HYPERSUMS OF POWERS OF INTEGERS VIA THE STOLZ-CESÀRO LEMMA

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Abstract: In this paper, we consider the hypersum polynomial $P_k^{(m)}(n) = \sum_{r=1}^{k+m+1} c_{k,m}^r n^r$, which is a generalization of the polynomial associated with the sums of powers of integers $P_k^{(0)}(n) = 1^k + 2^k + \cdots + n^k$. Using the Stolz-Cesàro lemma, we derive an explicit formula for the set of coefficients $\{c_{k,m}^r\}_{r=1}^{k+m+1}$ in terms of $\{c_{k,m-1}^r\}_{r=1}^{k+m}$ and the Bernoulli numbers. This allows us to obtain the m -th order hypersum $P_k^{(m)}(n)$ once the $(m-1)$ -th order hypersum $P_k^{(m-1)}(n)$ is known. Moreover, we show how to obtain $P_k^{(m)}(n)$ from $P_{k-1}^{(m)}(n)$.

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1. Introduction

The Stolz-Cesàro lemma (for a review see e.g. [1, Appendix B]) constitutes one of the most powerful tools of analysis for evaluating limits of sequences. In one of its simplest form, this lemma refers to the existence of the limits

$$l_1 = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} \quad \text{and} \quad l_2 = \lim_{n \rightarrow \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}},$$

where $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of real numbers, with $(b_n)_{n=1}^{\infty}$ being

strictly increasing and unbounded above. The lemma asserts that, if the limit l_2 exists, then the limit l_1 also exists, and moreover, $l_1 = l_2$.

A striking application of the Stolz-Cesàro lemma was made by Kung [2] to determine the coefficients of the polynomial related to the sum of powers of integers, $P_k^{(0)}(n) = 1^k + 2^k + \cdots + n^k$. Specifically, assuming that the polynomial is $P_k^{(0)}(n) = \sum_{r=1}^{k+1} c_{k,0}^r n^r$, Kung used the lemma to show that

$$c_{k,0}^{k+1} = \frac{1}{k+1},$$

and, for $j = 1, 2, \dots, k$,

$$c_{k,0}^j = \frac{1}{j} \sum_{t=j+1}^{k+1} (-1)^{t-j-1} \binom{t}{j-1} c_{k,0}^t.$$

For convenience, we rewrite these equations in the unified form

$$\sum_{t=j}^{k+1} (-1)^{t-j} \binom{t}{j-1} c_{k,0}^t = \delta_{j,k+1}, \quad 1 \leq j \leq k+1, \quad (1)$$

where $\delta_{i,j}$ is the classical Kronecker symbol which equals 1 if $i = j$ and 0 otherwise. The equations in (1) constitute a triangular system of $k+1$ equations in the unknowns $\{c_{k,0}^r\}_{r=1}^{k+1}$. This system was apparently first formulated by Schultz [3, Equation (6)].

In this paper, we shall be concerned with a generalization of the sums of powers of integers, namely the hypersums (that is, sums of sums) of powers of integers $P_k^{(m)}(n)$. For non-negative integers k and m , and positive integer n , they are defined recursively as follows [4]: $P_k^{(0)}(n) = \sum_{i=1}^n i^k$, $P_k^{(1)}(n) = \sum_{i=1}^n P_k^{(0)}(i)$, $P_k^{(2)}(n) = \sum_{i=1}^n P_k^{(1)}(i)$, $\dots$, and, generally

$$P_k^{(m)}(n) = \sum_{i=1}^n P_k^{(m-1)}(i).$$

For example, the n -th triangular, tetrahedral, and pentatope number [5, Chapter 2] corresponds, respectively, to $P_1^{(0)}(n)$, $P_1^{(1)}(n)$, and $P_1^{(2)}(n)$. Moreover, since $P_k^{(0)}(n)$ is a polynomial in n of degree $k+1$ with zero constant term, inductively $P_k^{(m)}(n)$ is a polynomial in n of degree $k+m+1$ with zero constant term:

$$P_k^{(m)}(n) = \sum_{r=1}^{k+m+1} c_{k,m}^r n^r. \quad (2)$$

Following Kung's approach, in Section 2 we use the Stolz-Cesàro lemma to determine the coefficients of the hypersum polynomial (2). More precisely, by applying the lemma, we derive the following system of equations involving the sets of coefficients $\{c_{k,m}^r\}_{r=1}^{k+m+1}$ and $\{c_{k,m-1}^r\}_{r=1}^{k+m}$:

$$\sum_{t=j}^{k+m+1} (-1)^{t-j} \binom{t}{j-1} c_{k,m}^t = c_{k,m-1}^{j-1}, \quad 1 \leq j \leq k+m+1, \quad (3)$$

which is the generalization to arbitrary $m \geq 1$ of the system (1). The system (3) now involves $k+m+1$ equations in the unknowns $\{c_{k,m}^r\}_{r=1}^{k+m+1}$. Solving this system gives us $\{c_{k,m}^r\}_{r=1}^{k+m+1}$ in terms of $\{c_{k,m-1}^r\}_{r=1}^{k+m}$, and thus allows us to obtain the m -th order hypersum $P_k^{(m)}(n)$ once the $(m-1)$ -th order hypersum $P_k^{(m-1)}(n)$ is known.

In Section 3, we provide the solution to the systems of equations (1) and (3). In particular, we give the explicit expression for the coefficients $\{c_{k,m}^r\}_{r=1}^{k+m+1}$ in terms of $\{c_{k,m-1}^r\}_{r=1}^{k+m}$ and the Bernoulli numbers. We illustrate our findings by means of a simple numerical example consisting in first calculating $P_6^{(1)}(n)$ from $P_6^{(0)}(n)$, and then $P_6^{(2)}(n)$ from $P_6^{(1)}(n)$. Finally, in Section 4, we show how to obtain $P_k^{(m)}(n)$ from $P_{k-1}^{(m)}(n)$.

2. The Stolz-Cesàro Lemma in Action

Let us first note that, from the recursive definition of $P_k^{(m)}(n)$, it trivially follows that

$$P_k^{(m)}(n) - P_k^{(m)}(n-1) = P_k^{(m-1)}(n). \quad (4)$$

On the other hand, for $j = 1, 2, \dots, k+m+1$, from (2) we have

$$c_{k,m}^j = \frac{1}{n^j} \left(P_k^{(m)}(n) - \sum_{\substack{i=1 \\ i \neq j}}^{k+m+1} c_{k,m}^i n^i \right).$$

Since the coefficients $\{c_{k,m}^r\}_{r=1}^{k+m+1}$ are independent of n , it must be that

$$c_{k,m}^j = \lim_{n \rightarrow \infty} \frac{P_k^{(m)}(n) - \sum_{i=j+1}^{k+m+1} c_{k,m}^i n^i}{n^j},$$

where the summation in the numerator is zero if $j = k + m + 1$. Clearly, $(n^j)_{n=1}^\infty$ is a strictly increasing and unbounded sequence, and then we can apply the Stolz-Cesàro lemma to get

$$c_{k,m}^j = \lim_{n \rightarrow \infty} \frac{P_k^{(m-1)}(n) - \sum_{i=j+1}^{k+m+1} c_{k,m}^i (n^i - (n-1)^i)}{n^j - (n-1)^j},$$

where we have used (4). Expanding the binomial $(n-1)^j$, we readily see that the highest term in the denominator is jn^{j-1} . Since the limit $c_{k,m}^j$ exists, it thus suffices to concentrate on the $(j-1)$ -th order term in the numerator, which is given by

$$\begin{aligned} & c_{k,m-1}^{k+m} n^{k+m}, \text{ for } j = m + k + 1; \\ & \left[c_{k,m-1}^{j-1} + \sum_{t=j+1}^{k+m+1} (-1)^{t-j-1} \binom{t}{j-1} c_{k,m}^t \right] n^{j-1}, \text{ for } j = 1, 2, \dots, k + m. \end{aligned}$$

It is therefore concluded that

$$c_{k,m}^{k+m+1} = \frac{1}{k+m+1} c_{k,m-1}^{k+m}, \quad (5a)$$

and

$$c_{k,m}^j = \frac{1}{j} \left[c_{k,m-1}^{j-1} + \sum_{t=j+1}^{k+m+1} (-1)^{t-j-1} \binom{t}{j-1} c_{k,m}^t \right], \quad j = 1, 2, \dots, k + m. \quad (5b)$$

