's sequence.

(iii) Assume $(t,a) = (3,6)$. Fix a general $Y \in L(3,0,0)$. Take $L \in |\mathcal{O}_Q(1,0)|$ and a general $S \subset Q$ with $\sharp(S) = 6$. Set $X := Y \cup 2L \cup 2S$. It is sufficient to prove that $h^0(\mathcal{I}_X(5)) = 0$. Since $h^1(Q,\mathcal{I}_{\{2S,Q\}}(3,5)) = 0$ ([14, Proposition 5.2 and Theorem 7.2]) and $Y \cap Q$ is a general union of 6 points of Q, we have $h^i(Q,\mathcal{I}_{X\cap Q}(5)) = 0$, $i = 0,1$. We have $\mathrm{Res}_Q(X) = Y \cup L \cup S$. Since $Y \cup L$ may be considered as a general element of $L(4,0,0)$, we have $h^0(\mathcal{I}_{Y\cup L}(3)) = 4$ and $h^0(\mathcal{I}_{Y\cup L}(1)) = 0$. Since S is a general union of 6 points of Q, Lemma 3 gives $h^0(\mathcal{I}_{\mathrm{Res}_Q(X)}(3)) = 0$. Apply the Castelnuovo's sequence.

(iv) Assume $(t,a) = (1,9)$. Fix $L, R \in \mathcal{O}_Q(1,0)|$ such that $L \neq R$, a general $Z \in L(0,3,0)$ and a general $S \subset Q$ such that $\sharp(S) = 6$. Set $X := 2L\cup R\cup Z\cup 2S$. It is sufficient to prove that $h^0(\mathcal{I}_X(5)) = 0$. Since $h^i(\mathcal{I}_{\{2S,Q\}}(2,5)) = 0$, $i = 0,1$ ([14, Proposition 5.2 and Theorem 7.2]), we have $h^i(Q,\mathcal{I}_{X\cap Q}(5)) = 0$, $i = 0,1$. We have $\mathrm{Res}_Q(X) = L \cup Z \cup S$. Since $h^1(\mathcal{I}_Z(3)) = 0$ (Lemma 1), we have $h^0(\mathcal{I}_{Z\cup L}(3)) = 4$. Obviously $h^0(\mathcal{I}_Z(1)) = 0$. Since $\mathrm{Res}_Q(Z \cup L) = Z$ and S is a general union of 6 points of Q, Lemma 3 gives $h^0(\mathcal{I}_{Z\cup L\cup S}(3)) = 0$. The Castelnuovo's sequence gives $h^0(\mathcal{I}_X(5)) = 0$. □

Lemma 31. *We have $h^0(\mathcal{I}_{X(t,a,2)}(5)) = 0$ for all $(a,t) \in \mathbb{N}^2$ such that $6t + 4a + 32 \geq 56$.*

Proof. Set $\beta' := 6t+4a-24$. Notice that β' is even. Increasing if necessary a we reduce to the case $0 \leq \beta' \leq 3$. Since all cases with $\beta' = 0$ are covered by Lemma 29, we may assume $\beta' = 2$. Hence either $(t,a) = (3,2)$ or $(t,a) = (1,5)$.

(i) Assume $(t,a) = (3,2)$. Fix lines $L, R \in |\mathcal{O}_Q(1,0)|$, a general $S \subset Q$ with $\sharp(S) = 2$ and a general $Y \in L(3,0,0)$. Set $X := 2L \cup 2R \cup Y \cup 2S$. It is sufficient to prove that $h^0(\mathcal{I}_X(5)) = 0$. Since $h^1(Q,\mathcal{I}_{\{2S,Q\}}(1,5)) = 0$ ([14, Proposition 5.2 and Theorem 7.2]) and $Y \cap Q$ is a general union of 6 points, we have $h^i(Q,\mathcal{I}_{X\cap Q}(5)) = 0$, $i = 0,1$. Since (for a general Q) $Y \cup L$ is a general union of 4 lines, we have $h^0(\mathcal{I}_{Y\cup L}(3)) = 0$. Obviously $h^0(\mathcal{I}_Y(1)) = 0$. Since $\mathrm{Res}_Q(X) = Y \cup L \cup S$ and S is a general union of 6 points of Q, Lemma 3 gives $h^0(\mathcal{I}_{Y\cup L\cup S}(3)) = 0$. Apply the Castelnuovo's sequence.

(ii) Assume $(t,a) = (1,5)$. Fix lines $L, R \in |\mathcal{O}_Q(1,0)|$, a general $S \subset Q$ with $\sharp(S) = 4$ and a general $Y \in L(1,1,0)$. Set $X := Y \cup 2S \cup 2L \cup 2R$. It is sufficient to prove that $h^0(\mathcal{I}_X(5)) = 0$. Since $h^i(\mathcal{I}_{\{2S,Q\}}(1,5)) = 0$, $i = 0,1$ ([14, Proposition 5.2 and Theorem 7.2]), we have $h^0(Q,\mathcal{I}_{X\cap Q}(5)) = 0$. We have $\mathrm{Res}_Q(X) = Y \cup L \cup R \cup S$. Since any two disjoint lines are contained in a smooth quadric, $Y \cup L \cup R$ may be considered as a general element of $L(3,1,0)$. Hence $h^0(\mathcal{I}_Y(3) = 4$ (Lemma 1). Since $h^0(\mathcal{I}_Y(1)) = 0$, Lemma 3 gives $h^0(\mathcal{I}_{Y \cup L \cup R \cup S}(3)) = 0$. Use the Castelnuovo's sequence. □

Lemma 32. *We have* $h^0(\mathcal{I}_{X(t,a,3)}(5)) = h^0(\mathcal{I}_{X(t,a,0)}(3))$ *and* $h^1(\mathcal{I}_{X(t,a,3)}(5)) \geq 2t$. *If either* $1 \leq t \leq 4$ *and* $t + a \leq 4$ *or* $(t,a) = (2,3)$, *then*

$$h^0(\mathcal{I}_{X(t,a,3)}(5)) \cdot h^0(\mathcal{I}_{X(t,a,3)}(5)) > 0.$$

If $5t + 4a \leq 35$, *then* $h^1(\mathcal{I}_{X(t,a,3)}(x)) = 0$ *for all* $x \geq 6$.

Proof. Write $X(t,a,c) = 2L_1 \sqcup 2L_2 \sqcup 2L_3 \sqcup B \sqcup A$ with L_1, L_2, L_3 skew lines, B general in $L(t,0,0)$ and A general in $L(0,a,0)$. Let Q' be the unique smooth quadric containing $L_1 \cup L_2 \cup L_3$, say as an element of $|\mathcal{O}_Q(3,0)|$. Since $(2L_1 \cup 2L_2 \cup 2L_3) \cap Q \in |\mathcal{O}_Q(6,0)|$ and $B \cap Q$ is a general union of $2t$ points of Q and $A \cap Q = \emptyset$, we have $h^0(Q, \mathcal{I}_{X(t,a,c)}(5)) = 0$ and $h^1(Q, \mathcal{I}_{X(t,a,c)}(5)) = 2t$. Since $\mathrm{Res}_Q(X(t,a,c)) = B \cup A$ is a general element of $L(t,a,0)$ and $h^2(\mathcal{I}_{A \cup B}(3)) = h^1(\mathcal{O}_{A \cup B}(3)) = 0$, the Castelnuovo's sequence gives

$$h^0(\mathcal{I}_{X(t,a,3)}(5)) = h^0(\mathcal{I}_{X(t,a,0)}(3))$$

and

$$h^1(\mathcal{I}_{X(t,a,3)}(5)) \geq 2t.$$

The second part follows from Lemma 1, i.e. from [4]. The third part follows from the Castelnuovo's sequence, because $h^1(\mathcal{I}_{X(t,a,0)}(x-2)) = 0$ (Lemma 1). □

Lemma 33. *We have*

$$h^0(\mathcal{I}_{X(0,0,4)}(5)) = 0,$$

$$h^1(\mathcal{I}_{X(0,0,4)}(6)) = h^0(\mathcal{I}_{X(0,0,4)}(6)) = 2$$

and

$$h^1(\mathcal{I}_{X(0,0,4)}(x)) = 0$$

for all $x \geq 7$.

