HYPERSUMS OF POWERS OF INTEGERS VIA THE STOLZ-CESÀRO LEMMA

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Abstract: In this paper, we consider the hypersum polynomial $P^{(m)}_k(n) = \sum_{r=1}^{k+m+1} c_{k,m}^r n^r$, which is a generalization of the polynomial associated with the sums of powers of integers $P^{(0)}_k(n) = 1^k + 2^k + \cdots + n^k$. Using the Stolz-Cesàro lemma, we derive an explicit formula for the set of coefficients $\{c_{k,m}^r\}_{r=1}^{k+m+1}$ in terms of $\{c_{k,m-1}^r\}_{r=1}^{k+m}$ and the Bernoulli numbers. This allows us to obtain the $m$-th order hypersum $P^{(m)}_k(n)$ once the $(m-1)$-th order hypersum $P^{(m-1)}_k(n)$ is known. Moreover, we show how to obtain $P^{(m)}_k(n)$ from $P^{(m)}_{k-1}(n)$.

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1. Introduction

The Stolz-Cesàro lemma (for a review see e.g. [1, Appendix B]) constitutes one of the most powerful tools of analysis for evaluating limits of sequences. In one of its simplest form, this lemma refers to the existence of the limits

$$l_1 = \lim_{n \to \infty} \frac{a_n}{b_n} \quad \text{and} \quad l_2 = \lim_{n \to \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}},$$

where $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are sequences of real numbers, with $(b_n)_{n=1}^\infty$ being...
strictly increasing and unbounded above. The lemma asserts that, if the limit $l_2$ exists, then the limit $l_1$ also exists, and moreover, $l_1 = l_2$.

A striking application of the Stolz-Cesàro lemma was made by Kung [2] to determine the coefficients of the polynomial related to the sum of powers of integers, $P_k^{(0)}(n) = 1^k + 2^k + \ldots + n^k$. Specifically, assuming that the polynomial is $P_k^{(0)}(n) = \sum_{r=1}^{k+1} c_{k,0} r^n$, Kung used the lemma to show that

$$c_{k,0}^{k+1} = \frac{1}{k+1},$$

and, for $j = 1, 2, \ldots, k$,

$$c_{k,0}^j = \frac{1}{j} \sum_{t=j+1}^{k+1} (-1)^{t-j-1} \binom{t}{j-1} c_{k,0}^t.$$

For convenience, we rewrite these equations in the unified form

$$\sum_{t=j}^{k+1} (-1)^{t-j} \binom{t}{j-1} c_{k,0}^t = \delta_{j,k+1}, \quad 1 \leq j \leq k+1,$$

(1)

where $\delta_{i,j}$ is the classical Kronecker symbol which equals 1 if $i = j$ and 0 otherwise. The equations in (1) constitute a triangular system of $k+1$ equations in the unknowns $\{c_{k,0}^r\}_{r=1}^{k+1}$. This system was apparently first formulated by Schultz [3, Equation (6)].

In this paper, we shall be concerned with a generalization of the sums of powers of integers, namely the hypersums (that is, sums of sums) of powers of integers $P_k^{(m)}(n)$. For non-negative integers $k$ and $m$, and positive integer $n$, they are defined recursively as follows [4]: $P_k^{(0)}(n) = \sum_{i=1}^n i^k$, $P_k^{(1)}(n) = \sum_{i=1}^n P_k^{(0)}(i)$, $P_k^{(2)}(n) = \sum_{i=1}^n P_k^{(1)}(i)$, and, generally

$$P_k^{(m)}(n) = \sum_{i=1}^n P_k^{(m-1)}(i).$$

For example, the $n$-th triangular, tetrahedral, and pentatope number [5, Chapter 2] corresponds, respectively, to $P_1^{(0)}(n)$, $P_1^{(1)}(n)$, and $P_1^{(2)}(n)$. Moreover, since $P_k^{(0)}(n)$ is a polynomial in $n$ of degree $k + 1$ with zero constant term, inductively $P_k^{(m)}(n)$ is a polynomial in $n$ of degree $k + m + 1$ with zero constant term:

$$P_k^{(m)}(n) = \sum_{r=1}^{k+m+1} c_{k,m}^r n^r.$$

(2)
Following Kung’s approach, in Section 2 we use the Stolz-Cesàro lemma to determine the coefficients of the hypersum polynomial (2). More precisely, by applying the lemma, we derive the following system of equations involving the sets of coefficients \( \{ c_{k,m}^r \}_{r=1}^{k+m+1} \) and \( \{ c_{k,m-1}^r \}_{r=1}^{k+m+1} \):

\[
\sum_{t=j}^{k+m+1} (-1)^{t-j} \binom{t}{j-1} c_{k,m}^t = c_{k,m-1}^{j-1}, \quad 1 \leq j \leq k + m + 1, \tag{3}
\]

which is the generalization to arbitrary \( m \geq 1 \) of the system (1). The system (3) now involves \( k + m + 1 \) equations in the unknowns \( \{ c_{k,m}^r \}_{r=1}^{k+m+1} \). Solving this system gives us \( \{ c_{k,m}^r \}_{r=1}^{k+m+1} \) in terms of \( \{ c_{k,m-1}^r \}_{r=1}^{k+m+1} \), and thus allows us to obtain the \( m \)-th order hypersum \( P_{k}^{(m)}(n) \) once the \( (m-1) \)-th order hypersum \( P_{k}^{(m-1)}(n) \) is known.

In Section 3, we provide the solution to the systems of equations (1) and (3). In particular, we give the explicit expression for the coefficients \( \{ c_{k,m}^r \}_{r=1}^{k+m+1} \) in terms of \( \{ c_{k,m-1}^r \}_{r=1}^{k+m+1} \) and the Bernoulli numbers. We illustrate our findings by means of a simple numerical example consisting in first calculating \( P_{6}^{(1)}(n) \) from \( P_{6}^{(0)}(n) \), and then \( P_{6}^{(2)}(n) \) from \( P_{6}^{(1)}(n) \). Finally, in Section 4, we show how to obtain \( P_{k}^{(m)}(n) \) from \( P_{k-1}^{(m)}(n) \).

### 2. The Stolz-Cesàro Lemma in Action

Let us first note that, from the recursive definition of \( P_{k}^{(m)}(n) \), it trivially follows that

\[
P_{k}^{(m)}(n) - P_{k}^{(m)}(n-1) = P_{k}^{(m-1)}(n). \tag{4}
\]

On the other hand, for \( j = 1, 2, \ldots, k + m + 1 \), from (2) we have

\[
c_{k,m}^j = \frac{1}{n^j} \left( P_{k}^{(m)}(n) - \sum_{i=1}^{k+m+1} c_{k,m}^i n^i \right).
\]

Since the coefficients \( \{ c_{k,m}^r \}_{r=1}^{k+m+1} \) are independent of \( n \), it must be that

\[
c_{k,m}^j = \lim_{n \to \infty} \frac{P_{k}^{(m)}(n) - \sum_{i=j+1}^{k+m+1} c_{k,m}^i n^i}{n^j},
\]
where the summation in the numerator is zero if \( j = k + m + 1 \). Clearly, \((n^j)_{n=1}^{\infty}\) is a strictly increasing and unbounded sequence, and then we can apply the Stolz-Cesàro lemma to get

\[
\frac{c_{j,k,m}}{n^j} = \lim_{n \to \infty} \frac{P_k^{(m-1)}(n) - \sum_{i=j+1}^{k+m+1} c_{k,m}^i (n^i - (n-1)^i)}{n^j - (n-1)^j},
\]

where we have used (4). Expanding the binomial \((n-1)^j\), we readily see that the highest term in the denominator is \(jn^j\). Since the limit \(c_{j,k,m}\) exists, it thus suffices to concentrate on the \((j-1)\)-th order term in the numerator, which is given by

\[
c_{k,m}^{k+m} n^{k+m}, \text{ for } j = m + k + 1;
\]

\[
c_{k,m-1}^{j-1} + \sum_{t=j+1}^{k+m+1} (-1)^{t-j-1} \binom{t}{j} c_{k,m}^t n^{t-1}, \text{ for } j = 1, 2, \ldots, k + m.
\]

It is therefore concluded that

\[
c_{k,m}^{k+m+1} = \frac{1}{k + m + 1} c_{k,m-1}^{k+m}, \quad (5a)
\]

and

\[
c_{k,m}^j = \frac{1}{j} \left[ c_{k,m-1}^{j-1} + \sum_{t=j+1}^{k+m+1} (-1)^{t-j-1} \binom{t}{j} c_{k,m}^t \right], \quad j = 1, 2, \ldots, k + m. \quad (5b)
\]