Relations (5a) and (5b) can be finally combined to yield the system of equations (3). Furthermore, by applying (5a) repeatedly to itself, the leading coefficient $c_{k,m}^{k+m+1}$ is found to be

$$\begin{aligned} c_{k,m}^{k+m+1} &= \frac{1}{k+m+1} c_{k,m-1}^{k+m} \\ &= \left(\frac{1}{k+m+1} \right) \left(\frac{1}{k+m} \right) c_{k,m-2}^{k+m-1} \\ &= \left(\frac{1}{k+m+1} \right) \left(\frac{1}{k+m} \right) \left(\frac{1}{k+m-1} \right) \cdots \left(\frac{1}{k+2} \right) c_{k,0}^{k+1} \\ &= \frac{k!}{(k+m+1)!}, \end{aligned}$$

where we have used that $c_{k,0}^{k+1} = 1/(k+1)$.

3. Explicit Solution to the Systems of Equations

The system (1) can be written in matrix form as

$$\begin{pmatrix} k+1 & 0 & 0 & \dots & 0 & 0 \\ -\binom{k+1}{k-1} & k & 0 & \dots & 0 & 0 \\ \binom{k+1}{k-2} & -\binom{k}{k-2} & k-1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{k-1} \binom{k+1}{1} & (-1)^{k-2} \binom{k}{1} & (-1)^{k-3} \binom{k-1}{1} & \dots & 2 & 0 \\ (-1)^k & (-1)^{k-1} & (-1)^{k-2} & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} c_{k,0}^{k+1} \\ c_{k,0}^k \\ c_{k,0}^{k-1} \\ \vdots \\ c_{k,0}^2 \\ c_{k,0}^1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Pre-multiplying by the inverse of the coefficient matrix, we get

$$\begin{pmatrix} c_{k,0}^{k+1} \\ c_{k,0}^k \\ c_{k,0}^{k-1} \\ \vdots \\ c_{k,0}^2 \\ c_{k,0}^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{k+1} & 0 & 0 & \dots & 0 & 0 \\ \binom{k+1}{k} \frac{B_1}{k+1} & \frac{1}{k} & 0 & \dots & 0 & 0 \\ \binom{k+1}{k-1} \frac{B_2}{k+1} & \binom{k}{k-1} \frac{B_1}{k} & \frac{1}{k-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{k+1}{2} \frac{B_{k-1}}{k+1} & \binom{k}{2} \frac{B_{k-2}}{k} & \binom{k-1}{2} \frac{B_{k-3}}{k-1} & \dots & \frac{1}{2} & 0 \\ \binom{k+1}{1} \frac{B_k}{k+1} & \binom{k}{1} \frac{B_{k-1}}{k} & \binom{k-1}{1} \frac{B_{k-2}}{k-1} & \dots & \binom{2}{1} \frac{B_1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

where B_j is the j -th Bernoulli number [5, Chapter 4]. Thus, taking $B_1 = \frac{1}{2}$, we obtain the well-known relations (see e.g. [6]):

$$c_{k,0}^r = \frac{1}{k+1} \binom{k+1}{r} B_{k+1-r}, \quad r = 1, 2, \dots, k+1. \quad (6)$$

Similarly, inverting the coefficient matrix associated with the system of equations (3) gives, for $m \geq 1$,

$$\begin{pmatrix} c_{k,m}^{k+m+1} \\ c_{k,m}^{k+m} \\ c_{k,m}^{k+m-1} \\ \vdots \\ c_{k,m}^2 \\ c_{k,m}^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{k+m+1} & 0 & 0 & \dots & 0 & 0 \\ \binom{k+m+1}{k+m} \frac{B_1}{k+m+1} & \frac{1}{k+m} & 0 & \dots & 0 & 0 \\ \binom{k+m+1}{k+m-1} \frac{B_2}{k+m+1} & \binom{k+m}{k+m-1} \frac{B_1}{k+m} & \frac{1}{k+m-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{k+m+1}{2} \frac{B_{k+m-1}}{k+m+1} & \binom{k+m}{2} \frac{B_{k+m-2}}{k+m} & \binom{k+m-1}{2} \frac{B_{k+m-3}}{k+m-1} & \dots & \frac{1}{2} & 0 \\ \binom{k+m+1}{1} \frac{B_{k+m}}{k+m+1} & \binom{k+m}{1} \frac{B_{k+m-1}}{k+m} & \binom{k+m-1}{1} \frac{B_{k+m-2}}{k+m-1} & \dots & \binom{2}{1} \frac{B_1}{2} & 1 \end{pmatrix} \times \begin{pmatrix} c_{k,m-1}^{k+m} \\ c_{k,m-1}^{k+m-1} \\ c_{k,m-1}^{k+m-2} \\ c_{k,m-1}^{k+m-3} \\ \vdots \\ c_{k,m-1}^1 \\ 0 \end{pmatrix}, \quad (7)$$