Proof. Set $D := X(0,0,4)_{\mathrm{red}}$ and write $D = L \sqcup B$ with $B \in L(3,0,0)$. Let Q' be the only smooth quadric containing B, say as lines of type $(1,0)$. The scheme $X(0,0,4) \cap Q'$ is a disjoint union of a divisor of type $(6,0)$ and two general 2-points of Q. Hence $h^0(Q', \mathcal{I}_{Q' \cap X(0,0,4)}(5)) = 0$, $h^0(\mathcal{I}_{Q' \cap X(0,0,4)}(6)) = 2$,

$h^1(Q', \mathcal{I}_{Q'\cap X(0,0,4)}(6)) = 2$ and $h^1(Q', \mathcal{I}_{Q'\cap X(0,0,4)}(x)) = 0$ for all $x \geq 7$. The scheme $\mathrm{Res}_{Q'}(X(0,0,4)) = 2L \sqcup B$ is a general element of $L(3,0,1)$. We have $h^0(\mathcal{I}_{2L\cup B}(3)) = 0$ (Lemma 9) and $h^1(\mathcal{I}_{2L\cup B}(y)) = 0$ for all $y \geq 4$ (Remark 3 and Lemma 13). We have $h^2(\mathcal{I}_{2L\cup B}(4)) = 0$. Use several times the Castelnuovo's sequence. □

Lemma 34. *We have*

$$h^1(\mathcal{I}_{X(t,a,1)}(6)) = 0$$

for all $(t,a) \in \mathbb{N}^2$ *such that* $7t + 4a \leq 65$ *and*

$$h^0(\mathcal{I}_{X(t,a,1)}(6)) = 0$$

for all $(t,a) \in \mathbb{N}^2$ *such that* $7t + 4a \geq 65$ *(for the latter part we assume that either* $t \leq 4$ *or the characteristic is zero).*

Proof. In all cases by semicontinuity it is sufficient to find $X \in L(t,a,1)$ such that $h^0(\mathcal{I}_X(6)) \cdot h^1(\mathcal{I}_X(6)) = 0$.

(a) In this step we prove the first statement. Set $\beta := 65 - 7t - 4b$. Increasing if necessary b we reduce to the case $0 \leq \beta \leq 3$. We have the following triples (t,a,β): $(9,0,2)$, $(8,2,1)$, $(7,4,0)$, $(6,5,3)$, $(5,7,2)$, $(4,9,1)$, $(3,11,0)$, $(2,12,3)$, $(1,14,2)$, $(0,16,1)$.

(a1) Assume $(t,a) = (9,0)$. Take $L \in |\mathcal{O}_Q(1,0)|$ and $E \in |\mathcal{O}_Q(3,0)|$ with 3 connected components and $E \cap L = \emptyset$. Fix a general $Y \in L(6,0,0)$ and take $X := Y \cup E \cup 2L$. We have $\mathrm{Res}_Q(X) = Y \cup L$. Since $Y \cup L$ may be considered as a general union of 7 lines, we have $h^i(\mathcal{I}_{Y\cup L}(4)) = 0$, $i = 0,1$. Therefore by the Castelnuovo's sequence it is sufficient to prove that $h^1(Q, \mathcal{I}_{X\cap Q}(6)) = 0$. This is true, because $X \cap Q$ is the union of a divisor of type $(5,0)$ and 12 general points.

(a2) Assume $(t,a) = (8,2)$. Take $L \in |\mathcal{O}_Q(1,0)|$ and $E \in |\mathcal{O}_Q(3,0)|$ with 3 connected components and $E \cap L = \emptyset$. Fix a general $O \in Q$ and a general $Y \in L(5,1,0)$. Set $X := Y \cup E \cup 2O \cup 2L$. We have $\mathrm{Res}_Q(X) = Y \cup L \cup \{O\}$. Lemma 1 gives $h^1(\mathcal{I}_{Y\cup L}(4)) = 0$. Obviously $h^0(\mathcal{I}_Y(2)) = 0$. Since O is general in Q, we get $h^i(\mathcal{I}_{\mathrm{Res}_Q(X)}(4)) = 0$, $i = 0,1$ (Lemma 3). By the Castelnuovo's sequence it is sufficient to prove that $h^1(Q, \mathcal{I}_{X\cap Q}(6)) = 0$. This is true, because $X \cap Q$ is a general union of a divisor of type $(5,0)$, a general 2-point of Q and 10 general points of Q.

(a3) Assume $(t,a) = (7,4)$. Take $L \in |\mathcal{O}_Q(1,0)|$ and $E \in |\mathcal{O}_Q(3,0)|$ with 3 connected components and $E \cap L = \emptyset$. Fix a general $S \subset Q$ such that $\sharp(S) = 2$. Fix a general $Y \in L(4,2,0)$ and set $X := Y \cup 2L \cup E \cup 2S$. Since

$X \cap Q$ is a general union of a divisor of type $(5,0)$, two general 2-point of Q and 8 general points, we have $h^i(Q, \mathcal{I}_{X\cap Q}(6)) = 0$ ([14, Proposition 5.2 and Theorem 7.2]). By the Castelnuovo's sequence it is sufficient to prove that $h^i(\mathcal{I}_{\mathrm{Res}_Q(X)}(4)) = 0$, $i = 0,1$. We have $\mathrm{Res}_Q(X) = Y \cup L \cup S$. Since $Y \cup L$ may be considered as a general element of $L(5,2,0)$, we have $h^1(\mathcal{I}_{Y\cup L}(4)) = 0$. Obviously $h^0(\mathcal{I}_Y(2)) = 0$. Use Lemma 3.

(a4) Assume $0 \le t \le 6$. Set $\alpha_t := 3$ if $3 \le t \le 6$, $\alpha_2 = \alpha_1 = 1$ and $\alpha_0 = 0$. If $t \ne 2$, then set $e_t := \lfloor (7(5-\alpha_t) - 2(t-\alpha_t))/3 \rfloor$ and $f_t := 7(5-\alpha_t) - 3e_t - 2(t-\alpha_t)$. Set $e_2 := 6$ and $f_2 = 0$. We have $(e_6, f_6) = (2,2)$, $(e_5, f_5) = (3,1)$, $(e_4, f_4) = (4,0)$, $(e_3, f_3) = (4,2)$, $(e_1, f_1) = (9,1)$. In all cases we have $a \ge e_t + f_t$. Fix $L \in |\mathcal{O}_Q(1,0)|$ and $E \in |\mathcal{O}_Q(\alpha_t, 0)|$ with E with α_t connected components and $E \cap L = \emptyset$. Fix a general $S \cup S' \subset Q$ such that $\sharp(S) = \alpha_t$, $\sharp(S') = f_t$ and $S \cap S' = \emptyset$. Fix a general $Y \in L(t - \alpha_t, a - e_t - f_t, 0)$. Set $X_1 := 2L \cup Y \cup 2S$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5]), to prove that a general union X of X_1 and f_t 2-points (and hence to prove the lemma in this case) it is sufficient to prove that $h^1(Q, \mathcal{I}_{(X_1\cup Q)\cup S'}(6)) = 0$ and that $h^1(\mathcal{I}_{Y\cup L\cup S\cup\{2S',Q\}}(4)) = 0$. We first check that $h^1(\mathcal{I}_{Y\cup L\cup\{2S',Q\}}(4)) = 0$. By the last part of Remark 3 it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup L\cup 2S'}(4)) = 0$. In all cases we have $h^0(\mathcal{O}_{Y\cup L\cup 2S'}(4)) \le 35$. Hence Lemma 1 gives $h^1(\mathcal{I}_{Y\cup L\cup 2S'}(4)) = 0$. In all cases we have $h^0(\mathcal{O}_{(X_1\cap Q)\cup S'}(6)) \le 49$. In all cases we have $h^1(Q, \mathcal{I}_{(X_1\cap Q)\cup S'}(6)) = 0$ by [14, Proposition 5.2 and Theorem 7.2]; we chose the values of α_2, e_2 and f_2 to avoid the case $(2,6)$ with 7 2-points of Q, which in the list of exceptional cases.

(b) Now we prove the second statement. Set $\beta' := 7t + 4b - 65$. Decreasing if necessary a and, if $a = 0$, decreasing also t we reduce to the cases $0 \le \beta' \le 3$ and the cases $a = 0$ and $0 \le \beta' \le 7$. Since the first part covered all cases with $\beta' = 0$, we may also assume $\beta' > 0$. Hence we need to check the following triples (t, a, β'): $(10,0,5)$, $(9,1,2)$, $(8,3,3)$, $(6,6,1)$, $(5,8,2)$, $(4,10,3)$, $(2,13,1)$, $(1,14,2)$, $(0,17,3)$. However, since a 2-point contains a point, the case $(t,a,\beta') = (0,17,3)$ follows from the case $(t,a,\beta) = (0,16,1)$, the case $(t,a,\beta') = (8,3,3)$ follows from the case $(t,a,\beta) = (8,2,2)$ and the case $(t,a,\beta') = (4,10,3)$ follows from the case $(t,a,\beta) = (4,9,1)$ (cases proved in step (a)). A 2-point contains a tangent vector. Hence in characteristic zero all cases with $\beta' = 2$ are true by [12] or [8, Lemma 1.8]; to cover Theorem 1 in arbitrary characteristic we check in arbitrary characteristic the case $(t,a) = (1,14)$. Any two points of $\mathbb{P}^3$ are contained in a line. Hence the case $(t,a,\beta') = (10,0,5)$ of part (b) follows from the case $(t,a,\beta) = (9,0,2)$ of part (a).