Relations (5a) and (5b) can be finally combined to yield the system of equations (3). Furthermore, by applying (5a) repeatedly to itself, the leading coefficient \(c_{k,m}^{k+m+1}\) is found to be

\[
c_{k,m}^{k+m+1} = \frac{1}{k + m + 1} c_{k,m-1}^{k+m}
\]

\[
= \left( \frac{1}{k + m + 1} \right) \left( \frac{1}{k + m} \right) c_{k,m-2}^{k+m-1}
\]

\[
= \left( \frac{1}{k + m + 1} \right) \left( \frac{1}{k + m} \right) \left( \frac{1}{k + m - 1} \right) \cdots \left( \frac{1}{2} \right) c_{k,0}^{k+1}
\]

\[
= \frac{1}{(k + m + 1)!},
\]

where we have used that \(c_{k,0}^{k+1} = 1/(k + 1)\).
3. Explicit Solution to the Systems of Equations

The system (1) can be written in matrix form as

\[
\begin{pmatrix}
\begin{array}{cccccc}
k + 1 & 0 & 0 & \ldots & 0 & 0 \\
-\left(\frac{k+1}{k-1}\right) & k & 0 & \ldots & 0 & 0 \\
\frac{k+1}{k-2} & -\left(\frac{k}{k-2}\right) & k - 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^{k-1} \binom{k+1}{1} & (-1)^{k-2} \binom{k}{1} & (-1)^{k-3} \binom{k}{1} & \ldots & 2 & 0 \\
(-1)^k & (-1)^{k-1} & (-1)^{k-2} & \ldots & -1 & 1
\end{array}
\end{pmatrix}
\begin{pmatrix}
c_{k,0}^{k+1} \\
c_{k,0}^k \\
c_{k,0}^{k-1} \\
\vdots \\
c_{k,0}^2 \\
c_{k,0}^1
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}
\]

Pre-multiplying by the inverse of the coefficient matrix, we get

\[
\begin{pmatrix}
k_{k,0}^{k+1} \\
k_{k,0}^k \\
k_{k,0}^{k-1} \\
\vdots \\
k_{k,0}^2 \\
k_{k,0}^1
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{k+1} & 0 & 0 & \ldots & 0 & 0 \\
\binom{k+1}{k} \frac{B_k}{k+1} & \frac{1}{k} & 0 & \ldots & 0 & 0 \\
\binom{k+1}{k-1} \frac{B_k}{k+1} & \frac{k}{k-1} \frac{B_k}{k} & \frac{1}{k-1} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\binom{k+1}{2} \frac{B_{k-1}}{k+1} & \frac{k}{2} \frac{B_{k-2}}{k} & \frac{k-1}{2} \frac{B_{k-3}}{k-1} & \ldots & \frac{1}{2} & 0 \\
\binom{k+1}{1} \frac{B_k}{k+1} & \frac{k}{1} \frac{B_{k-1}}{k} & \frac{k-1}{1} \frac{B_{k-2}}{k-1} & \ldots & \left(\frac{1}{2}\right) \frac{B_1}{2} & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}
\]

where \( B_j \) is the \( j \)-th Bernoulli number [5, Chapter 4]. Thus, taking \( B_1 = \frac{1}{2} \), we obtain the well-known relations (see e.g. [6]):

\[
c_{k,0}^r = \frac{1}{k+1} \binom{k+1}{r} B_{k+1-r}, \quad r = 1, 2, \ldots, k + 1.
\]
Similarly, inverting the coefficient matrix associated with the system of equations (3) gives, for $m \geq 1$,

$$
\begin{pmatrix}
  c_{k,m}^{k+m+1} \\
  \vdots \\
  c_{k,m}^{1}
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{k+m+1} & 0 & 0 & \cdots & 0 & 0 \\
  (k+m+1)B_1 & \frac{1}{k+m} & 0 & \cdots & 0 & 0 \\
  (k+m+1)B_2 & (k+m+1)B_1 & \frac{1}{k+m} & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  (k+m+1)B_k & (k+m+1)B_{k-1} & \cdots & \frac{1}{k+m} & \cdots & 0 \\
  (k+m+1)B_{k+1} & (k+m+1)B_k & \cdots & (k+m+1)B_{k-2} & \cdots & \frac{1}{k+m} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  (k+m+1)B_m & (k+m+1)B_{m-1} & \cdots & (k+m+1)B_{m-2} & \cdots & \frac{1}{k+m} \\
  \end{pmatrix} \begin{pmatrix}
  c_{k,m}^{k+m} \\
  \vdots \\
  c_{k,m}^{0}
\end{pmatrix}
$$

