VARIOUS OPERATIONS AND FUZZY PREORDERS
INDUCED BY ALEXANDROV TOPOLOGIES

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Abstract: In this paper, we investigate the properties of various operations and fuzzy preorders induced by Alexandrov topologies in complete residuated lattices. Moreover, we give their examples.

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1. Introduction


On the other hand, Kortelainen [6] investigated the relation between topologies and rough sets. The relationships between Alexandrov topologies and fuzzy rough sets are studied [3-5,7,10]. Algebraic structures of fuzzy rough sets are developed in many directions [10-12].
In this paper, we investigate the properties of various operations and fuzzy preorders induced by Alexandrov topologies in complete residuated lattices. Moreover, we give their examples.

### 2. Preliminaries

**Definition 2.1.** [1,2] An algebra \((L, \wedge, \vee, \odot, \rightarrow, \bot, \top)\) is called a complete residuated lattice if it satisfies the following conditions:

(C1) \(L = (L, \leq, \vee, \wedge, \bot, \top)\) is a complete lattice with the greatest element \(\top\) and the least element \(\bot\);

(C2) \((L, \odot, \top)\) is a commutative monoid;

(C3) \(x \odot y \leq z\) iff \(x \leq y \rightarrow z\) for \(x, y, z \in L\).

In this paper, we assume \((L, \wedge, \vee, \odot, \rightarrow, * \bot, \top)\) is a complete residuated lattice with a strong negation; i.e. \(x^{**} = x\). For \(\alpha \in L, A, \top \in L^X, (\alpha \rightarrow A)(x) = \alpha \rightarrow A(x), (\alpha \odot A)(x) = \alpha \odot A(x)\) and \(\top_x(x) = \top, \top_x(x) = \bot,\) otherwise.

**Lemma 2.2.** [1,2] For each \(x, y, z, x_i, y_i \in L\), the following properties hold.

1. If \(y \leq z\), then \(x \odot y \leq x \odot z\).
2. If \(y \leq z\), then \(x \rightarrow y \leq x \rightarrow z\) and \(z \rightarrow x \leq y \rightarrow x\).
3. \(x \rightarrow y = \top\) iff \(x \leq y\).
4. \(x \odot (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma} (x \odot y_i)\).
5. \(x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i)\) and \((\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)\).
6. \(\bigvee_{i \in \Gamma} x_i \rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)\) and \(\bigwedge_{i \in \Gamma} x_i \rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)\).
7. \((x \rightarrow y) \odot x \leq y\) and \((y \rightarrow z) \odot (x \rightarrow y) \leq (x \rightarrow z)\).
8. \(x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)\) and \(x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)\).
9. \(\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*\) and \(\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*\).
10. \((x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)\) and \((x \odot y)^* = x \rightarrow y^*\).
11. \(x^* \rightarrow y^* = y \rightarrow x\) and \((x \rightarrow y)^* = x \odot y^*\).
12. \(z \leq x \odot y \rightarrow x \odot z\).
13. \(x \rightarrow y \odot z \geq (x \rightarrow y) \odot z\) and \((x \rightarrow y) \rightarrow z \geq x \odot (y \rightarrow z)\).

**Definition 2.3.** [4,5,7] A subset \(\tau \subset L^X\) is called an Alexandrov topology if it satisfies:

(T1) \(\bot_X, \top_X \in \tau\) where \(\top_X(x) = \top\) and \(\bot_X(x) = \bot\) for \(x \in X\).

(T2) If \(A_i \in \tau\) for \(i \in \Gamma\), \(\bigvee_{i \in \Gamma} A_i, \bigwedge_{i \in \Gamma} A_i \in \tau\).

(T3) \(\alpha \odot A \in \tau\) for all \(\alpha \in L\) and \(A \in \tau\).

(T4) \(\alpha \rightarrow A \in \tau\) for all \(\alpha \in L\) and \(A \in \tau\).
Definition 2.4. [1,3,7] Let $X$ be a set. A function $e_X : X \times X \to L$ is called:

- (E1) reflexive if $e_X(x, x) = \top$ for all $x \in X$,
- (E2) transitive if $e_X(x, y) \sqcap e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,
- (E3) anti-symmetric if $e_X(x, y) = e_X(y, x) = \top$, then $x = y$.

If $e$ satisfies (E1) and (E2), $(X, e_X)$ is a fuzzy preordered set. If $e$ satisfies (E1), (E2) and (E3), $(X, e_X)$ is a fuzzy partially order set.

Example 2.5. (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then $(L, e_L)$ is a fuzzy partially order set.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X} (A(x) \to B(x))$. Then $(L^X, e_{L^X})$ is a fuzzy partially order set from Lemma 1.2 (7).

Definition 2.6. [11,12] Let $(X, e_X)$ be a fuzzy partially ordered set and $A \in L^X$.

- (1) A point $x_0$ is called a join of $A$, denoted by $x_0 = \sqcup A$, if it satisfies
  - (J1) $A(x) \leq e_X(x, x_0)$,
  - (J2) $\bigwedge_{x \in X} (A(x) \to e_X(x, y)) \leq e_X(x_0, y)$.
- (2) A point $x_1$ is called a meet of $A$, denoted by $x_1 = \sqcap A$, if it satisfies
  - (M1) $A(x) \leq e_X(x, x_1)$,
  - (M2) $\bigwedge_{x \in X} (A(x) \to e_X(y, x)) \leq e_X(y, x_1)$.

Remark 2.7. [11,12] Let $(X, e_X)$ be a fuzzy partially ordered set and $A \in L^X$.