(b1) Take $(t,a,\beta') = (6,6,1)$. Fix $L \in \mathcal{O}_Q(1,0)$, a general $E \in |\mathcal{O}_Q(2,0)|$, a general $S \subset Q$ with $\sharp(S) = 4$, a general $O \in Q$ and a general $Y \in L(4,1,0)$. Set $X' := Y \cup E \cup 2L \cup 2S$. We have $\mathrm{Res}_Q(X') = Y \cup L \cup S$. Moreover we have $h^i(Q, \mathcal{I}_{(Q\cap X')\cup\{O\}}(6)) = 0$, because $\{O\} \cup (Y \cap Q)$ is a general union of 9 points of Q and $h^0(Q, \mathcal{I}_{\{2S,Q\}}(2,6)) = 9$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5]) to prove that a general union of X' and a 2-point (and hence to prove the lemma in this case), it is sufficient to prove that $h^0(\mathcal{I}_{Y\cup L\cup S\cup\{2O,Q\}}(4)) = 0$. Lemma 1 gives $h^1(\mathcal{I}_{Y\cup L\cup 2O}(4)) = 0$ and hence $h^1(\mathcal{I}_{Y\cup L\cup\{2O,Q\}}(4)) = 0$ (last part of Remark 3), i.e. $h^0(\mathcal{I}_{Y\cup L\cup\{2O,Q\}}(4)) = 3$. Obviously $h^0(\mathcal{I}_Y(2)) = 0$. Since S is general, Lemma 3 gives

$$h^0(\mathcal{I}_{Y\cup L\cup S\cup\{2O,Q\}}(4)) = 0.$$

(b2) Assume $(t, a, \beta') = (2, 13, 1)$. Fix $L, R \in |\mathcal{O}_Q(1,0)|$ with $L \neq R$. Fix a general $S \subset Q$ such that $\sharp(S) = 9$ and a general $Y \in L(1,4,0)$. Set $X := Y \cup 2L \cup R \cup M \cup 2S$. We have $\mathrm{Res}_Q(X) = Y \cup L \cup S$. Lemma 1 gives $h^1(\mathcal{I}_{Y\cup L}(4)) = 0$ and $h^0(\mathcal{I}_{Y\cup L}(4)) = 9$. Obviously $h^0(\mathcal{I}_Y(2)) = 0$. Since S is general in Q, part (i) of Lemma 3 gives $h^0(\mathcal{I}_{\mathrm{Res}_Q(X)}(4)) = 0$. We have $h^i(\mathcal{I}_{X\cap Q}(6)) = 0$, $i = 0,1$, because $h^1(\mathcal{I}_{\{2S,Q\}}(3,6)) = 0$ ([14, Proposition 5.2 and Theorem 7.2]), i.e. $h^0(\mathcal{I}_{\{2S,Q\}}(3,6)) = 1$, and $Y \cap Q$ is a general union of two points of Q.

(b3) Assume $(t, a, \beta') = (1, 14, 2)$. Fix $L, R \in |\mathcal{O}_Q(1,0)|$ with $L \neq R$. Fix a general $S \subset Q$ such that $\sharp(S) = 10$ and a general $Y \in L(0,4,0)$. Set $X := Y \cup 2L \cup 2S$. We have $h^0(Q, \mathcal{O}_{Q\cap X}(6)) = 0$, because $h^0(Q, \mathcal{I}_{\{2S,Q\}}(3,6)) = 0$ ([14, Proposition 5.2 and Theorem 7.2]). By the Castelnuovo's sequence it is sufficient to prove that $h^i(\mathcal{I}_{\mathrm{Res}_Q(X)}(4)) = 0$, $i = 0,1$. We have $\mathrm{Res}_Q(X) = Y \cup L \cup S$. Lemma 1 gives $h^1(\mathcal{I}_{Y\cup L}(4)) = 0$ and hence $h^0(\mathcal{I}_{Y\cup L}(4)) = 10$. Since the singular locus of a quadric is a proper linear subspace, we have $h^0(\mathcal{I}_Y(2)) = 0$. Since the finite set S is general in Q and $\sharp(S) = 10$, Lemma 3 gives $h^i(\mathcal{I}_{Y\cup L\cup S}(4)) = 0$, $i = 0,1$. □

Lemma 35. *We have* $h^1(\mathcal{I}_{X(t,a,c)}(6)) \geq 2c - 6$.

Proof. We may assume $c \geq 4$. Write $X(t,a,c) = A \sqcup L_1 \sqcup \cdots \sqcup L_c$ with $A \in L(t,a,0)$ and $L_1, \dots, L_c$ lines. Let Q be the smooth quadric containing $L_1 \cup L_2 \cup L_3$. Call $|\mathcal{O}_Q(1,0)|$ the ruling of Q containing L_1. The generality of $X(t,a,c)$ gives that $X(t,a,c) \cap Q$ is a general union of a divisor of type $(6,0)$, $2c-6$ 2-points and $2t$ points. Hence $h^1(Q, \mathcal{I}_{X(t,0,c)\cap Q}(6)) \geq 2c - 6$. Use the Castelnuovo's sequence and that $h^2(\mathcal{I}_{X(t+3,a,c-3)}(4)) = 0$. □

Remark 5. By Lemmas 13 and 35 we have $h^0(\mathcal{I}_{X(t,a,c)}(6)) \cdot h^1(\mathcal{I}_{X(t,a,c)}(6)) > 0$ if either $(t,a,c) = (0,0,4)$ or $(t,a,c) = (0,1,4)$.

4. Proof of Theorem 1

Consider the following assertion H_k, $k \ge 5$:

Assertion H_k: Fix nonnegative integers t, c, a such that $t \le \lfloor (k+2)^2/64 \rfloor - \lfloor k^2/64 \rfloor$, $3t + c \le \lfloor k^2/64 \rfloor$ and $4a + t(k+1) + (3k+5)c \le \binom{k+3}{3}$. Then there exists $W \in L(t,a,c)$ such that $h^1(\mathcal{I}_W(k)) = 0$, the t lines of W are contained in a smooth quadric Q and two of the 2-points of W have support contained in the same quadric Q.

Proof of Theorem 1 and of Assertion H_k: Recall that $k \ge 5$. Fix any $A \in L(t,a,c)$ and assume $h^1(\mathcal{I}_A(k)) = 0$. Since $k \ge 2$, we have $h^2(\mathcal{I}_A(k-1)) = h^1(\mathcal{O}_A(k-1)) = 0$. Hence Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_A(x)) = 0$ for all $x > k$. See step (iii) for the case $6 \le k \le 19$. Before step (iii) we assume $k \ge 20$ and that $H_{k'}$ and Theorem 1 are true for all integers $k' < k$. In step (i) we prove H_k and part (i) of Theorem 1, while in step (ii) we consider part (ii) of Theorem 1.

(i) In this step we assume $t(k+1) + c(3k+1) + 4a \le \binom{k+3}{3}$. Set $\beta := \binom{k+3}{3} - (k+1)t - 4a - (3k+1)c$. Adding if necessary several 2-points we see that it is sufficient to do the cases with $0 \le \beta \le 3$. Since Theorem 1 is true if $k' < k$, sometimes we assume $3t + c > \lfloor (k-1)^2/64 \rfloor$ (not always, because to get H_k if $3t + c \le 3t + c > \lfloor (k-1)^2/64 \rfloor$ we need an additional argument).

(i1) Assume $c \le \lfloor (k-2)^2/64 \rfloor$. Set $t_1 := \lfloor (\lfloor (k-2)^2/64 \rfloor - c)/3 \rfloor$, i.e. let t_1 be the maximal integer such that $3t_1 + c \le \lfloor (k-2)^2/64 \rfloor$. Our assumption on c implies $t_1 \ge 0$ and $0 \le \lfloor (k-2)^2/64 \rfloor - c - 3t_1 \le 2$. Since $3t + c > \lfloor (k-1)^2/64 \rfloor$, we have $t > t_1$ (for H_k see step (i1.1)).

Claim 1: We have $t - t_1 \le k - 3$.

Proof of Claim 1: Assume $t - t_1 \ge k - 2$. Since $3t + c \le \lfloor k^2/64 \rfloor$ and $3t_1 + c \ge \lfloor (k-2)^2/64 \rfloor - 2$, we get $3(k-2) + 2 \ge \lfloor k^2/64 \rfloor - \lfloor (k-2)^2/64 \rfloor$, which is false for all $k \ge 5$.

Let $U \subset Q$ be a general union of $t - t_1$ lines of type $(1,0)$. Set $u := \lfloor ((k+1)(k+1+t_1-t) - 2t_1 - 6c)/3 \rfloor$ and $v := (k+1)(k+1+t_1-t) - 2t_1 - 6c$. We have $0 \le v \le 2$ and $3u + v + 2t_1 + 6c = (k+1)^2$.

Claim 2: We have $u \ge 2$.

Proof of Claim 2: Assume $u \le 1$. Since $v \le 2$, we get $5 + 2t_1 + 6c \ge (k+1)^2$. Since $t_1 \le t$ and $3t + c \le k^2/64$, we get a contradiction.

Fix a general $S \cup S' \subset Q$ such that $\sharp(S) = u$, $\sharp(S') = v$ and $S \cap S' = \emptyset$ (we are using that $u \ge 0$ and this inequality is true by Claim 2).

Claim 3: We have $a \ge u + v$.

