REFINEMENTS OF RAMANUJAN FORMULA FOR GAMMA FUNCTION

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1. Introduction

Many researchers have paid great attention to the problem of approximating the gamma function. Undeniedly the most known formula is Stirling's formula, e.g., [1],

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x.$$

Slightly better results give Burnside formula [3]

$$\Gamma(x+1) \sim \sqrt{2\pi} \left(\frac{x + \frac{1}{2}}{e}\right)^{x + \frac{1}{2}}$$

and Gosper's formula [4]

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$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \sqrt{1 + \frac{1}{6x}}.$$

However Ramanujan formula, e.g., [2], [5]-[11]

$$\Gamma(x+1) \sim \rho(x) := \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \sqrt[6]{1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}} \quad (1)$$

is more accurate. We propose in this note the replacing the terms in powers in x^{-k} by powers in $(x-m)^{-k}$, where m is certain real number. More precisely, we introduce the following class of approximations for every $m, b, c \in \mathbb{R}$:

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \sqrt[6]{1 + \frac{1}{2(x-m)} + \frac{b}{(x-m)^2} + \frac{c}{(x-m)^3}}. \quad (2)$$

Note that Ramanujan's formula is obtained in $m = 0, b = \frac{1}{8}, c = \frac{1}{240}$ case. We show that there are other values of m, b, c for which better approximations can be obtained.

2. The Results

Some individual computations we made motivated us to introduce the following class of approximations for every $a \in \mathbb{R}$:

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \sqrt[6]{1 + \frac{1}{2(x+2a-\frac{1}{4})} + \frac{a}{(x+2a-\frac{1}{4})^2} + \frac{2a^2 - \frac{13}{480}}{(x+2a-\frac{1}{4})^3}}, \quad (3)$$

which is a sub-family of (2).

Lemma 1. *Let $a > \frac{1}{8}$. If (3) is an upper-approximation, then it is more accurate than Ramanujan's formula.*

Proof. Known fact, Ramanujan's formula (1) is an upper-approximation of gamma function. Our assertion follows if we prove that

$$\sqrt[6]{1 + \frac{1}{2(x+2a-\frac{1}{4})} + \frac{a}{(x+2a-\frac{1}{4})^2} + \frac{2a^2 - \frac{13}{480}}{(x+2a-\frac{1}{4})^3}}$$

$$< \sqrt[6]{1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}}.$$

In this sense, we have

$$\begin{aligned} & \left(1 + \frac{1}{2\left(x + 2a - \frac{1}{4}\right)} + \frac{a}{\left(x + 2a - \frac{1}{4}\right)^2} + \frac{2a^2 - \frac{13}{480}}{\left(x + 2a - \frac{1}{4}\right)^3} \right) \\ & - \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} \right) \\ & = -\frac{(8a-1)T(x)}{240x^3(4x+8a-1)^3} \\ & < 0, \end{aligned}$$

where

$$T(x) = (7680a^2 + 960a - 192)x^2 + (1920a^2 - 384a + 18)x + (64a^2 - 16a + 1).$$

As $a > \frac{1}{8}$, all coefficients of the polynomial T are positive, so $T > 0$. The proof is now completed. $\square$

For a concrete comparison, let us take $a = \frac{1}{7}$ in (3) to get

$$\begin{aligned} \frac{\Gamma(x+1)}{\sqrt{2\pi x} \left(\frac{x}{e}\right)^x} & \sim \beta(x) \\ & := \sqrt[6]{1 + \frac{1}{2\left(x + \frac{1}{28}\right)} + \frac{1}{7\left(x + \frac{1}{28}\right)^2} + \frac{323}{23520\left(x + \frac{1}{28}\right)^3}}. \quad (4) \end{aligned}$$

The next table shows us the superiority of $\Gamma(x+1) \sim \beta(x)$ over Ramanujan's formula $\Gamma(x+1) \sim \rho(x)$.

x	$\ln \{\rho(x) / \Gamma(x+1)\}$	$\ln \{\beta(x) / \Gamma(x+1)\}$
10	8.5872×10^{-8}	7.0976×10^{-8}
25	2.3455×10^{-9}	1.9506×10^{-9}
100	9.4519×10^{-12}	7.8819×10^{-12}
500	1.5247×10^{-14}	1.2723×10^{-14}

By replacing 323 by 321 in (4), an under-approximation is obtained,

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x} \left(\frac{x}{e}\right)^x} \sim \alpha(x) := \sqrt[6]{1 + \frac{1}{2\left(x + \frac{1}{28}\right)} + \frac{1}{7\left(x + \frac{1}{28}\right)^2} + \frac{321}{23520\left(x + \frac{1}{28}\right)^3}},$$

as we can see from the following theorem.

Theorem 1. *The following inequalities are valid for every $x \geq 75$ in the left-hand side and for every $x \geq 2$ in the right-hand side:*

$$\alpha(x) < \frac{\Gamma(x+1)}{\sqrt{2\pi x} \left(\frac{x}{e}\right)^x} < \beta(x).$$

Proof. We use the following inequalities:

$$\exp\{u(x)\} < \frac{\Gamma(x+1)}{\sqrt{2\pi x} \left(\frac{x}{e}\right)^x} < \exp\{v(x)\},$$

where

$$u(x) = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7},$$

$$v(x) = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9}.$$

It remains to show that $f < 0$ and $g < 0$, where

$$f(x) = -u(x) + \ln \alpha(x), \quad g(x) = v(x) - \ln \beta(x).$$

This is true, since f, g are strictly increasing on $[75, \infty)$ and $[2, \infty)$ respectively, with $f(\infty) = g(\infty) = 0$. For the monotonicity,

$$f(x) = \frac{P(x)}{5040x^8(28x+1)(20020x+66640x^2+109760x^3+2133)},$$

$$g(x) = \frac{Q(x)}{55440x^{10}(28x+1)(60060x+199920x^2+329280x^3+6427)},$$

where P and Q are polynomials with positive leading coefficient having the largest real roots $74.398 \dots$ and $1.372 \dots$, respectively. $\square$

Finally we pose the open problem of finding the values of m, a, b, c which provide the most accurate approximations (2) and (3).

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