(7)

from which we obtain the desired relationships

$$
c_{k,m}^{k+m+1-r} = \sum_{j=0}^{k+m+1-r} \frac{1}{j+r} \binom{j+r}{r} B_j c_{k,m-1}^{j+r-1}, \quad r = 1, 2, \ldots, k + m + 1.
$$

(8)

Incidentally, for $r = 1$ we get the symmetric relation $c_{k,m}^{1} = \sum_{j=1}^{k+m} B_j c_{k,m-1}^{j}$ (note that in this case the summation starts at $j = 1$ since $c_{k,m-1}^{0} = 0$.)

As a simple numerical example, let us evaluate $P_{6}^{(1)}(n)$ from $P_{6}^{(0)}(n)$, and then $P_{6}^{(2)}(n)$ from $P_{6}^{(1)}(n)$. Using (6), we readily obtain

$$
P_{6}^{(0)}(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n.
$$

Thus, letting $k = 6$ and $m = 1$, and plugging in the coefficients $c_{6,0}^{7} = \frac{1}{7}$, $c_{6,0}^{6} = \frac{1}{2}$, $c_{6,0}^{5} = \frac{1}{2}$, $c_{6,0}^{4} = 0$, $c_{6,0}^{3} = -\frac{1}{6}$, $c_{6,0}^{2} = 0$, and $c_{6,0}^{1} = \frac{1}{42}$, the matrix
equation (7) becomes

\[
\begin{pmatrix}
  c^8_{6,1} \\
  c^7_{6,1} \\
  c^6_{6,1} \\
  c^5_{6,1} \\
  c^4_{6,1} \\
  c^3_{6,1} \\
  c^2_{6,1} \\
  c^1_{6,1}
\end{pmatrix} =
\begin{pmatrix}
  \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \frac{1}{12} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
  \frac{7}{12} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
  0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 \\
  -\frac{7}{24} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\
  0 & -\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & 0 & 0 \\
  \frac{1}{12} & 0 & -\frac{1}{12} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{12} \\
  0 & \frac{1}{12} & 0 & -\frac{1}{30} & 0 & \frac{1}{6} & \frac{1}{2} & 1
\end{pmatrix}
\begin{pmatrix}
  \frac{1}{7} \\
  \frac{1}{2} \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

By making the matrix multiplication on the right-hand side, we deduce that

\[
P^{(1)}_6(n) = \frac{1}{56} n^8 + \frac{1}{7} n^7 + \frac{5}{12} n^6 + \frac{1}{2} n^5 + \frac{1}{8} n^4 - \frac{1}{6} n^3 - \frac{5}{84} n^2 + \frac{1}{42} n.
\]

Alternatively, to derive \( P^{(2)}_6(n) \), we can use the explicit solution (8). Indeed, letting \( k = 6 \) and \( m = 2 \), and setting \( c^8_{6,1} = \frac{1}{56}, c^7_{6,1} = \frac{1}{7}, c^6_{6,1} = \frac{5}{12}, c^5_{6,1} = \frac{1}{2}, c^4_{6,1} = \frac{1}{8}, c^3_{6,1} = -\frac{1}{6}, c^2_{6,1} = -\frac{5}{84}, \) and \( c^1_{6,1} = \frac{1}{12} \) in the right-hand side of (8) yields

\[
P^{(2)}_6(n) = \frac{1}{504} n^9 + \frac{1}{112} n^8 + \frac{1}{8} n^6
\]

\[
+ \frac{19}{40} n^5 + \frac{3}{16} n^4 - \frac{5}{63} n^3 - \frac{5}{56} n^2 + \frac{1}{140} n.
\]

We note that the sum of the coefficients of the hypersum polynomial \( P^{(m)}_k(n) \) is equal to unity since, for all non-negative integers \( k \) and \( m \), \( P^{(m)}_k(1) = 1 \).