- (1) $x_0$ is a join of $A$ iff $\bigwedge_{x \in X} (A(x) \to e_X(x, y)) = e_X(x_0, y)$.
- (2) $x_1$ is a meet of $A$ iff $\bigwedge_{x \in X} (A(x) \to e_X(y, x)) = e_X(y, x_1)$.
- (3) If $x_0$ is a join of $A$, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of $A$ exist, then it is unique.

Remark 2.8. [11,12] Let $(L^X, e_{L^X})$ be a fuzzy partially ordered set and $\Phi \in L^{L^X}$.

- (1) If $e_{L^X}(A, B) = e_{L^X}(C, B)$ for all $B \in L^X$, for $B = \top^*_X$, $A = C$.
- (2) If $e_{L^X}(A, B) = e_{L^X}(A, C)$ for all $B \in L^X$, for $A = \top^*_X$, $B = C$.
- (3) Since $\bigwedge_{A \in L^X} (\Phi(A) \to e_{L^X}(A, B)) = e_{L^X}(\bigvee_{A \in L^X} (\Phi(A) \sqcup A), B) = e_{L^X}(\sqcup \Phi, B)$, then $\sqcup \Phi = \bigvee_{A \in L^X} (\Phi(A) \sqcup A)$.
- (4) Since $\bigwedge_{A \in L^X} (\Phi(A) \to e_{L^X}(B, A)) = \bigvee_{A \in L^X} e_{L^X}(B, (\Phi(A) \to A)) = e_{L^X}(B, \bigwedge_{A \in L^X} (\Phi(A) \to A))$, then $\sqcap \Phi = \bigwedge_{A \in L^X} (\Phi(A) \to A)$.

Theorem 2.9. [4] A structure $\tau$ is an Alexandrov topology on $X$ iff $\tau^* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on $X$. 
3. Various Operations and Fuzzy Preorders Induced by Alexandrov Topologies

**Theorem 3.1.** Let $\tau$ be an Alexandrov topology on $X$. Define $\mathcal{K}_\tau : L^X \to L^X$ as follows:

$$\mathcal{K}_\tau(A) = \bigvee_{B \in \tau} (e_{L^X}(B, A^*) \odot B).$$

Then the following properties hold.

1. $e_{L^X}(A, B) \leq e_{L^X}(K_\tau(B), K_\tau(A))$, for all $A, B \in L^X$.
2. $K_\tau(A) \leq A^*$ for all $A \in L^X$.
3. $K_\tau(K_\tau(A)) = K_\tau(A)$ for all $A \in L^X$.
4. $K_\tau(\alpha \odot A) = \alpha \to K_\tau(A)$ for all $\alpha \in L, A \in L^X$.
5. $K_\tau(\bigvee_{i \in I} A_i) = \bigwedge_{i \in I} K_\tau(A_i)$ for all $A_i \in L^X$.
6. $K_\tau(\sqcup \Phi) = \sqcap K_\tau(\Phi)$ for each $\Phi : L^X \to L$ where $K_\tau : L^L \to L^L$ defined as $K_\tau(\Phi)(B) = \bigvee_{K_\tau(A)=B} \Phi(A)$.
7. $K_\tau(A) = \bigvee \{B \in L^X \mid B \leq A^*, B \in \tau\}$.
8. Define $\tau_{K_\tau} = \{A \mid A^* = K_\tau(A)\} = \{K_\tau^*(A) \mid A \in L^X\}$. Then $\tau = \tau_{K_\tau}^*$.
9. Define $e_\tau : X \times X \to L$ as

$$e_\tau(x, y) = \bigwedge_{A \in \tau} (A(x) \to A(y))$$

Then $e_\tau$ is a fuzzy preorder such that

$$e_\tau(x, y) = \bigwedge_{z \in X} (K_\tau(\top z)(x) \to K_\tau(\top z)(y)).$$

10. There exists a fuzzy preorder $e_{K_\tau} : X \times X \to L$ such that

$$K_\tau(A)(y) = \bigwedge_{x \in X} (e_{K_\tau}(x, y) \to A^*(x)).$$

Moreover, $e_{K_\tau} = e_\tau^{-1}$.

**Proof.** (1) By Lemma 2.2 (6,8,12), we have

$$e_{L^X}(K_\tau(B), K_\tau(A))$$
$$\geq \bigwedge_{x \in X} \bigwedge_{C \in \tau} ((e_{L^X}(C, B^*) \odot C(x)) \to (e_{L^X}(C, A^*) \odot C(x)))$$
$$\geq e_{L^X}(B^*, A^*) = e_{L^X}(A, B)$$
(2) Since $\varepsilon_{LX}(C, A^*) \circ C \leq A^*$ from Lemma 2.2 (7), $\mathcal{K}_\tau(A) \leq A^*$. 
(3) Since $\mathcal{K}_\tau(A) \in \tau$, then $\mathcal{K}_\tau(K^*_\tau(A)) \geq e_{LX}(\mathcal{K}_\tau(A), \mathcal{K}_\tau(A)) \circ \mathcal{K}_\tau(A) = \mathcal{K}_\tau(A)$. By (2), $\mathcal{K}_\tau(K^*_\tau(A)) = \mathcal{K}_\tau(A)$. 
(4) Since $\alpha \to \mathcal{K}_\tau(A) \leq \alpha \to A^* = (\alpha \odot A^*)$ and $\alpha \to \mathcal{K}_\tau(A) \in \tau$, 
\[ \mathcal{K}_\tau(\alpha \odot A) \geq e_{LX}(\alpha \to \mathcal{K}_\tau(A), (\alpha \odot A^*) \circ (\alpha \to \mathcal{K}_\tau(A)) \]
\[ = \alpha \to \mathcal{K}_\tau(A), \]
\[ \mathcal{K}_\tau(\alpha \odot A) = \bigvee_{B \in \tau}(e_{LX}(B, \alpha \to A^*) \circ B) \]
\[ = \bigvee_{B \in \tau}((\alpha \to e_{LX}(B, A^*)) \circ B) \]
\[ \geq \alpha \to \bigvee_{B \in \tau}(e_{LX}(B, A^*)) \circ B \text{ (by Lemma 2.2 (13))} \]
\[ = \alpha \to \mathcal{K}_\tau(A). \]

Thus, $\mathcal{K}_\tau(\alpha \odot A) = \alpha \to \mathcal{K}_\tau(A)$.