from which we obtain the desired relationships

$$c_{k,m}^r = \sum_{j=0}^{k+m+1-r} \frac{1}{j+r} \binom{j+r}{r} B_j c_{k,m-1}^{j+r-1}, \quad r = 1, 2, \dots, k+m+1. \quad (8)$$

Incidentally, for $r = 1$ we get the symmetric relation $c_{k,m}^1 = \sum_{j=1}^{k+m} B_j c_{k,m-1}^j$ (note that in this case the summation starts at $j = 1$ since $c_{k,m-1}^0 = 0$.)

As a simple numerical example, let us evaluate $P_6^{(1)}(n)$ from $P_6^{(0)}(n)$, and then $P_6^{(2)}(n)$ from $P_6^{(1)}(n)$. Using (6), we readily obtain

$$P_6^{(0)}(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n.$$

Thus, letting $k = 6$ and $m = 1$, and plugging in the coefficients $c_{6,0}^7 = \frac{1}{7}$, $c_{6,0}^6 = \frac{1}{2}$, $c_{6,0}^5 = \frac{1}{2}$, $c_{6,0}^4 = 0$, $c_{6,0}^3 = -\frac{1}{6}$, $c_{6,0}^2 = 0$, and $c_{6,0}^1 = \frac{1}{42}$, the matrix

equation (7) becomes

$$\begin{pmatrix} c_{6,1}^8 \\ c_{6,1}^7 \\ c_{6,1}^6 \\ c_{6,1}^5 \\ c_{6,1}^4 \\ c_{6,1}^3 \\ c_{6,1}^2 \\ c_{6,1}^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & & & & & & & \\ & \frac{1}{2} & \frac{1}{7} & & & & & \\ & \frac{7}{12} & \frac{1}{2} & \frac{1}{6} & & & & \\ & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{5} & & & \\ -\frac{7}{24} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{4} & & & \\ 0 & -\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & & \\ \frac{1}{12} & 0 & -\frac{1}{12} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \\ 0 & \frac{1}{42} & 0 & -\frac{1}{30} & 0 & \frac{1}{6} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{7} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ -\frac{1}{6} \\ 0 \\ \frac{1}{42} \\ 0 \end{pmatrix}.$$

By making the matrix multiplication on the right-hand side, we deduce that

$$P_6^{(1)}(n) = \frac{1}{56}n^8 + \frac{1}{7}n^7 + \frac{5}{12}n^6 + \frac{1}{2}n^5 + \frac{1}{8}n^4 - \frac{1}{6}n^3 - \frac{5}{84}n^2 + \frac{1}{42}n.$$

Alternatively, to derive $P_6^{(2)}(n)$, we can use the explicit solution (8). Indeed, letting $k = 6$ and $m = 2$, and setting $c_{6,1}^8 = \frac{1}{56}$, $c_{6,1}^7 = \frac{1}{7}$, $c_{6,1}^6 = \frac{5}{12}$, $c_{6,1}^5 = \frac{1}{2}$, $c_{6,1}^4 = \frac{1}{8}$, $c_{6,1}^3 = -\frac{1}{6}$, $c_{6,1}^2 = -\frac{5}{84}$, and $c_{6,1}^1 = \frac{1}{42}$ in the right-hand side of (8) yields

$$P_6^{(2)}(n) = \frac{1}{504}n^9 + \frac{3}{112}n^8 + \frac{1}{7}n^7 + \frac{3}{8}n^6 + \frac{19}{40}n^5 + \frac{3}{16}n^4 - \frac{8}{63}n^3 - \frac{5}{56}n^2 + \frac{1}{140}n.$$

We note that the sum of the coefficients of the hypersum polynomial $P_k^{(m)}(n)$ is equal to unity since, for all non-negative integers k and m , $P_k^{(m)}(1) = 1$.