Proof of Claim 3: Write u_k and v_k if we need to make explicit the dependence on k. Since $3u+v+2t_1+6c = (k+1)^2$ and $3t_1+c \ge (k-2)^2/64-3$, we have $3u+v \le (k+1)^2-(k-2)^2/96+2$. We get $3u_{10}+v_{10} \le 122$, $3u_{11}+v_{11} \le 145$, $3u_{12}+v_{12} \le 170$. Since $\beta \le 3$ and $3t+c \le k^2/64$, we have $4a+(3k+1)k^2/64 \ge \binom{k+3}{3} - 3$; if $a \le u+v-1$, then $4u+4v+(3k+1)k^2/64 \ge \binom{k+3}{3}+1$. We get $4u_{10}+4v_{10} \ge 238$, $4u_{11}+4v_{11} \ge 301$, $4u_{12}+4v_{12} \ge 373$. Since $v \le 2$, we get a contradiction if $k = 10, 11, 12$. Since $v \le 2$ the study of the function $\psi(k) = (\binom{k+3}{3}+1-(3k+1)k^2/64-8)/4-((k+1)^2-(k-2)^2/96+2)/3$ gives a contradiction for all $k \ge 13$, concluding the proof of Claim 3.

Fix a general $Y \in L(t_1, a-u-v, c)$ (it is defined, because $a-u-v \ge 0$ by Claim 3).

By Claim 1 we have $k-t+t_1 \ge 3$. Since $Y \cap Q$ is a general union of $2t_1+6c$ 2-points of Q, $S \cup S'$ is general in Q and $3u+v = (k+1)(k+1+t_1-t)$, we get $h^i(Q, \mathcal{I}_{U\cup(Y\cap Q)\cup S'\cup\{2S;Q\}}(k)) = 0$. By the differential Horace lemma for double points ([1], [11, Lemma 5]) to prove that $h^1(\mathcal{I}_M(k)) = 0$ for a general union M of $Y \cup U \cup 2S$ and v general 2-points it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup S\cup\{2S',Q\}}(k-2)) = 0$.

Claim 4: $h^1(\mathcal{I}_{Y\cup\{2S',Q\}}(k-2)) = 0$.

Proof of Claim 4: It is sufficient to prove that $h^1(\mathcal{I}_{Y\cup 2S'}(k-2)) = 0$. Since $v \le 2$ and any two points of $\mathbb{P}^3$ are contained in a smooth quadric, $Y \cup 2S$ has the Hilbert function of $X(t_1, a-u, c)$. We have $h^0(\mathcal{O}_{X(t_1,a-u,c)}(k-2)) = \binom{k+1}{3} - \beta - u + v$. Claim 2 gives $h^0(\mathcal{O}_{X(t_1,a-u,c)}(k-2)) \le \binom{k+1}{3}$. The inductive assumption gives $h^1(\mathcal{I}_{Y\cup 2S'}(k-2)) = 0$, proving Claim 4.

By Claim 4 we have $h^0(\mathcal{I}_{Y\cup\{2S',Q\}}(k-2)) = \beta + u$. By Lemma 3 and Claim 4 to prove that $h^1(\mathcal{I}_{Y\cup S\cup\{2S',Q\}}(k-2)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_Y(k-4)) \le \beta$. We have

$$(k-1)t_1 + (3k-5)c + 4(a-u-v) + u + 3v = \binom{k+1}{3} - \beta \tag{3}$$

Recall that $(k-2)^2/64-2 \le 3t_1+c \le (k-2)^2/64$. Hence there are integers t', c' such that $0 \le t' \le t_1$, $0 \le c' \le c$ and $(k-4)^2/64-3 < 3t'+c' \le (k-4)^2/64$. Take $M \in L(t', a-u-v, c')$ such that $M \subseteq Y$. Since $k \ge 9$, the inductive assumption gives that either $h^0(\mathcal{I}_M(k-4)) = 0$ or $h^1(\mathcal{I}_M(k-4)) = 0$. In the former case we get $h^0(\mathcal{I}_M(k-4)) = 0$, concluding the proof in this case. Assume $h^1(\mathcal{I}_M(k-4)) = 0$. We have $h^0(\mathcal{I}_M(k-4)) = \binom{k-1}{3} - 4(a-u-v) - (k-3)t' - (3k-11)c'$. Since $h^0(\mathcal{I}_Y(k-4)) \le h^0(\mathcal{I}_M(k-4))$, from (3) we get that it is sufficient to prove that

$$(k-1)(t-t') + (3k-5)(c-c') + 2t' + 6c' \le (k-1)^2 \tag{4}$$

We have $3(t_1 - t') + c - c' \le (k-2)^2/64 - (k-4)^2/64 + 3 = (k+36)/16$. We have $2t' + 6c' \le 6(k-4)^2/64$. Hence it is sufficient to have $(3k-5)(k+36)/16 + 3(k-4)^2/32 \le (k-1)^2$. This inequality is true for all $k \ge 9$.

(i1.1) Now we consider H_k in the set-up of step (i1). Hence $t \le \lfloor (k+2)^2/64 \rfloor - \lfloor k^2/64 \rfloor \le (k+20)/16$. We take Q as the smooth quadric. Let $V \subset Q$ be a general union of t lines of type $(1,0)$. Set $u' := \lfloor ((k+1)(k+1-t) - 6c)/3 \rfloor$ and $v' := (k+1)(k+1-t) - 6c - 3u'$. We have $0 \le v' \le 2$. Since $t \le (k+20)/16$ and $k \ge 20$, the analogous of Claim 1 holds true in this case. Since $t \le (k+20)/16$, we have $u' \ge 2$, i.e. the analogous of Claim 2 is true in this case. Fix a general $S_1 \cup S_1' \subset Q$ such that $\sharp(S_1) = u'$, $\sharp(S_1') = v'$ and $S_1 \cap S_1' = \emptyset$.

Claim 5: We have $a \ge u' + v'$.

Proof of Claim 5: We have

$$(3k-5)c + 4(a - u' - v') + u' + 3v' = \binom{k+1}{3} - \beta \tag{5}$$

and $3u' + v' + 6c = (k+1)(k+1-t)$ and hence $3u' + v' \le (k+1)^2$. Since $u \ge v'$, we have $u' + 3v' \le 3u' + v'$ and hence $u' + 3v' \le (k+1)^2$. Since $c \le (k-2)^2/64$ and $\beta \le 3$, (5) gives $a - u' - v' \ge 0$ for all $k \ge 10$.

Fix a general $Y' \in L(0, a - u' - v', c)$. Since $u' \ge 2$ to prove H_k for the integers t, c it is sufficient to prove that $h^1(\mathcal{I}_W(k)) = 0$ for a general union W of $Y \cup V \cup 2S_1$ and v' 2-points. Since $k + 1 - t \le k - 3$, [14, Proposition 5.2 and Theorem 7.2] gives $h^i(Q, \mathcal{I}_{(Y \cap Q) \cup \{2S_1, Q\} \cup S_1'}(k, k-t)) = 0$, $i = 0, 1$. By the differential Horace lemma for double points ([1], [11, Lemma 5]) to prove that $h^1(\mathcal{I}_W(k)) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Y' \cup S_1 \cup \{2S_1', Q\}}(k-2)) = 0$. We first check that $h^1(\mathcal{I}_{Y' \cup S' \cup \{2S_1', Q\}}(k-2)) = 0$. It is sufficient to check that $h^1(\mathcal{I}_{Y' \cup 2S_1'}(k-2)) = 0$. Since any two points of $\mathbb{P}^3$ are contained in a smooth quadric, the schemes $X(0, a-u', c)$ and $Y' \cup 2S_1'$ have the same Hilbert function. By (5) we have $h^0(\mathcal{I}_{Y' \cup 2S_1'}(k-2)) = \binom{k+1}{3} - \beta + v' - u'$. Since $u' \ge v'$ and $c \le \lfloor (k-2)^2/64 \rfloor$, the inductive assumption gives $h^1(\mathcal{I}_{Y' \cup 2S_1'}(k-2)) = 0$ and hence $h^1(\mathcal{I}_{Y' \cup \{2S_1', Q\}}(k-2)) = 0$, i.e. $h^0(\mathcal{I}_{Y' \cup \{2S_1', Q\}}(k-2)) = \beta + u'$. By Lemma 3 it is sufficient to prove that $h^0(\mathcal{I}_{Y'}(k-4)) \le \beta$.

Assume for the moment $c \ge \lfloor (k-4)^2/64 \rfloor$. Fix $M' \in L(0, a-u'-v', \lfloor (k-4)^2/64 \rfloor)$ such that $M' \subseteq Y'$. By the inductive assumption either $h^0(\mathcal{I}_{M'}(k-4)) = 0$ (and hence $h^0(\mathcal{I}_{Y'}(k-4)) = 0$) or $h^1(\mathcal{I}_{M'}(k-4)) = 0$, i.e. $h^0(\mathcal{I}_{M'}(k-4)) = \binom{k-1}{3} - (3k-11)c - 4(a'-u'-v')$. Since $h^0(\mathcal{I}_{M'}(k-4)) \ge h^0(\mathcal{I}_{Y'}(k-4))$, in order to obtain a contradiction we may assume $\binom{k-1}{3} - (3k-11)c - 4(a'-u'-v') \ge \beta + 1$. From (5) we get $6c + u' + 3v' \ge k^2 - 2k$. Since $6c + t(k+1) + 3u' + v' = (k+1)^2$ and $u' \ge v'$, we get $t < 4$, i.e. $t \le 3$. First assume $t = 3$. We get

$6c + 3u' + v' = k^2 - k + 2$, while $6c + u' + 3v' \ge k^2 - 2k$. Since $v' \le 2$, to get a contradiction it is sufficient to have $2u' \ge k + 1$. Assume $2u' \le k$. Since $6c \le 3(k-2)^2/32$ and $6c + 3u' + v' = k^2 - k + 2$, we get a contradiction. Now assume $t \le 2$. Since the union of two disjoint lines and any two points is contained in a smooth quadric, in this case H_k for the given (t, a, c) follows from part (i) of Theorem 1 for the same triple (t, a, c).