(5) By (1), since $\mathcal{K}_\tau(B) \leq \mathcal{K}_\tau(A)$ for $A \leq B$, $\mathcal{K}_\tau(\bigvee_{i \in \Gamma} A_i) \leq \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i)$. Since $\bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i) \leq \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i) \in \tau$, we have 
\[ \mathcal{K}_\tau(\bigvee_{i \in \Gamma} A_i) \geq e_{LX}(\bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i), \bigwedge_{i \in \Gamma} A_i^*) \circ \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i) \]
\[ = \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i). \]

Hence $\mathcal{K}_\tau(\bigvee_{i \in \Gamma} A_i) = \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i)$.

(6) For each $\Phi : LX \to L$, put $Q = \cap \mathcal{K}_\tau^\to(\Phi)$. Since $\mathcal{K}_\tau^\to(\Phi) : LX \to L$ is a map, we have $\cap \mathcal{K}_\tau^\to(\Phi) = \bigwedge_{C \in LX} (\mathcal{K}_\tau^\to(\Phi)(C) \to C)$ and $Q = \cap \mathcal{K}_\tau^\to(\Phi) = \mathcal{K}_\tau(\bigvee \Phi)$ from:
\[ e_{LX}(B, Q) = \bigwedge_{C \in LX} (\mathcal{K}_\tau^\to(\Phi)(C) \to e_{LX}(B, C)) \]
\[ = e_{LX}(B, \bigwedge_{C \in LX} (\mathcal{K}_\tau^\to(\Phi)(C) \to C))) \]
\[ = e_{LX}(B, \bigwedge_{A \in LX} (\Phi(A) \to \mathcal{K}_\tau(A)))) \]
\[ = e_{LX}(B, \mathcal{K}_\tau(\bigvee_{A \in LX} (\Phi(A) \odot A)))) \text{ (by (4) and (5))} \]
\[ = e_{LX}(B, \mathcal{K}_\tau(\bigvee \Phi)). \]

(7) Put $K(A) = \bigvee\{B \in LX \mid B \leq A^*, B \in \tau\}$. Since $K(A) \leq A^*$ and $K(A) \in \tau$, we have 
\[ \mathcal{K}_\tau(A) = \bigvee_{B \in \tau} (e_{LX}(B, A^*) \circ B) \geq e_{LX}(K(A), A^*) \circ K(A) = K(A). \]

Since $\mathcal{K}_\tau(A) \leq A$ and $\mathcal{K}_\tau(A) \in \tau$, we have $K(A) \geq \mathcal{K}_\tau(A)$. Hence $\mathcal{K}_\tau = K$.

(8) Put $\tau_1 = \{K^*_\tau(A) \mid A \in LX\}$. Let $A \in \tau_\mathcal{K}$. Then $A = K^*_\tau(A) \in \tau_1$. Let $K^*_\tau(A) \in \tau_1$. By (3), $\mathcal{K}_\tau(K^*_\tau(A)) = \mathcal{K}_\tau(A)$. So, $K^*_\tau(A) \in \tau_{K_\tau}$. Thus $\tau_{K_\tau} = \tau_1$. 

Moreover, $\tau = \tau_{K_{\tau}}^*$ from:

$$A \in \tau \text{ iff } K_{\tau}(A^*) = A \text{ iff } A \in \tau_{K_{\tau}}^*.$$  

(9) Since $A = \bigvee_{x \in X} (A(x) \odot T_x)$, by (4) and (5), $K_{\tau}(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow K_{\tau}(T_x)(y))$. Since $\tau = \{K_{\tau}(A) \mid A \in L^X\}$, we have

$$e_\tau(x, y) = \bigwedge_{A \in \tau} (A(x) \rightarrow A(y)) = \bigwedge_{A \in L^X} (K_{\tau}(A)(x) \rightarrow K_{\tau}(A)(y))$$

$$\leq \bigwedge_{x \in X} (K_{\tau}(T_z)(x) \rightarrow K_{\tau}(T_z)(y)).$$

$$e_{K_{\tau}}(x, x) = \bigvee_{y \in X} (e_{K_{\tau}}(x, y) \odot e_{K_{\tau}}(y, x)) \leq e_{K_{\tau}}(x, x)$$

iff $\bigvee_{y \in X} (K_{\tau}(T_x)(y) \odot K_{\tau}(T_y)(z)) \leq K_{\tau}(T_x)(z)$

iff $\bigwedge_{y \in X} (K_{\tau}(T_y)(z)) \geq K_{\tau}(T_z)(z)$

iff $K_{\tau}(y, z) \geq K_{\tau}(T_z)(z)$

Thus $e_{K_{\tau}}$ is a fuzzy preorder.

