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VARIOUS OPERATIONS AND FUZZY PREORDERS INDUCED BY ALEXANDROV TOPOLOGIES

Yong Chan Kim
Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo, 210-702, KOREA

Abstract: In this paper, we investigate the properties of various operations and fuzzy preorders induced by Alexandrov topologies in complete residuated lattices. Moreover, we give their examples.

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1. Introduction

Pawlak [8,9] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices.

On the other hand, Kortelainen [6] investigated the relation between topologies and rough sets. The relationships between Alexandrov topologies and fuzzy rough sets are studied [3-5,7,10]. Algebraic structures of fuzzy rough sets are developed in many directions [10-12].

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In this paper, we investigate the properties of various operations and fuzzy preorders induced by Alexandrov topologies in complete residuated lattices. Moreover, we give their examples.

2. Preliminaries

Definition 2.1. [1,2] An algebra $(L, \wedge, \vee, \odot, \to, \bot, \top)$ is called a *complete residuated lattice* if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, ^* \bot, \top)$ is a complete residuated lattice with a strong negation;i.e. $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \to A)(x) = \alpha \to A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(x) = \bot$, otherwise.

Lemma 2.2. [1,2] For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) If $y \leq z$, then $x \odot y \leq x \odot z$.

(2) If $y \leq z$, then $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(3) $x \to y = \top$ iff $x \leq y$.

(4) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$.

(5) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(6) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(7) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.

(8) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(9) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(10) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(11) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

(12) $y \to z \leq x \odot y \to x \odot z$.

(13) $x \to y \odot z \geq (x \to y) \odot z$ and $(x \to y) \to z \geq x \odot (y \to z)$.

Definition 2.3. [4,5,7] A subset $\tau \subset L^X$ is called an *Alexandrov topology* if it satisfies:

(T1) $\bot_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\bot_X(x) = \bot$ for $x \in X$.

(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.

(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(T4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Definition 2.4. [1,3,7] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) *reflexive* if $e_X(x,x) = \top$ for all $x \in X$,

(E2) *transitive* if $e_X(x,y) \odot e_X(y,z) \leq e_X(x,z)$, for all $x,y,z \in X$,

(E3) *anti-symmetric* if $e_X(x,y) = e_X(y,x) = \top$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a *fuzzy preordered set.* If e satisfies (E1), (E2) and (E3), (X, e_X) is a *fuzzy partially order set.*

Example 2.5. (1) We define a function $e_L : L \times L \to L$ as $e_L(x,y) = x \to y$. Then (L, e_L) is a fuzzy partially order set.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A,B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy partially order set from Lemma 1.2 (7).

Definition 2.6. [11,12] Let (X, e_X) be a fuzzy partially ordered set and $A \in L^X$.

(1) A point x_0 is called a *join* of A, denoted by $x_0 = \sqcup A$, if it satisfies

(J1) $A(x) \leq e_X(x, x_0)$,

(J2) $\bigwedge_{x\in X}(A(x) \to e_X(x,y)) \leq e_X(x_0, y)$.

A point x_1 is called a *meet* of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \leq e_X(x_1, x)$,

(M2) $\bigwedge_{x\in X}(A(x) \to e_X(y,x)) \leq e_X(y, x_1)$.

Remark 2.7. [11,12] Let (X, e_X) be a fuzzy partially ordered set and $A \in L^X$.

(1) x_0 is a join of A iff $\bigwedge_{x\in X}(A(x) \to e_X(x,y)) = e_X(x_0, y)$.

(2) x_1 is a meet of A iff $\bigwedge_{x\in X}(A(x) \to e_X(y,x)) = e_X(y, x_1)$.

(3) If x_0 is a join of A, then it is unique because $e_X(x_0,y) = e_X(y_0,y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0,y_0) = e_X(y_0,x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.

Remark 2.8. [11,12] Let (L^X, e_{L^X}) be a fuzzy partially ordered set and $\Phi \in L^{L^X}$.