4. Concluding Remarks

In this paper, we have addressed the problem of determining the coefficients of \( P^{(m)}_k(n) \) from the coefficients of \( P^{(m-1)}_k(n) \). Another related problem is to determine the coefficients of \( P^{(m)}_k(n) \) from those corresponding to \( P^{(m)}_{k-1}(n) \). To this end, the following recursive formula comes to rescue here (see [7, Proposition 9]):

\[
c^r_{k,m} = (m + 1)(c^r_{k-1,m} - c^r_{k-1,m+1}) + c^r_{k-1,m}, \tag{9}
\]
for $1 \leq r \leq k + m + 1$, $k \geq 1$, and $m \geq 0$. Now, from equation (8) we have

$$c_{k-1,m+1} = \sum_{j=0}^{k-r} \frac{1}{j+r} \binom{j+r}{r} B_j c_{k-1,m}^{j+r-1}.$$  \hfill (10)

Therefore, combining equations (9) and (10) gives

$$c_{k,m} = (m+1) \left( c_{k-1,m} - \sum_{j=0}^{k-r} \frac{1}{j+r} \binom{j+r}{r} B_j c_{k-1,m}^{j+r-1} \right) + c_{k-1,m}^{r-1}.$$  \hfill (11)

In particular, it follows that

$$c_{k,m}^{k+m+1} = \frac{k}{k+m+1} c_{k-1,m}^{k+m}.$$  \hfill (12)

Using equations (11) and (12), we can get $P^{(m)}_k(n)$ from $P^{(m)}_{k-1}(n)$. As an example, let us evaluate $P^{(2)}_7(n)$ from $P^{(2)}_6(n)$. For $k = 7$ and $m = 2$, from (12) we have $c_{7,2}^{10} = \frac{7}{10} c_{6,2}^9 = \frac{7}{10} \frac{1}{504} = \frac{1}{720}$. Likewise, for $k = 7$, $m = 2$, and $r = 9$, from (11) we have (recall that we are taking $B_1 = \frac{1}{2}$)

$$c_{7,2}^9 = 3 \left( c_{6,2}^9 - \frac{1}{9} c_{6,2}^8 - \frac{1}{2} c_{6,2}^9 + c_{6,2}^8 \right) = 3 \left( \frac{1}{504} - \frac{1}{9} \frac{1}{112} - \frac{1}{2} \frac{1}{504} \right) + \frac{3}{112} = \frac{1}{48}.$$

Proceeding in the same way for the remaining coefficients $c_{7,2}^8, \ldots, c_{7,2}^1$, one arrives at the polynomial

$$P^{(2)}_7(n) = \frac{1}{720} n^{10} + \frac{1}{48} n^9 + \frac{1}{8} n^8 + \frac{3}{8} n^7 + \frac{133}{240} n^6 + \frac{21}{80} n^5 - \frac{2}{9} n^4 - \frac{5}{24} n^3 + \frac{1}{24} n^2 + \frac{1}{20} n.$$

On the other hand, as it should be, by applying reiteratively the recurrence relation (12), we retrieve the expression previously obtained for $c_{k,m}^{k+m+1}$:

$$c_{k,m}^{k+m+1} = \frac{k}{k+m+1} c_{k-1,m}^{k+m} = \left( \frac{k}{k+m+1} \right) \left( \frac{k-1}{k+m} \right) c_{k-2,m}^{k+m-1}.$$
\[
\frac{k}{k + m + 1} \left( \frac{k - 1}{k + m} \right) \left( \frac{k - 2}{k + m - 1} \right) \cdots \left( \frac{1}{m + 2} \right) c_{0,m}^{m+1} \\
= \frac{k!}{(k + m + 1)!},
\]

where we have used that \( c_{0,m}^{m+1} = 1/(m + 1)! \) ([7, Proposition 2]).

Finally, we emphasize that, by means of our algorithm to get \( P_k^{(m)}(n) \) from \( P_k^{(m-1)}(n) \) (equation (8)), and the one to get \( P_k^{(m)}(n) \) from \( P_{k-1}^{(m)}(n) \) (equation (11)), one could in principle calculate any hypersum polynomial \( P_k^{(m)}(n) \) starting from \( P_0^{(0)}(n) = n \).

### References