4. Concluding Remarks

In this paper, we have addressed the problem of determining the coefficients of $P_k^{(m)}(n)$ from the coefficients of $P_k^{(m-1)}(n)$. Another related problem is to determine the coefficients of $P_k^{(m)}(n)$ from those corresponding to $P_{k-1}^{(m)}(n)$. To this end, the following recursive formula comes to rescue here (see [7, Proposition 9]):

$$c_{k,m}^r = (m+1)(c_{k-1,m}^r - c_{k-1,m+1}^r) + c_{k-1,m}^{r-1}, \quad (9)$$

for $1 \leq r \leq k + m + 1$, $k \geq 1$, and $m \geq 0$. Now, from equation (8) we have

$$c_{k-1,m+1}^r = \sum_{j=0}^{k+m+1-r} \frac{1}{j+r} \binom{j+r}{r} B_j c_{k-1,m}^{j+r-1}. \quad (10)$$

Therefore, combining equations (9) and (10) gives

$$c_{k,m}^r = (m+1) \left(c_{k-1,m}^r - \sum_{j=0}^{k+m+1-r} \frac{1}{j+r} \binom{j+r}{r} B_j c_{k-1,m}^{j+r-1} \right) + c_{k-1,m}^{r-1}. \quad (11)$$

In particular, it follows that

$$c_{k,m}^{k+m+1} = \frac{k}{k+m+1} c_{k-1,m}^{k+m}. \quad (12)$$

Using equations (11) and (12), we can get $P_k^{(m)}(n)$ from $P_{k-1}^{(m)}(n)$. As an example, let us evaluate $P_7^{(2)}(n)$ from $P_6^{(2)}(n)$. For $k = 7$ and $m = 2$, from (12) we have $c_{7,2}^{10} = \frac{7}{10} c_{6,2}^9 = \frac{7}{10} \frac{1}{504} = \frac{1}{720}$. Likewise, for $k = 7$, $m = 2$, and $r = 9$, from (11) we have (recall that we are taking $B_1 = \frac{1}{2}$)

$$\begin{aligned} c_{7,2}^9 &= 3 \left(c_{6,2}^9 - \frac{1}{9} c_{6,2}^8 - \frac{1}{2} c_{6,2}^9 \right) + c_{6,2}^8 \\ &= 3 \left(\frac{1}{504} - \frac{1}{9} \frac{3}{112} - \frac{1}{2} \frac{1}{504} \right) + \frac{3}{112} = \frac{1}{48}. \end{aligned}$$

Proceeding in the same way for the remaining coefficients $c_{7,2}^8, \dots, c_{7,2}^1$, one arrives at the polynomial

$$\begin{aligned} P_7^{(2)}(n) &= \frac{1}{720} n^{10} + \frac{1}{48} n^9 + \frac{1}{8} n^8 + \frac{3}{8} n^7 \\ &\quad + \frac{133}{240} n^6 + \frac{21}{80} n^5 - \frac{2}{9} n^4 - \frac{5}{24} n^3 + \frac{1}{24} n^2 + \frac{1}{20} n. \end{aligned}$$

On the other hand, as it should be, by applying reiteratively the recurrence relation (12), we retrieve the expression previously obtained for $c_{k,m}^{k+m+1}$:

$$\begin{aligned} c_{k,m}^{k+m+1} &= \frac{k}{k+m+1} c_{k-1,m}^{k+m} \\ &= \left(\frac{k}{k+m+1} \right) \left(\frac{k-1}{k+m} \right) c_{k-2,m}^{k+m-1} \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{k}{k+m+1} \right) \left(\frac{k-1}{k+m} \right) \left(\frac{k-2}{k+m-1} \right) \cdots \left(\frac{1}{m+2} \right) c_{0,m}^{m+1} \\
 &= \frac{k!}{(k+m+1)!},
 \end{aligned}$$

where we have used that $c_{0,m}^{m+1} = 1/(m+1)!$ ([7, Proposition 2]).

Finally, we emphasize that, by means of our algorithm to get $P_k^{(m)}(n)$ from $P_k^{(m-1)}(n)$ (equation (8)), and the one to get $P_k^{(m)}(n)$ from $P_{k-1}^{(m)}(n)$ (equation (11)), one could in principle calculate any hypersum polynomial $P_k^{(m)}(n)$ starting from $P_0^{(0)}(n) = n$.

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