(1) If $e_{L^X}(A,B) = e_{L^X}(C,B)$ for all $B \in L^X$, for $B = \top_X^*$, $A = C$.

(2) If $e_{L^X}(A,B) = e_{L^X}(A,C)$ for all $B \in L^X$, for $A = \top_X$, $B = C$.

(3) Since $\bigwedge_{A\in L^X}(\Phi(A) \to e_{L^X}(A,B)) = e_{L^X}(\bigvee_{A\in L^X}(\Phi(A) \odot A), B) = e_{L^X}(\sqcup\Phi, B)$, then $\sqcup\Phi = \bigvee_{A\in L^X}(\Phi(A) \odot A)$.

(4) Since $\bigwedge_{A\in L^X}(\Phi(A) \to e_{L^X}(B,A) = \bigwedge_{A\in L^X} e_{L^X}(B, (\Phi(A) \to A)) = e_{L^X}(B, \bigwedge_{A\in L^X}(\Phi(A) \to A))$, then $\sqcap\Phi = \bigwedge_{A\in L^X}(\Phi(A) \to A)$.

Theorem 2.9. [4] A structure τ is an Alexandrov topology on X iff $\tau^* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on X.

3. Various Operations and Fuzzy Preorders Induced by Alexandrov Topologies

Theorem 3.1. Let τ be an Alexandrov topology on X. Define $\mathcal{K}_\tau : L^X \to L^X$ as follows:

$$\mathcal{K}_\tau(A) = \bigvee_{B\in\tau} (e_{L^X}(B, A^*) \odot B).$$

Then the following properties hold.

(1) $e_{L^X}(A, B) \le e_{L^X}(\mathcal{K}_\tau(B), \mathcal{K}_\tau(A))$, for all $A, B \in L^X$.

(2) $\mathcal{K}_\tau(A) \le A^*$ for all $A \in L^X$.

(3) $\mathcal{K}_\tau(\mathcal{K}^*_\tau(A)) = \mathcal{K}_\tau(A)$ for all $A \in L^X$.

(4) $\mathcal{K}_\tau(\alpha \odot A) = \alpha \to \mathcal{K}_\tau(A)$ for all $\alpha \in L, A \in L^X$.

(5) $\mathcal{K}_\tau(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{K}_\tau(A_i)$ for all $A_i \in L^X$.

(6) $\mathcal{K}_\tau(\sqcup\Phi) = \sqcap\mathcal{K}^{\to}_\tau(\Phi)$ for each $\Phi : L^X \to L$ where $\mathcal{K}^{\to}_\tau : L^{L^X} \to L^{L^X}$ defined as $\mathcal{K}^{\to}_\tau(\Phi)(B) = \bigvee_{\mathcal{K}_\tau(A)=B} \Phi(A)$.

(7) $\mathcal{K}_\tau(A) = \bigvee\{B \in L^X \mid B \le A^*, B \in \tau\}$.

(8) Define $\tau_{\mathcal{K}_\tau} = \{A \mid A^* = \mathcal{K}_\tau(A)\} = \{\mathcal{K}^*_\tau(A) \mid A \in L^X\}$. Then $\tau = \tau^*_{\mathcal{K}_\tau}$.

(9) Define $e_\tau : X \times X \to L$ as

$$e_\tau(x, y) = \bigwedge_{A\in\tau} (A(x) \to A(y))$$

Then e_τ is a fuzzy preorder such that

$$e_\tau(x, y) = \bigwedge_{z\in X} (\mathcal{K}_\tau(\top_z)(x) \to \mathcal{K}_\tau(\top_z)(y).$$

(10) There exists a fuzzy preorder $e_{K_\tau} : X \times X \to L$ such that

$$\mathcal{K}_\tau(A)(y) = \bigwedge_{x\in X} (e_{K_\tau}(x, y) \to A^*(x)).$$

Moreover, $e_{K_\tau} = e^{-1}_\tau$.