Now assume $c < \lfloor (k-4)^2/64 \rfloor$. By the inductive assumption either $h^0(\mathcal{I}_{Y'}(k-4)) = 0$ or $h^1(\mathcal{I}_{Y'}(k-4))$. We may assume that the latter occurs and that $h^0(\mathcal{I}_{Y'}(k-4)) \ge \beta + 1$, i.e. $4(a - u' - v') + (3k-11)c \le \binom{k-1}{3} - \beta - 1$. Therefore (5) gives $6c + u' + 3v' \ge (k-1)^2 - 1$. Since $6c + 3u' + v' + t(k+1) = (k+1)^2$ and $u' \ge v'$, we get $t \le 3$. Since $c < \lfloor (k-4)^2/64 \rfloor$ and $3t + c > \lfloor (k-1)^2/64 \rfloor$, we get a contradiction.

(i2) Assume $c > \lfloor (k-2)^2/64 \rfloor$. Set $m := c - \lfloor (k-2)^2/64 \rfloor$. We have $3t + m \le k^2/64 - (k-2)^2/64 + 1 = (k-1)/16 + 1 = (k+15)/16$.

Claim 6: We have $t + 2m \le k - 3$.

Proof of Claim 6: We just saw that $3t + m \le (k+15)/16$. We use that $k \ge 20$.

Let $F \subset Q$ be a disjoint union of t elements of type $|\mathcal{O}_Q(1,0)|$ and m connected elements of $|\mathcal{O}_Q(2,0)|$. The scheme F is a disjoint union of t lines and m 2-lines of Q and hence $h^0(\mathcal{O}_F(k)) = (t+2m)(k+1)$. Write $F = F_1 \sqcup F_2$ with F_2 the union of the degree two connected components and set $G := (F_2)_{\mathrm{red}}$, $E := F_1 \sqcup 2G$. We have $E \in L(t, 0, c - \lfloor (k-2)^2/64 \rfloor)$, $E \cap Q = F$ and $\mathrm{Res}_Q(E) = G$. Set $e := \lfloor (k+1)(k+1-t-2m) - 6\lfloor (k-2)^2/64 \rfloor)/3 \rfloor$ and $f := (k+1)(k+1-t-2m) - 6\lfloor (k-2)^2/64 \rfloor - 3e$. We have $0 \le f \le 2$.

Claim 7: We have $e \ge 2$.

Proof of Claim 7: Assume $e \le 1$. Since $f \le 2$, we get $(k+1)(k+1-t-2m) - 6\lfloor (k-2)^2/64 \rfloor \le 5$. Since $3t+m \le (k+15)/16$, we have $t+2m \le (k+15)/8$. Hence $(k+1)(k+1-t-2m) \ge 7(k+1)(k-14)/8 > 6\lfloor (k-2)^2/64 \rfloor + 5$ for all $k \ge 20$.

Claim 8: We have $a \ge e + f$.

Proof of Claim 8: We have $4a + (k+1)t + (3k+1)c \ge \binom{k+3}{3} - 3$. Since $3t + c \le \lfloor k^2/64 \rfloor$, we get $4a \ge \binom{k+3}{3} - 3 - (3k+1)k^2/64$. Since $t + m > 0$ and $f \le 2$, we have $3e + f \le (k+1)k - 6\lfloor (k-2)^2/64 \rfloor$. See the proof of Claim 3.

By Claim 8 we have $a - e - f \ge 0$. Fix a general $Y \in L(0, a-e-f, \lfloor (k-2)^2/64 \rfloor$. The scheme $Y \cap Q$ is a general union of $2\lfloor (k-2)^2/64 \rfloor$ 2-points of Q.

By Claim 7 we have $e \ge 0$. Fix a general $S \cup S' \subset Q$ such that $\sharp(S) = e$, $\sharp(S') = f$ and $S \cup S'$. Set $X' := Y \cup E \cup 2S$. We have $X' \in L(t, a-f, c)$. Hence to prove Theorem 1 it is sufficient to prove that a general union, X'', of X' and f 2-points satisfies $h^1(\mathcal{I}_{X''}(k)) = 0$. We have $3e + f = (k+1)(k+1-$

$2m-t)-6\lfloor (k-2)^2/64\rfloor$. Since $t+2m\le k-3$ (Claim 5), [14, Proposition 5.2 and Theorem 7.2] gives $h^i(Q,\mathcal{I}_{(Y\cap Q)\cup\{2S,Q\}\cup S'}(k)(-F))=0$, $i=0,1$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5]) to prove that $h^1(\mathcal{I}_{X''}(k))=0$ (and hence to prove this part of Theorem 1 in this case) it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup G\cup S\cup\{2S',Q\}}(k-2))=0$.

Claim 9: $h^1(\mathcal{I}_{Y\cup G\cup\{2S',Q\}}(k-2))=0$.

Proof of Claim 9: It is sufficient to prove that $h^1(\mathcal{I}_{Y\cup G\cup 2S'}(k-2))=0$. We have $h^0(\mathcal{O}_{Y\cup G\cup 2S'}(k-2))=\binom{k+1}{3}-\beta-e+f\le\binom{k+1}{3}$ (Claim 8). Since $\deg(G)\le\lfloor k^2/64\rfloor-\lfloor (k-2)^2/64\rfloor$ and $\sharp(S')=f\le 2$, to prove Claim 10 it is sufficient to apply the case $h=\deg(G)$ and $g=\sharp(S')$ of H_{k-2}.

Hence to conclude the proof part (i) of Theorem 1 in this case it is sufficient to prove the following claim.

Claim 10: $h^0(\mathcal{I}_Y(k-4))\le\beta$.

Proof of Claim 10: Take $M\in L(0,a-e-f,\lfloor (k-4)^2/64\rfloor)$ such that $M\subseteq Y$. By the inductive assumption either $h^0(\mathcal{I}_M(k-4))=0$ (and hence $h^0(\mathcal{I}_Y(k-4))=0$) or $h^1(\mathcal{I}_M(k-4))=0$. We may assume that the latter case occurs and hence $h^0(\mathcal{I}_M(k-4))=\binom{k-1}{3}-4(a-e-f)-(3k-11)\lfloor (k-4)^2/64\rfloor$. We have $3e+f+6\lfloor (k-2)^2/64\rfloor=(k+1)^2-(k+1)t-2(k+1)(c-\lfloor (k-2)^2/64\rfloor)$ and $4a+(k+1)t+(3k+1)c=\binom{k+3}{3}-\beta$. To prove Claim 10 it is sufficient to prove the inequality $4(e+f)+(k+1)t+(3k+1)(c-\lfloor (k-4)^2/64\rfloor)+6\lfloor (k-2)^2/64\rfloor\le (k+1)^2+(k-1)^2=2k^2+2$. Recall that $3e+f+6\lfloor (k-2)^2/64\rfloor+(t+2(c-\lfloor (k-2)^2/64\rfloor))=(k+1)^2$. Set $\eta:=\lfloor (k-2)^2/64\rfloor-\lfloor (k-4)^2/64\rfloor$. We have $|\eta-(3k-3)/16|<2$. It is sufficient to prove that $(k-1)(c-\lfloor (k-2)^2/64\rfloor)+3f+6\lfloor (k-4)^2/64\rfloor+(3k-5)\eta\le (k-1)^2$. We have $c-\lfloor (k-2)^2/64\rfloor\le\lfloor k^2/64\rfloor-\lfloor (k-2)^2/64\rfloor\le (k-1)/16$. We have $(k-1)^2/16+6+3(k-4)^2/32+(3k-5)(3k-3)/64+(3k-5)/32\le (k-1)^2$

In this case we put all t lines inside Q and Q contains $e\ge 2$ of the points in the support of the 2-points of our solution. Hence H_k is proved in this case.

(ii) Assume $(k+1)t+4a+(3k+1)c\ge\binom{k+3}{3}$. Set $\beta':=\binom{k+3}{3}-(k+1)t-4a-(3k+1)c$. By the case $\beta=0$ of step (i) we may assume $\beta'>0$. Since $(k+1)t+(3k+1)c\le\binom{k+3}{3}$, we have $a>0$. Hence decreasing if necessary a we may assume $0<\beta'\le 3$. Repeat the proofs of step (i), avoiding step (i1.1).