By (9),

$$e_{K_{\tau}}(x, y) = \bigwedge_{z \in X} (K_{\tau}(T_z)(x) \rightarrow K_{\tau}(T_z)(y))$$

$$= \bigwedge_{z \in X} (K_{\tau}(T_z)(x) \rightarrow K_{\tau}(T_z)(y))$$

$$= \bigwedge_{z \in X} (e_{K_{\tau}}(z, y) \rightarrow e_{K_{\tau}}(z, x))$$

$$\leq e_{K_{\tau}}(y, y) \rightarrow e_{K_{\tau}}(y, x) = e_{K_{\tau}}(y, x).$$

Since $e_{K_{\tau}}$ is a fuzzy preorder, $e_{K_{\tau}}(z, y) \odot e_{K_{\tau}}(y, x) \leq e_{K_{\tau}}(z, x)$. Thus

$$e_{K_{\tau}}(y, x) \leq \bigwedge_{z \in X} (e_{K_{\tau}}(z, y) \rightarrow e_{K_{\tau}}(z, x)) = e_{K_{\tau}}(x, y).$$

Hence $e_{K_{\tau}}(y, x) = e_{\tau}(y, x)$. 
**Theorem 3.2.** Let $\tau$ be an Alexandrov topology on $X$. Define $\mathcal{M}_\tau : L^X \to L^X$ as follows:

$$M_\tau(A) = \bigwedge_{B \in \tau} (e_{L^X}(A^*, B) \to B).$$

Then the following properties hold.

1. $e_{L^X}(A, B) \leq e_{L^X}(M_\tau(B), M_\tau(A))$, for all $A, B \in L^X$.
2. $A^* \leq M_\tau(A)$ for all $A \in L^X$.
3. $M_\tau(M_\tau(A)) = M_\tau(A)$ for all $A \in L^X$.
4. $M_\tau(\alpha \to A) = \alpha \circ M_\tau(A)$ for all $\alpha \in L, A \in L^X$.
5. $M_\tau(\bigwedge_{i \in I} A_i) = \bigvee_{i \in I} M_\tau(A_i)$ for all $A_i \in L^X$.
6. $M_\tau(\bigcap \Phi) = \bigcup M_\tau^\tau(\Phi)$ for each $\Phi : L^X \to L$ where $M_\tau^\tau : L^{L^X} \to L^{L^X}$ defined as $M_\tau^\tau(\Phi)(B) = \bigvee\{M_\tau(A) = B(\Phi(A))\}$.
7. $M_\tau(A) = \bigwedge\{B \in L^X \mid A^* \leq B, B \in \tau\}$.
8. Define $\tau_{M_\tau} = \{A \mid A = M_\tau(A^*)\}$. Then $\tau = \tau_{M_\tau} = \{M_\tau(A^*) \mid A \in L^X\}$.
9. Define $e_\tau : X \times X \to L$ as follows

$$e_\tau(x, y) = \bigwedge_{A \in \tau} (A(x) \to A(y)).$$

Then $e_\tau$ is a fuzzy preorder such that

$$e_\tau(x, y) = \bigwedge_{A \in \tau} (M_\tau(T_\tau^*(x)) \to M_\tau(T_\tau^*(y))).$$

Moreover, $e_\tau^*(x, y) = e_\tau(y, x) = e_\tau^{-1}(x, y)$.

10. $(M_\tau(A^*))^* = K_\tau^*(A)$ for all $A \in L^X$.
11. There exists a fuzzy preorder $e_{M_\tau} : X \times X \to L$ such that

$$M_\tau(A)(y) = \bigvee_{x \in X} (e_{M_\tau}(x, y) \circ A^*(x)),
\quad
K_\tau^*(A)(y) = \bigwedge_{x \in X} (e_{M_\tau}(x, y) \to A^*(x)).$$

Moreover, $e_{M_\tau}(x, y) = e_{K_\tau^*}(y, x) = e_{M_\tau^*}(y, x) = e_{K_\tau^*}(x, y) = e_{\tau^*}(x, y)$ for all $x, y \in X$.

**Proof.** (1) By Lemma 2.2 (6,8), we have

$$e_{L^X}(M_\tau(B), M_\tau(A))
\geq \bigwedge_{x \in X} \bigwedge_{C \in \tau} (e_{L^X}(B^*, C) \to C(x)) \to \bigwedge_{D \in \tau} (e_{L^X}(A^*, D) \to D(x))
\geq \bigwedge_{x \in X} \bigwedge_{C \in \tau} (e_{L^X}(B^*, C) \to C(x)) \to (e_{L^X}(A^*, C) \to C(x))
\geq \bigwedge_{C \in \tau} (e_{L^X}(A^*, C) \to (e_{L^X}(B^*, C)))
\geq e_{L^X}(B^*, A^*) = e_{L^X}(A, B).$$
(2) Since \( e_{L^X}(A^*, B) \odot A^* \leq B \) iff \( A^* \leq e_{L^X}(A^*, B) \rightarrow B \), then \( A^* \leq M_\tau(A) \).

(3) Since \( M_\tau(A) \in \tau \), then \( M_\tau(M_\tau^*(A)) \leq e_{L^X}(M_\tau(A), M_\tau(A)) \rightarrow M_\tau(A) = M_\tau(A) \). By (2), \( M_\tau(M_\tau^*(A)) = M_\tau(A) \).

(4) Since \( \alpha \odot A^* \leq \alpha \odot M_\tau(A) \) and \( \alpha \odot M_\tau(A) \in \tau \),

\[
M_\tau(\alpha \rightarrow A) \leq e_{L^X}(\alpha \odot A^*, \alpha \odot M_\tau(A)) \rightarrow \alpha \odot M_\tau(A) = \alpha \odot M_\tau(A).
\]

(5) By (1), since \( M_\tau(B) \leq M_\tau(A) \) for \( A \leq B \), \( \bigvee_{i \in \Gamma} M_\tau(A_i) \leq M_\tau(\bigwedge_{i \in \Gamma} A_i) \).