Proof. (1) By Lemma 2.2 (6,8,12), we have

$$\begin{aligned}
&e_{L^X}(\mathcal{K}_\tau(B), \mathcal{K}_\tau(A)) \\
&= \bigwedge_{x\in X}(\bigvee_{C\in\tau}(e_{L^X}(C, B^*) \odot C(x)) \to \bigvee_{D\in\tau}(e_{L^X}(D, A^*) \odot D(x)) \\
&\ge \bigwedge_{x\in X} \bigwedge_{C\in\tau}((e_{L^X}(C, B^*) \odot C(x)) \to (e_{L^X}(C, A^*) \odot C(x)) \\
&\ge \bigwedge_{C\in\tau}((e_{L^X}(C, B^*) \to (e_{L^X}(C, A^*)) \\
&\ge e_{L^X}(B^*, A^*) = e_{L^X}(A, B)
\end{aligned}$$

(2) Since $e_{L^X}(C, A^*) \odot C \leq A^*$ from Lemma 2.2 (7), $\mathcal{K}_\tau(A) \leq A^*$.

(3) Since $\mathcal{K}_\tau(A) \in \tau$, then $\mathcal{K}_\tau(\mathcal{K}^*_\tau(A)) \geq e_{L^X}(\mathcal{K}_\tau(A), \mathcal{K}_\tau(A)) \odot \mathcal{K}_\tau(A) = \mathcal{K}_\tau(A)$. By (2), $\mathcal{K}_\tau(\mathcal{K}^*_\tau(A)) = \mathcal{K}_\tau(A)$.

(4) Since $\alpha \to \mathcal{K}_\tau(A) \leq \alpha \to A^* = (\alpha \odot A)^*$ and $\alpha \to \mathcal{K}_\tau(A) \in \tau$,

$$\begin{aligned} \mathcal{K}_\tau(\alpha \odot A) &\geq e_{L^X}(\alpha \to \mathcal{K}_\tau(A), (\alpha \odot A)^*) \odot (\alpha \to \mathcal{K}_\tau(A)) \\ &= \alpha \to \mathcal{K}_\tau(A), \end{aligned}$$

$$\begin{aligned} \mathcal{K}_\tau(\alpha \odot A) &= \bigvee_{B \in \tau}(e_{L^X}(B, \alpha \to A^*) \odot B) \\ &= \bigvee_{B \in \tau}((\alpha \to e_{L^X}(B, A^*)) \odot B) \\ &\leq \alpha \to \bigvee_{B \in \tau}(e_{L^X}(B, A^*)) \odot B) \text{ (by Lemma 2.2 (13))} \\ &= \alpha \to \mathcal{K}_\tau(A). \end{aligned}$$

Thus, $\mathcal{K}_\tau(\alpha \odot A) == \alpha \to \mathcal{K}_\tau(A)$.

(5) By (1), since $\mathcal{K}_\tau(B) \leq \mathcal{K}_\tau(A)$ for $A \leq B$, $\mathcal{K}_\tau(\bigvee_{i \in \Gamma} A_i) \leq \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i)$. Since $\bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i) \leq \bigwedge_{i \in \Gamma} A^*_i$ and $\bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i) \in \tau$, we have

$$\begin{aligned} \mathcal{K}_\tau(\bigvee_{i \in \Gamma} A_i) &\geq e_{L^X}(\bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i), \bigwedge_{i \in \Gamma} A^*_i) \odot \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i) \\ &= \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i). \end{aligned}$$

Hence $\mathcal{K}_\tau(\bigvee_{i \in \Gamma} A_i) == \bigwedge_{i \in \Gamma} \mathcal{K}_\tau(A_i)$.