(iii) Assume $k\le 19$. We have $3t+c\le 0$ if $k\le 7$, $3t+c\le 1$ if $k\le 11$, $3t+c\le 2$ if $k=12,13$, $3t+c\le 3$ if $k=14,15$, $3t+c\le 4$ if $k=16,17$, $3t+c\le 5$ if $k=19$. Since the case $c=0$ is true by [4], we may assume $c>0$. Hence $t=0$. All cases are done as in step (i2) taking as G the union of c general connected elements of $|\mathcal{O}_Q(2,0)|$. □

5. Proofs of Propositions 1 and 2

Set $u_{r,k} := \lceil \binom{r+k}{r}/(k+1)\rceil$ and $v_{r,k} := (k+1)u_{r,k} - \binom{r+k}{r}$. We have $0 \le v_{r,k} \le k$ and $(k+1)u_{r,k} - v_{r,k} = \binom{r+k}{r}$.

Lemma 36. *We have $u_{r,k} - u_{r,k-1} \ge 2k$ if either $r = 4$ and $k \ge 12$ or $r \ge 5$ and $k \ge 2$. We have $u_{5,1} = 3$, $v_{5,1} = 0$, $u_{5,2} = 7$, $v_{5,2} = 0$, $u_{5,3} = 14$, $v_{5,3} = 0$, $u_{4,1} = 3$, $v_{4,1} = 1$, $u_{4,2} = 5$, $v_{4,2} = 0$, $u_{4,3} = 9$, $v_{4,3} = 1$, $u_{4,4} = 14$, $v_{4,4} = 0$, $u_{4,5} = 26$, $v_{4,5} = 4$, $u_{4,6} = 30$, $v_{4,6} = 0$, $u_{4,7} = 42$, $v_{4,7} = 6$, $u_{4,8} = 55$, $v_{4,8} = 0$, $u_{4,9} = 72$, $v_{4,9} = 5$, $u_{4,10} = 91$, $v_{4,10} = 0$, $u_{4,11} = 114$, $v_{4,11} = 3$.*

Proof. Since $u_{r,k} - u_{r,k-1} \ge \binom{r+k}{r}/(k+1) - \binom{r+k-1}{r}/k - 1$, it is sufficient to check when $\binom{r+k-1}{r}(r-1) \ge k(k+1)(2k+1)$. The last inequality is true if either $r = 4$ and $k \ge 12$ or $r = 5$ and $k \ge 4$ or $r \ge 6$ and $k \ge 2$. The explicit values are easily computed. □

Lemma 37. *For all integers $r \ge 4$ and $k \ge 2$ we have $u_{r,k} - 2v_{r,k} \ge 0$ and $h^i(\mathcal{I}_X(k)) = 0$, $i = 0, 1$, for a general union $X \subset \mathbb{P}^r$ of $u_{r,k} - 2v_{r,k}$ lines and $v_{r,k}$ reducible conics whose singular point is contained in H.*

Proof. The lemma says that Assertion $H'_{k,r}$ of [13, page 182] is true. Use [13, §3] (if k is low and $r = 4, 5$, one can also use Lemma 36). □

Lemma 38. *Fix integers $k \ge 3r \ge 4$, $t \ge 0$ and e such that $0 \le e \le k$ and $t(k+1) + (2k+1)e \le \binom{r+k}{r}$. Then $h^1(\mathcal{I}_X(k)) = 0$ for a general union $X \subset \mathbb{P}^r$ of t lines and e reducible conics whose singular point is contained in H.*

Proof. Set $\beta := \binom{r+k}{r} - (k+1)t - e(2k+1)$. Increasing if necessary t we may assume that $0 \le \beta \le k$. We have $t = u_{r,k} - 2e$ if $e \ge v_{r,k}$ and $t = u_{r,k} - 2e - 1$ if $e < v_{r,k}$. First assume $e \le v_{r,k-1}$. Let $Y \subset \mathbb{P}^r$ be a general union of $u_{r,k-1} - 2v_{r,k-1}$ lines and $v_{r,k-1}$ reducible conics with singular points contained in H. Lemma 37 gives $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$. For a general Y we may assume that $Y \cap H$ is a general union of $u_{r,k-1} - 2v_{r,k-1}$ points and $v_{r,k-1}$ tangent vectors of H. Write $Y = Y_1 \sqcup Y_2$ with Y_2 the union of the degree two connected components of H. Since $e \le k$ and $k \ge r$, Lemma 36 implies $u_{r,k-1} - 2v_{r,k-1} \ge e - v_{r,k-1}$. Hence we may write $Y_1 = Y' \sqcup Y''$ with $\deg(Y') = e - v_{r,k-1}$. For each connected component L of Y'' take a general line $R_L \subset H$ through the point $L \cap H$. Set $G := \cup_{L \subseteq Y'} R_L$. Since $t \ge u_{r,k} - v_{r,k}$, Lemma 36 gives $t - \deg(Y) \ge e - v_{r,k-1}$. Let $F \subset H$ be a general union of $t - \deg(Y) - \deg(G)$ lines. Set $X := Y \cup G \cup F$. X is a disjoint union of t lines and e reducible conics with singular point contained in H. Hence to prove the lemma in this case it is sufficient to prove that $h^1(\mathcal{I}_X(k)) = 0$. By [13] we have $h^1(H, \mathcal{I}_{F\cup G}(k)) = 0$. Since $Y \cap H$ is a general union of points and tangent vectors and $\beta \geq 0$, we get $h^1(H, \mathcal{I}_{X\cap H}(k)) = 0$, by the case $q'' = t = 0$, $d = v_{r,k-1}$ of [13, $H''_{k-1,r}$]. Now assume $v_{r,k-1} > e$. Let $E \subset H$ be a general union of $t - u_{r,k-1}$ lines. Write $Y_2 = U \sqcup V$ with U union of e reducible conics and V union of $v_{r,k-1} - e$ reducible conics. For each $P \in \mathrm{Sing}(V)$ fix a 3-dimensional space $W_P \subset \mathbb{P}^r$ containing the connected component D_P of V containing P and set $w_P := \{2P, W_P\}$. Set $\mathbf{w} := \cup_{P\in \mathrm{Sing}(V)} w_P$. The scheme $X' := E \cup Y \cup \mathbf{w} \cup Z$ is a flat limit of a family of disjoint unions of U and t lines. By the semicontinuity theorem for cohomology it is sufficient to prove that $h^1(\mathcal{I}_{X'}(k)) = 0$. We have $\mathrm{Res}_H(X') = Y$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_H(X')}(k)) = 0$. The scheme $X' \cap H$ is a general union of E, e tangent vectors and $v_{r,k-1} - e$ degree 3 zero-dimensional schemes called triple points in [13]. Apply [13, $H''_{k,r}$] with $q'' = 0$, $d = e$ and $t = v_{r,k-1} - e$ and get $h^1(H, \mathcal{I}_{X'\cap H}(k)) = 0$. Apply the Castelnuovo's inequality. □

Proof of Proposition 1: We fix $z \in \mathbb{N}$ and a zero-dimensional scheme $Z \subset \mathbb{P}^r$ with $\deg(Z) \leq z$. We take any hyperplane $H \subset \mathbb{P}^r$ such that $Z \cap H = \emptyset$. Hence in H we only need the case $z = 0$. If $r = 4$, then we use [4]. By [6] we may assume $r \geq 6$. Assume that Proposition 1 for $z = 0$ is true in $\mathbb{P}^{r-1}$ (if $r = 4$ we need [6] instead of the inductive assumption) and call $\beta_{r-1,0}$ the corresponding integer (we may take $\beta_{r-1,0} = 0$ ([4], [6], [7]), but we do not need it). Increasing if necessary $\beta_{r-1,0}$ we may assume that $\beta_{r-1,0} \geq r$. Set $\beta_{r,z} := \binom{4r+\beta_{r-1,0}+z}{r}$. Fix $(t, a) \in \mathbb{N}^2$ and let k be its critical value with respect to z, i.e. the first integer $k > 0$ such that $z + t(k+1) + (r+1)a \leq \binom{r+k}{r}$. By the Castelnuovo-Mumford's lemma a scheme $X = W \sqcup Z$ with $W \in L(t, a, 0; r)$ has maximal rank if and only if $h^1(\mathcal{I}_X(k)) = 0$ and $h^0(\mathcal{I}_X(k-1)) = 0$. Since $L(t, a, 0; r)$ is irreducible, by the semicontinuity theorem for cohomology it is sufficient to find $A, B \in L(t, a, 0; r)$ such that $A \cap Z = B \cap Z = \emptyset$, $h^1(\mathcal{I}_{A\cup Z}(k)) = 0$ and $h^0(\mathcal{I}_{A\cup Z}(k-1)) = 0$. Since $\beta_{r,z} \geq \binom{4r+\beta_{r-1,0}}{r}$, we have $k \geq 3r+\beta_{r-1,0}+z$. Hence for this fixed choice of $\beta_{r,z}$ Proposition 1 is trivially true if $k < 3r + \beta_{r-1,0} + z$ (there is no such datum (t, a)). Hence we use induction on k.

(a) In this step we prove the existence of A. Set $\beta := \binom{r+k}{r} - t(k+1) - (r+1)a - z$.