Since \( \bigvee_{i \in \Gamma} A_i^* \leq \bigvee_{i \in \Gamma} M_\tau(A_i) \) and \( \bigvee_{i \in \Gamma} M_\tau(A_i) \in \tau \), we have

\[
M_\tau(\bigwedge_{i \in \Gamma} A_i) \leq e_{L^X}(\bigvee_{i \in \Gamma} A_i^*, \bigvee_{i \in \Gamma} M_\tau(A_i)) \rightarrow \bigvee_{i \in \Gamma} M_\tau(A_i).
\]

Hence \( M_\tau(\bigwedge_{i \in \Gamma} A_i) = \bigvee_{i \in \Gamma} M_\tau(A_i) \).

(6) For each \( \Phi : L^X \rightarrow L \), put \( P = \bigcap M_\tau(\Phi) \). Since \( M_\tau(\Phi) : L^X \rightarrow L \) is a map, we have \( \bigcup M_\tau(\Phi) = \bigvee_{i \in L^X} (M_\tau(\Phi)(C) \odot C) \) and \( P = \bigcup M_\tau(\Phi) = M_\tau(\bigcap \Phi) \) from:

\[
\begin{align*}
e_{L^X}(P, B) &= \bigwedge_{C \in L^X} (M_\tau(\Phi)(C) \rightarrow e_{L^X}(C, B)) \\
&= \bigwedge_{C \in L^X} e_{L^X}(M_\tau(\Phi)(C) \odot C, B) \\
&= e_{L^X}(\bigvee_{C \in L^X} (M_\tau(\Phi)(C) \odot C), B) \\
&= e_{L^X}(\bigvee_{A \in L^X} (\Phi(A) \odot M_\tau(A)), B) \\
&= e_{L^X}(M_\tau(\bigwedge_{A \in L^X} (\Phi(A) \rightarrow A)), B) \quad \text{(by (4) and (5))} \\
&= e_{L^X}(M_\tau(\bigcap \Phi), B)
\end{align*}
\]

(7) Put \( M(A) = \bigwedge \{ B \in L^X \mid A^* \leq B, B \in \tau \} \). Since \( A^* \leq M(A) \) and \( M(A) \in \tau \), we have

\[
M_\tau(A) = \bigwedge_{B \in \tau} (e_{L^X}(A^*, B) \rightarrow B) \leq e_{L^X}(A^*, M(A)) \rightarrow M(A) = M(A).
\]

Since \( A^* \leq M_\tau(A) \) and \( M_\tau(A) \in \tau \), we have \( M(A) \leq M_\tau(A) \). Hence \( M_\tau = M \).

(8) We have \( \tau = \tau_{M_\tau} \) from \( A \in \tau \) iff \( M_\tau(A^*) = A \) iff \( A \in \tau_{M_\tau} \). Put \( \tau = \{ M_\tau(A^*) \mid A \in L^X \} \). Let \( A \in \tau \). Then \( A = M_\tau(A^*) \). Let \( M_\tau(A^*) \in \tau_1 \). Then \( M_\tau(M_\tau^*(A^*)) = M_\tau(A^*) \in \tau_{M_\tau} = \tau \). Hence \( \tau = \tau_1 \).
(9) By (8), since \( \mathcal{M}_\tau(T^*_z) \in \tau \),
\[
e_\tau(x, y) = \bigwedge_{A \in \tau} (A(x) \to A(y)) \leq \bigwedge_{x \in X} (\mathcal{M}_\tau(T^*_z)(x) \to \mathcal{M}_\tau(T^*_z)(y)).
\]
Since \( A^* = \bigwedge_{z \in X} (A(z) \to T^*_z) \), by (4) and (5), \( \mathcal{M}_\tau(A^*)(x) = \bigvee_{z \in X} (A(z) \circ \mathcal{M}_\tau(T^*_z)(x)) \).
\[
e_\tau(x, y) = \bigwedge_{A \in \mathcal{S}} (A(x) \to A(y)) = \bigwedge_{A \in \mathcal{T}} (\mathcal{M}_\tau(A^*)(x) \to \mathcal{M}_\tau(A^*)(y))
\]
\[
= \bigwedge_{A \in \mathcal{T}} (A(z) \circ \mathcal{M}_\tau(T^*_z)(x)) \to \bigwedge_{w \in X} (A(w) \circ \mathcal{M}_\tau(T^*_w)(y))
\]
\[
= \bigwedge_{A \in \mathcal{T}} \bigwedge_{z \in X} ((A(z) \circ \mathcal{M}_\tau(T^*_z)(x)) \to (A(z) \circ \mathcal{M}_\tau(T^*_w)(y)))
\]
\[
\geq \bigwedge_{z \in X} (\mathcal{M}_\tau(T^*_z)(x) \to \mathcal{M}_\tau(T^*_z)(y)).
\]