(a1) Assume $(k+1)t + \beta \geq \binom{r+k-1}{r-1}$. Let x be the minimal integer such that $(t - x) + x(k+1) + \beta \geq \binom{r+k-1}{r-1}$ (it exists, because $k > 0$). By assumption we have $x \leq t$. Set $\gamma := \binom{r+k-1}{r-1} - \beta - (t - x) - x(k+1)$. The minimality of the integer x implies that $\gamma \leq k$. Since $(k+1)t + \beta \leq \binom{r+k}{r}$, we have $x \geq 0$. Since $k \geq 3r$, we also have $x \geq 2k$. Hence $x \geq 2\gamma$. Let

$E \subset H$ be a general union of $x - 2\gamma$ lines and γ reducible conics. Write $E = E_1 \sqcup E_2$ with E_i the union of the degree i connected components of E. For each $P \in \mathrm{Sing}(E_2)$ let $V_P \subset \mathbb{P}^r$ be a 3-dimensional linear subspace such that $V_P \cap H$ is a plane spanned by the connected component of E_2 containing P. Set $v_P := \{2P, V_P\}$ and $\mathbf{v} := \cup_{P \in \mathrm{Sing}(E_2)} v_P$. Notice that $(E \cup \mathbf{v}) \cap H = E$ (scheme-theoretic intersection). Fix a general $Y \in L(t-x, a, 0; r)$. Since Y is general, no component of Y_{red} is contained in H, $Z \cap Y = \emptyset$, $Y \cap E = \emptyset$ and $Y \cap H$ is a general union of $t-x$ general points of H. $E_2 \cup \mathbf{v}$ is a disjoint union of γ sundial in the sense of and hence it is a flat limit of a family of disjoint unions of 2γ lines ([13, Example 2.1.1], [9]). Hence $X := Z \cup Y \cup E \cup \mathbf{v}$ is a flat limit of a family of elements of $L(t, a, 0; r)$. Hence to prove the existence of A in this case it is sufficient to prove that $h^1(\mathcal{I}_X(k)) = 0$. We have $\mathrm{Res}_H(X) = Z \cup Y \cup \mathrm{Sing}(E_2)$. Since $\sharp(\mathrm{Sing}(E_2)) = \gamma$ and $Y \cap E = Y \cap Z = \emptyset$, the definitions of β and γ give $h^0(\mathcal{O}_{Z \cup Y \cup \mathrm{Sing}(E_2)}(k-1)) = \binom{r+k-1}{r}$. The inductive assumption on k gives $h^1(\mathcal{I}_{Z \cup Y}(k-1)) = 0$ and hence $h^0(\mathcal{I}_{Z \cup Y}(k-1)) = \gamma$. Since $\mathrm{Sing}(E_2)$ is a general subset of H with cardinality γ and $\mathrm{Res}_H(Y) = Y$, to prove that $h^i(\mathcal{I}_{Z \cup Y \cup \mathrm{Sing}(E_2)}(k-1)) = 0$, it is sufficient to check that $h^0(\mathcal{I}_{Z \cup Y}(k-2)) = 0$. This is true by the inductive assumption on k. Since $\beta \geq 0$ and $r \geq 5$, Lemma 36 gives $h^1(H, \mathcal{I}_E(k)) = 0$ and $h^0(H, \mathcal{I}_E(k)) = \beta + t - x$. Since $Y \cap H$ is a general subset of H with cardinality $t - x$ and $\beta \geq 0$, we get $h^1(H, \mathcal{I}_{X \cap H}(k)) = 0$. The Castelnuovo's sequence gives $h^1(\mathcal{I}_X(k)) = 0$.

(a2) Assume $(k+1)t + \beta < \binom{r+k-1}{r-1}$.

(a2.1) First assume $(k+1)t + r\beta_{r-1,0} \leq \binom{r+k-1}{r-1}$. Set $e := \lfloor (\binom{r+k-1}{r-1} - (k+1)t - r\beta_{r-1,0})/r \rfloor$ and $f := \binom{r+k-1}{r-1} - (k+1)t - r\beta_{r-1,0} - re$. We have $0 \leq f \leq r-1$. Let $M \subset H$ be a general union of t lines. Fix a general $S \cup S' \subset H$ such that $\sharp(S) = \beta_{r-1,0} + e$, $\sharp(S') = f$ and $S \cap S' = \emptyset$. We have $h^0(\mathcal{O}_{M \cup \{2S, H\} \cup S'}(k)) = \binom{r+k-1}{r-1} - \beta$. Since $e \geq 0$, we have $\sharp(S) \geq \beta_{r-1,0}$. Hence the inductive assumption on r (using [4] if $r - 1 = 3$) gives $h^1(H, \mathcal{O}_{M \cup \{2S, H\} \cup S'}(k)) = 0$.

Claim 1: We have $a \geq \beta_{r-1,z} + e + f$.

Proof of Claim 1: We have $(k+1)t + r(\beta_{r-1,0} + e) + f = \binom{r+k-1}{r-1} - \beta$ and $(k+1)t + (r+1)a = \binom{r+k}{r} - \beta$. Use that $f \leq r$ and that $k \geq 3r$.

Since $\beta_{r,z} \geq \beta_{r-1,z} + r - 1 \geq \beta_{r-1,0} + f$, we have $a \geq \beta_{r-1,0} + e + f$.

By Claim 1 we have $a \geq \beta_{r,z} + e + f$. Fix a general $J \in L(0, a - e - f, 0; r)$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5], [2] in characteristic $\neq 2$) to prove that a general union X'' of $Z \cup J \cup S$ and f 2-points satisfies $h^1(\mathcal{I}_{X''}(k-1)) = 0$ (and hence to prove the proposition in this case) it is sufficient to prove that $h^1(\mathcal{I}_{Z \cup J \cup S \cup \{2S', H\}}(k-1)) = 0$.

Claim 2: $h^1(\mathcal{I}_{Z \cup J \cup \{2S', H\}}(k-1)) = 0$.

Proof of Claim 2: Set $U' := J \cup 2S'$. Since $\sharp(S') = f \le r-1$ and any r points of $\mathbb{P}^r$ are contained in a hyperplane, U' may be considered as a general union of $a-e-\beta_{r-1,0}$ 2-points. Since $\beta_{r-1,0} \ge r$, we have $\sharp(S) \ge \sharp(S')$. Since $h^0(\mathcal{O}_{Z\cup Y_3\cup S\cup\{2S',H\}}(k-1)) = \binom{r+k-1}{r}$, we get $h^0(\mathcal{O}_{Z\cup U'}(k-1)) \le \binom{r+k-1}{r}$. Hence $h^1(\mathcal{I}_{Z\cup U'}(k-1)) = 0$ by the inductive assumption on the critical value. Since $Z\cup J\cup\{2S',H\} \subseteq Z\cup U'$ and $Z\cup U'$ and $Z\cup J\cup\{2S',H\}$ have the same positive -dimensional connected components, the restriction map $H^0(\mathcal{O}_{Z\cup U'}(k-1)) \to H^0(\mathcal{O}_{Z\cup W}(k-1))$ is surjective. Hence $h^1(\mathcal{I}_{Z\cup J\cup\{2S',H\}}(k-1)) = 0$, proving Claim 2.

By the definitions of β, e and f we have $h^0(\mathcal{O}_{Z\cup J\cup S\cup\{2S',H\}}(k-1)) = \binom{r+k-1}{r}$. Since J is general, we have $\mathrm{Res}_H(Z\cup J\cup\{2S',H\}) = Z\cup J$. Since S is general in H, Claim 2 shows that to prove that $h^1(\mathcal{I}_{Z\cup J\cup S\cup\{2S',H\}}(k-1)) = 0$ (and hence to prove the proposition in this case) it is sufficient to prove that $h^0(\mathcal{I}_J(k-2)) = 0$. Since $k-2 \ge 5$ and J is general, the inductive assumption on k shows that it is sufficient to check that $(r+1)(a-\beta_{r-1,0}-e-f) \ge \binom{r+k-2}{r}$. By the definitions of β, e and f we have we have $(r+1)(a-\beta_{r-1,0}-e-f)+\beta_{r-1,0}+e+rf = \binom{r+k-1}{r}$. Hence it is sufficient to check that $\beta_{r-1,0}+e+rf \le \binom{r+k-2}{r-1}$. Since $t \ge 0$, we have $z + r(\beta_{r-1,0}+e)+f \le \binom{r+k-1}{r-1}$. Use that $k > r$.

(a2.2) Now assume $(k+1)t + r\beta_{r-1,0} > \binom{r+k-1}{r-1}$. Let t' be the maximal integer such that $(k+1)t' + r\beta_{r-1,0} \le \binom{r+k-1}{r-1}$. By assumption we have $t' < t$.

Claim 3: We have $t' \ge 0$.

Proof of Claim 3: By the definition of $\beta_{r,z}$ we have $k \ge \beta_{r-1,0}$. Hence the inequality $t' \ge 0$ is obvious.