(10) Since \( A = \bigwedge_{z \in X} (A^*(z) \to T^*_z) \), by (4) and (5),
\[
\mathcal{M}_\tau(A)(x) = \mathcal{M}_\tau(\bigwedge_{z \in X} (A^*(z) \to T^*_z))(x) = \bigvee_{z \in X} (A^*(z) \circ \mathcal{M}_\tau(T^*_z)(x)).
\]
Put \( e_{M_\tau}(x, y) = \mathcal{M}_\tau(T^*_x)(y) \). Then
\[
\mathcal{M}_\tau(A)(y) = \bigvee_{x \in X} (e_{M_\tau}(x, y) \circ A^*(x)).
\]
\[
e_{M_\tau}(x, x) = \mathcal{M}_\tau(T^*_x)(x) \geq T_x(x) = T
\]
\[
\text{iff } \bigvee_{y \in X} (e_{M_\tau}(x, y) \circ e_{M_\tau}(y, z)) \leq e_{M_\tau}(x, z)
\]
\[
= \mathcal{M}_\tau(T^*_x)(y) \leq \mathcal{M}_\tau(T^*_x)(z)
\]
\[
= \mathcal{M}_\tau(\bigwedge_{y \in X} (e_{M_\tau}(T^*_y)(y) \to T^*_y))(z) \leq \mathcal{M}_\tau(T^*_x)(z)
\]
\[
= \mathcal{M}_\tau(\bigwedge_{y \in X} (e_{M_\tau}(T^*_y)(y) \to T^*_y))(z) \leq \mathcal{M}_\tau(T^*_x)(z)
\]
Hence \( e_{M_\tau} \) is a fuzzy preorder. Since \( e_{M_\tau}(x, y) = \mathcal{M}_\tau(T^*_x)(y) = K^*_{T x}(y) \),
by Theorem 3.1(10),
\[
K^*_{T x}(A)(y) = \bigwedge_{x \in X} (e_{M_\tau}(x, y) \to A^*(x)).
\]
Moreover, by Theorem 3.1(10), \( e_{M_\tau}(x, y) = e_{K_\tau}(y, x) = e_\tau(x, y) = e_{M_\tau^*}(y, x) = e_{K_\tau^*}(x, y) = e_\tau^*(y, x) \) for all \( x, y \in X \).
Example 3.3. Let \((L = [0, 1], \circ, \to, \ast)\) be a complete residuated lattice with a strong negation defined by

\[
x \circ y = (x + y - 1) \vee 0, \quad x \rightarrow y = (1 - x + y) \wedge 1, \quad x^* = 1 - x.
\]

Let \(X = \{x, y, z\}\) be a set and \(B \in L^X\) as follows:

\[
B(x) = 0.5, B(y) = 0.2, B(z) = 0.9.
\]

Define \(e_B(x, y) = B(x) \rightarrow B(y)\) such that

\[
e_B = \begin{pmatrix}
1 & 0.7 & 1 \\
1 & 1 & 1 \\
0.6 & 0.3 & 1
\end{pmatrix}.
\]

(1) We define

\[
\tau = \{e_B(A)(y) = \bigvee_{y \in X} (e_B(x, y) \circ A(x)) \mid A \in L^X\}.
\]

(T1) For \(\bot_X \in L^X\), \(e_B(\bot_X) = \bot_X \in \tau\). For \(\top_X \in L^X\), \(e_B(\top_X) = \top_X \in \tau\).

(T2) For \(e_B(A_i) \in \tau\) for each \(i \in \Gamma\), \(\bigvee_{i \in \Gamma} e_B(A_i) = e_B(\bigvee_{i \in \Gamma} A_i) \in \tau\).

Moreover, since \(e_B(A)(x) \geq e_B(x, x) \circ A(x) = A(x)\) and \(e_B(e_X(A)) = e_X(A)\),

\[
\bigwedge_{i \in \Gamma} e_B(A_i) \leq e_B(\bigwedge_{i \in \Gamma} e_B(A_i)) \leq \bigwedge_{i \in \Gamma} e_B(e_B(A_i)) = \bigwedge_{i \in \Gamma} e_B(A_i).
\]

Hence \(\bigwedge_{i \in \Gamma} e_B(A_i) = e_B(\bigwedge_{i \in \Gamma} A_i) \in \tau\).

(T3) For \(e_B(A) \in \tau\), \(\alpha \circ e_B(A) = e_B(\alpha \circ A) \in \tau\).

(T4) Since \(\alpha \circ e_B(\alpha \to e_B(A)) \leq e_B(e_B(A)) = e_B(A)\), we have

\[
\alpha \to e_B(A) \leq e_B(\alpha \to e_B(A)) \leq \alpha \to e_B(A)
\]

Hence, for \(e_B(A) \in \tau\), \(\alpha \to e_B(A) = e_B(\alpha \to e_B(A)) \in \tau\). Hence \(\tau\) is an Alexandrov topology on \(X\).

(2) Since \(B(x) \rightarrow B(y) \leq (\alpha \circ B)(x) \rightarrow (\alpha \circ B)(y)\), \(B(x) \rightarrow B(y) \leq (\alpha \rightarrow B)(x) \rightarrow (\alpha \rightarrow B)(y)\), we have \(e_\tau(x, y) = e_B(x, y)\). Moreover, \(e_{\tau^*}(x, y) = e_B(y, x)\).

Since

\[
\tau = \{(x \circ (1, 0.7, 1)) \lor (y \circ (1, 1, 1)) \lor (z \circ (0.6, 0.3, 1)) \mid (x, y, z) \in L^X\},
\]
by Theorem 3.1 (7), we obtain:

\[ \mathcal{K}_\tau(1_x) = \bigvee \{ A \in \tau \mid A = (x \circ (1, 0.7, 1)) \lor (y \circ (1, 1, 1)) \lor (z \circ (0.6, 0.3, 1)) \leq (0, 1, 1) \} = (0, 0, 0.4) \]

\[ \mathcal{K}_\tau(1_y) = \bigvee \{ A \in \tau \mid A = (x \circ (1, 0.7, 1)) \lor (y \circ (1, 1, 1)) \lor (z \circ (0.6, 0.3, 1)) \leq (1, 0, 1) \} = (0.3, 0.0, 0.7) \]

\[ \mathcal{K}_\tau(1_z) = \bigvee \{ A \in \tau \mid A = (x \circ (1, 0.7, 1)) \lor (y \circ (1, 1, 1)) \lor (z \circ (0.6, 0.3, 1)) \leq (1, 1, 0) \} = (0, 0, 0) \]

Put \( e_{K_\tau}(x, y) = \mathcal{K}_\tau^*(1_x)(y) \), then

\[ e_{K_\tau}(x, y) = \mathcal{K}_\tau^*(1_x)(y) = e_\tau(y, x) = e_B(y, x) = e_{\tau^*}(x, y). \]

(3) By Theorem 3.2 (7), we obtain:

\[ \mathcal{M}_\tau(1_y^*) = \bigwedge \{ A \in \tau \mid (1, 0, 0) \leq A = (x \circ (1, 0.7, 1)) \lor (y \circ (1, 1, 1)) \lor (z \circ (0.6, 0.3, 1)) \} = (1, 0.7, 1) \]

\[ \mathcal{M}_\tau(1_y^*) = \bigwedge \{ A \in \tau \mid (0, 1, 0) \leq A = (x \circ (1, 0.7, 1)) \lor (y \circ (1, 1, 1)) \lor (z \circ (0.6, 0.3, 1)) \} = (1, 1, 1) \]

\[ \mathcal{M}_\tau(1_z^*) = \bigwedge \{ A \in \tau \mid (0, 0, 1) \leq A = (x \circ (1, 0.7, 1)) \lor (y \circ (1, 1, 1)) \lor (z \circ (0.6, 0.3, 1)) \} = (0.6, 0.3, 1) \]

Moreover, we have \( e_{M_\tau}(y, x) = \mathcal{M}_\tau(1_y^*) = e_{K_\tau}(x, y) = \mathcal{K}_\tau^*(1_x)(y) = e_\tau(y, x) = e_B(y, x) = e_{\tau^*}(x, y). \)

(4) For \( D_1 = (0.5, 0.3, 0.7), D_2 = (0.4, 0.8, 0.5) \), we obtain

\[ \mathcal{K}_\tau(D_1)(y) = \bigwedge_{x \in X} (e_{K_\tau}(x, y) \rightarrow D_1^*(x)) = \bigwedge_{x \in X} (e_B(y, x) \rightarrow D_1^*(x)), \]

\[ \mathcal{M}_\tau(D_1)(y) = \bigvee_{x \in X} (e_{M_\tau}(x, y) \circ D_1^*(x)) = \bigvee_{x \in X} (e_B(x, y) \circ D_1^*(x)). \]

\[ \mathcal{K}_\tau(D_1) = (0.3, 0.3, 0.3), \mathcal{K}_\tau(D_2) = (0.5, 0.2, 0.5), \]

\[ \mathcal{M}_\tau(D_1) = (0.7, 0.7, 0.7), \mathcal{M}_\tau(D_2) = (0.6, 0.3, 0.6). \]

Let \( \Phi : \mathcal{L} \rightarrow \mathcal{L} \) as follows

\[
\Phi(B) = \begin{cases} 
0.7, & \text{if } B = B_1, \\
0.8, & \text{if } B = B_2, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
\bigwedge_{A \in \mathcal{L}} (\Phi(A) \rightarrow A) = (\Phi(D_1) \rightarrow D_1) \land (\Phi(D_2) \rightarrow D_2) = (0.7 \rightarrow (0.5, 0.3, 0.7)) \land (0.8 \rightarrow (0.4, 0.8, 0.5)) = (0.6, 0.6, 0.7)
\]
\[ \Box \Phi = \bigvee_{A \in L^X} (\Phi(A) \circ A) = (\Phi(D_1) \circ D_1) \vee (\Phi(D_2) \circ D_2) = (0.7 \circ (0.5, 0.3, 0.7)) \vee (0.8 \circ (0.4, 0.8, 0.5)) = (0.2, 0.6, 0.4) \]

\[ \mathcal{K}_\tau(\Box \Phi) = (0.6, 0.4, 0.6). \]
\[ \nabla \mathcal{K}^\tau_\tau(\Phi) = \bigwedge_{A \in L^X} (\Phi(A) \rightarrow \mathcal{K}_\tau(A)) \]
\[ = (\Phi(D_1) \rightarrow \mathcal{K}_\tau(D_1)) \wedge (\Phi(D_2) \rightarrow \mathcal{K}_\tau(D_2)) \]
\[ = (0.7 \rightarrow (0.3, 0.3, 0.3)) \wedge (0.8 \rightarrow (0.5, 0.2, 0.5)) \]
\[ = (0.6, 0.4, 0.6) \]

we have \( \mathcal{K}_\tau(\Box \Phi) = \nabla \mathcal{K}^\tau_\tau(\Phi) \).

\[ \mathcal{M}_\tau(\Box \Phi) = (0.4, 0.4, 0.4). \]
\[ \nabla \mathcal{M}^\tau_\tau(\Phi) = \bigwedge_{A \in L^X} (\Phi(A) \circ \mathcal{M}_\tau(A)) \]
\[ = (\Phi(D_1) \circ \mathcal{M}_\tau(D_1)) \vee (\Phi(D_2) \circ \mathcal{M}_\tau(D_2)) \]
\[ = (0.7 \circ (0.7, 0.7, 0.7)) \vee (0.8 \circ (0.6, 0.3, 0.6)) \]
\[ = (0.4, 0.4, 0.4) \]

Thus, \( \mathcal{M}_\tau(\Box \Phi) = \nabla \mathcal{M}^\tau_\tau(\Phi) \).