Set $e' := \lfloor (\binom{r+k-1}{r-1} - (k+1)t' - r\beta_{r-1,0})/r \rfloor$ and $f' := \binom{r+k-1}{r-1} - (k+1)t - r\beta_{r-1,0} - re'$. We have $0 \le f' \le r-1$. Let $M' \subset H$ be a general union of t lines. Fix a general $S_1 \cup S_1' \subset H$ such that $\sharp(S_1) = \beta_{r-1,0}+e'$, $\sharp(S_1') = f'$ and $S_1 \cap S_1' = \emptyset$. We have $h^0(\mathcal{O}_{M'\cup\{2S_1,H\}\cup S_1'}(k)) = \binom{r+k-1}{r-1} - \beta$. Since $e' \ge 0$, we have $\sharp(S_1) \ge \beta_{r-1,0}$. Hence the inductive assumption on r (or [4], [6], [7]) gives $h^1(H, \mathcal{I}_{M'\cup\{2S_1,H\}\cup S_1'}(k)) = 0$. As in the proof of Claim 1 we see that $a \ge e'+f'$. Fix a general $Y_1 \in L(t-t', a-e'-f', 0; r)$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5]) to prove that a general union X_2 of $Z\cup Y_1\cup S_1$ and f 2-points satisfies $h^1(\mathcal{I}_{X_2}(k-1)) = 0$ (and hence to prove the existence of A in this case) it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup Y_1\cup S_1\cup\{2S_1,H\}}(k-1)) = 0$. Assume for the moment $h^1(\mathcal{I}_{Y_1\cup\{2S_1',H\}}(k-1)) = 0$. We have $h^0(\mathcal{I}_{Z\cup Y_1\cup S_1\cup\{2S_1',H\}}(k-1)) = \binom{r+k-1}{r}$. Since S_1 is general in H, to prove that $h^1(\mathcal{I}_{Z\cup Y_1\cup S_1\cup\{2S_1',H\}}(k-1)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{Z\cup Y_1}(k-2)) = 0$. See step (a2.2.2).

Claim 4: We have $k(t-t') + r\beta_{r-1,0} \le \binom{r+k-2}{r-1}$.

Proof of Claim 4: The maximality property of the integer t' gives $(k+1)t' + r\beta_{r-1,0} \geq \binom{r+k-1}{r-1} - k$. Since $(k+1)t + \beta < \binom{r+k-1}{r-1}$, we get $(k+1)(t-t') + \beta - r\beta_{r-1,0} \leq k-1$. Since $\beta \geq 0$, it is sufficient to check that $2r\beta_{r-1,0} + k - 1 \leq \binom{r+k-2}{r-1}$. This is true, because our assumption on β_r implies $k \geq 3r + \beta_{r-1,0}$.

(a2.2.1) Now we check that $h^1(\mathcal{I}_{Z\cup Y_1\cup\{2S'_1,H\}}(k-1)) = 0$. By the semicontinuity theorem for cohomology it is sufficient to prove this h^1-vanishing for a specialization Y_2 of Y_1. Set $e'' := \lfloor (\binom{r+k-2}{r-1} - k(t-t') - rf')/r \rfloor$. Since $f' \leq r$, we have $f' \leq \beta_{r-1,0}$. Claim 4 gives $e'' \geq \beta_{r-1}$. Fix a general union $E \subset H$ of $t-t'$ lines. Fix a general $S_2 \cup S'_2 \subset H$ such that $\sharp(S_2) = e'' - f'$, $\sharp(S'_2) = f''$ and $S_2 \cap S'_2 = \emptyset$. We have $h^0(\mathcal{O}_{E\cup\{2S'_1,H\}\cup\{2S_2,H\}\cup S'_2}(k-1)) = \binom{r+k-2}{r-1}$. Since $\sharp(S'_1) + \sharp(S_2) \geq \beta_{r-1,0}$, either the inductive assumption on r or [4], [6], [7] give $h^i(H, \mathcal{I}_{E\cup\{2S'_1,H\}\cup\{2S_2,H\}\cup S'_2}(k-1)) = 0$, $i = 0, 1$. Assume for the moment $e - e' - e'' - f'' \geq 0$. In this case we take as a specialization of Y_1 a general union Y_2 of f'' 2-points and $W' := W \cup 2S_2$ with W general in $L(0, a - e' - e'' - f'', 0; r)$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5]) to prove that $h^1(\mathcal{I}_{Z\cup Y_2\cup\{2S'_1,H\}}(k-1)) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z\cup W\cup S_2\cup\{2S'_2,H\}}(k-2)) = 0$. See the last part of step (a1) with $k-2$ instead of $k-1$. Now assume $a-e-e'-e''-f'' < 0$. If $a-e-e'-e'' \geq 0$, take as a specialization of Y_1 a general union of $2S_2$ and $a-e-e'$ If $a-e-e'-e'' < 0$, take $S_3 \subset S_2$ with $\sharp(S_3) = a - e - e'$ and take $2S_3$ as a specialization of Y_1.

(a2.2.2) Now we check that $h^0(\mathcal{I}_{Z\cup Y_1}(k-2)) = 0$. We repeat step (a2.2.1), i.e. we specialize the union of the lines of Y_1 to a general union $E \subset H$ of $t-t'$ lines.

(b) In this step we prove the existence of B. Instead of β we use the integer $\beta' := z + kt + (r+1)a - \binom{r+k-1}{r}$. The definition of the integer k gives $\beta' > 0$. Decreasing if necessary t we may assume that either $t = 0$ or $\beta' \leq k-1$. We use $k-1$ instead of k, $k-2$ instead of $k-1$ and $k-3$ instead of $k-2$. In this step we use the assumption $k-3 \geq 5$ and that $k-1 \geq 3r$ (as in the proof of Claim 1). □

Proof of Proposition 2: We could easily modify the proof of Proposition 1, but we prefer to use the following quotations. Fix $(t,a) \in \mathbb{N}^2$ and a general $X \in L(t,a,0;r)$. If $a \geq \beta_{r,z}$, then X has maximal rank by Proposition 1. Hence we only need to test finitely many integers a, say $0 \leq a < \beta_{r,z}$. For any such a fix any union $Z_a \subset \mathbb{P}^r$ of a different 2-points. Use [3] for each of these schemes. □

Proposition 3. *Fix integers $r \geq 4$, $k \geq 6$ and $(t,c,a,t',a') \in \mathbb{N}^5$. Assume* $t' + t(k+1) + ra + ((r-1)k+1)c \leq \binom{k+r-1}{r-1}$, $(r+1)(t+t')(k-1) + (r+1)a' \leq$

$\binom{r+k-2}{r}$, $h^1(\mathbb{P}^{r-1},\mathcal{I}_F(k)) = 0$ *for a general* $F \in L(t,a,c;r-1)$. *If* $(t',a') \neq 0$, *then assume* $k \geq 7$. *Then* $h^1(\mathcal{I}_{Z\cup X}(k)) = 0$ *for a general* $X \in L(t+t',a+a',c;r)$.

Proof. Fix a hyperplane $H \subset \mathbb{P}^r$. Let $E \subset \mathbb{P}^r$ be a general union of t lines contained in H, a 2-points whose support in H and c 2-lines whose support is contained in H. Set $F := E \cap H$. Write $E = E_1 \sqcup 2S \sqcup 2A$ with E_1 union of t lines of H, S a general subset of H with cardinality a and A a general union of c lines of H. Fix a general $Y \in L(t',a',0;r)$. For a general Y we have $Y \cap E = \emptyset$ and $Y \cap H$ is a general union of t' points of H. Set $W := Y \cup E$. We have $\mathrm{Res}_H(W) = Y \cup A \cup S$. By the semicontinuity theorem for cohomology it is sufficient to prove that $h^1(\mathcal{I}_W(k)) = 0$. Since F is general in $L(t,a,c;r-1)$, we have $h^1(H,\mathcal{I}_F(k)) = 0$ ([4] if $r = 4$, [6] if $r = 5,6$ and [7] if $r \geq 7$). Since $t' \leq \binom{r+k-1}{r-1} - t(k+1) - ra - ((r-1)k+1)c = h^0(H,\mathcal{I}_F(k))$, we get $h^1(H,\mathcal{I}_{H\cap W}(k)) = 0$. Hence by the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup A\cup S}(k-1)) = 0$.

(a) Assume for the moment $(t',a') = (0,0)$, i.e. $Y = \emptyset$. Since $A \cup S \subset H$, we have $h^1(\mathcal{I}_{A\cup S}(k-1)) = h^1(H,\mathcal{I}_{A\cup S}(k-1))$. Since A is a general union of c lines of H, it has maximal rank ([13]). Since S is general in H, to conclude in this case it is sufficient to check that $kc+a \leq \binom{r+k-2}{r-1}$. We have $ra+((r-1)k+1)c \leq \binom{r+k-1}{r-1}$ and $\binom{r+k-1}{r-1}/\binom{r+k-2}{r-1} = (r+k-1)/(k-2) < r-1$.

(b) Assume $(t',a') \neq (0,0)$. In this case if $c > r/2$ the scheme $Y \cup A$ is not a general element of $L(t'+c,a',0;r)$ and hence we cannot just [6] and [7] to get that $L(t'+c,a',0;r)$ has maximal rank. Since $k-2 \geq 5$ and $(r+1)(t+t')(k-1) + (r+1)a' \leq \binom{r+k-2}{r}$, we have $h^1(\mathcal{I}_Y(k-2)) = 0$ ([6], [7]). Since $h^1(H,\mathcal{I}_{A\cup S}(k-1)) = 0$ by step (a), it is sufficient to apply the Castelnuovo's sequence. □

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