(5) Since \( \tau^* = \{A^* \mid A \in \tau\} \), we have
\[ \tau^* = \{ \bigwedge_{x \in X} (e_X(x, -) \rightarrow B(x)) \mid B \in L^X \}. \]

\[ \mathcal{K}_\tau^*(1_x) = \bigwedge\{A \in \tau^* \mid A = ((1, 0.7, 1) \rightarrow x) \wedge ((1, 1, 1) \rightarrow y) \wedge ((0.6, 0.3, 1) \rightarrow z) \leq (0, 1, 1)\} = (0, 0.3, 0) \]
\[ \mathcal{K}_\tau^*(1_y) = \bigwedge\{A \in \tau^* \mid A = ((1, 0.7, 1) \rightarrow x) \wedge ((1, 1, 1) \rightarrow y) \wedge ((0.6, 0.3, 1) \rightarrow z) \leq (1, 0, 1)\} = (0, 0) \]
\[ \mathcal{K}_\tau^*(1_z) = \bigwedge\{A \in \tau^* \mid A = ((1, 0.7, 1) \rightarrow x) \wedge ((1, 1, 1) \rightarrow y) \wedge ((0.6, 0.3, 1) \rightarrow z) \leq (1, 1, 0)\} = (0.4, 0.7, 0) \]

\[ \tau_1 = \{e_{B^*}(y) = \bigvee_{y \in X} (e_{B^*}(x, y) \circ A(x)) \mid A \in L^X \}. \]

\[ \mathcal{K}_{\tau_1}(1_x) = \bigvee\{A \in \tau_1 \mid A = (x \circ (1, 1, 0.6)) \vee (y \circ (0.7, 1, 0.3)) \vee (z \circ (1, 1, 1)) \leq (0, 1, 1)\} = (0, 0.3, 0) \]
\[ \mathcal{K}_{\tau_1}(1_y) = \bigvee\{A \in \tau_1 \mid A = (x \circ (1, 1, 0.6)) \vee (y \circ (0.7, 1, 0.3)) \vee (z \circ (1, 1, 1)) \leq (1, 0, 1)\} = (0, 0) \]
\[ \mathcal{K}_{\tau_1}(1_z) = \bigvee\{A \in \tau_1 \mid A = (x \circ (1, 1, 0.6)) \vee (y \circ (0.7, 1, 0.3)) \vee (z \circ (1, 1, 1)) \leq (1, 1, 0)\} = (0.4, 0.7, 0) \]

Hence \( \mathcal{K}_{\tau_1}(1_x)(y) = \mathcal{K}_{\tau^*}(1_x)(y) \) for all \( x, y \in X \), then \( \mathcal{K}_{\tau_1} = \mathcal{K}_{\tau^*} \). Thus,
\[ \tau^* = \tau_{\mathcal{K}_{\tau_1}}^* = \tau_{\mathcal{K}_{\tau^*}} = \tau_1. \]
For $D_1, D_2$ and $\Phi$ in (3), we obtain

$$K_{\tau^*}(D_i)(y) = \bigwedge_{x \in X} (e_{K_{\tau^*}}(x, y) \rightarrow D_i^*(x))$$

$$M_{\tau^*}(D_i)(y) = \bigvee_{x \in X} (e_{M_{\tau^*}}(x, y) \odot D_i^*(x)) = \bigvee_{x \in X} (e_{K_{\tau^*}}(y, x) \odot D_i^*(x)).$$

$$K_{\tau^*}(D_1) = (0.5, 0.7, 0.3), K_{\tau^*}(D_2) = (0.2, 0.2, 0.2),$$
$$M_{\tau^*}(D_1) = (0.5, 0.7, 0.3), M_{\tau^*}(D_2) = (0.6, 0.6, 0.5).$$

Since $\sqcup \Phi = (0.2, 0.6, 0.4)$ and

$$K_{\tau^*}(\sqcup \Phi) = (0.4, 0.4, 0.4).$$
$$\sqcap K_{\tau^*}(\Phi) = \bigwedge_{A \in L_X} (\Phi(A) \rightarrow K_{\tau^*}(A))$$
$$= (\Phi(D_1) \rightarrow K_{\tau^*}(D_1)) \land (\Phi(D_2) \rightarrow K_{\tau^*}(D_2))$$
$$= (0.7 \rightarrow (0.5, 0.7, 0.3)) \land (0.8 \rightarrow (0.2, 0.2, 0.2))$$
$$= (0.4, 0.4, 0.4)$$

we have $K_{\tau^*}(\sqcup \Phi) = \sqcap K_{\tau^*}(\Phi)$.

Since $\sqcap \Phi = (0.6, 0.6, 0.7)$ and

$$\sqcup M_{\tau^*}(\Phi) = (0.4, 0.4, 0.3).$$
$$\sqcap M_{\tau^*}(\Phi) = \bigvee_{A \in L_X} (\Phi(A) \odot M_{\tau^*}(A))$$
$$= (\Phi(D_1) \odot M_{\tau^*}(D_1)) \lor (\Phi(D_2) \odot M_{\tau^*}(D_2))$$
$$= (0.7 \odot (0.5, 0.7, 0.3)) \lor (0.8 \odot (0.6, 0.6, 0.5))$$
$$= (0.4, 0.4, 0.3),$$

then $M_{\tau^*}(\sqcap \Phi) = \sqcup M_{\tau^*}(\Phi)$.

References