(6) For each $\Phi : L^X \to L$, put $Q = \sqcap \mathcal{K}^{\to}_\tau(\Phi)$. Since $\mathcal{K}^{\to}_\tau(\Phi) : L^X \to L$ is a map, we have $\sqcap \mathcal{K}^{\to}_\tau(\Phi) = \bigwedge_{C \in L^X}(\mathcal{K}^{\to}_\tau(\Phi)(C) \to C)$ and $Q = \sqcap \mathcal{K}^{\to}_\tau(\Phi) = \mathcal{K}_\tau(\sqcup \Phi)$ from:

$$\begin{aligned} e_{L^X}(B, Q) &= \bigwedge_{C \in L^X}(\mathcal{K}^{\to}_\tau(\Phi)(C) \to e_{L^X}(B, C)) \\ &= e_{L^X}(B, \bigwedge_{C \in L^X}(\mathcal{K}^{\to}_\tau(\Phi)(C) \to C)) \\ &= e_{L^X}(B, \bigwedge_{C \in L^X}(\bigvee_{\mathcal{K}_\tau(A) = C} \Phi(A) \to C)) \\ &= e_{L^X}(B, \bigwedge_{A \in L^X}(\Phi(A) \to \mathcal{K}_\tau(A))) \\ &= e_{L^X}(B, \mathcal{K}_\tau(\bigvee_{A \in L^X}(\Phi(A) \odot A))) \text{ (by (4) and (5))} \\ &= e_{L^X}(B, \mathcal{K}_\tau(\sqcup \Phi)). \end{aligned}$$

(7) Put $K(A) = \bigvee\{B \in L^X \mid B \leq A^*, B \in \tau\}$. Since $K(A) \leq A^*$ and $K(A) \in \tau$, we have

$$\mathcal{K}_\tau(A) = \bigvee_{B \in \tau}(e_{L^X}(B, A^*) \odot B) \geq e_{L^X}(K(A), A^*) \odot K(A) = K(A).$$

Since $\mathcal{K}_\tau(A) \leq A$ and $\mathcal{K}_\tau(A) \in \tau$, we have $K(A) \geq \mathcal{K}_\tau(A)$. Hence $\mathcal{K}_\tau = K$.

(8) Put $\tau_1 = \{\mathcal{K}^*_\tau(A) \mid A \in L^X\}$. Let $A \in \tau_{\mathcal{K}_\tau}$. Then $A = \mathcal{K}^*_\tau(A) \in \tau_1$. Let $\mathcal{K}^*_\tau(A) \in \tau_1$. By (3), $\mathcal{K}_\tau(\mathcal{K}^*_\tau(A)) = \mathcal{K}_\tau(A)$. So, $\mathcal{K}^*_\tau(A) \in \tau_{\mathcal{K}_\tau}$. Thus $\tau_{\mathcal{K}_\tau} = \tau_1$.

Moreover, $\tau = \tau^*_{\mathcal{K}_\tau}$ from:

$$A \in \tau \text{ iff } \mathcal{K}_\tau(A^*) = A \text{ iff } A \in \tau^*_{\mathcal{K}_\tau}.$$

(9) Since $A = \bigvee_{x\in X}(A(x)\odot \top_x)$, by (4) and (5), $\mathcal{K}_\tau(A)(y) = \bigwedge_{x\in X}(A(x) \to \mathcal{K}_\tau(\top_x)(y))$. Since $\tau = \{\mathcal{K}_\tau(A) \mid A \in L^X\}$, we have

$$\begin{aligned} e_\tau(x,y) &= \textstyle\bigwedge_{A\in\tau}(A(x) \to A(y)) = \bigwedge_{A\in L^X}(\mathcal{K}_\tau(A)(x) \to \mathcal{K}_\tau(A)(y)) \\ &\leq \textstyle\bigwedge_{x\in X}(\mathcal{K}_\tau(\top_z)(x) \to \mathcal{K}_\tau(\top_z)(y)). \end{aligned}$$

$$\begin{aligned} e_\tau(x,y) &= \textstyle\bigwedge_{A\in\tau}(A(x) \to A(y)) = \bigwedge_{A\in L^X}(\mathcal{K}_\tau(A)(x) \to \mathcal{K}_\tau(A)(y)) \\ &= \textstyle\bigwedge_{A\in L^X}(\bigwedge_{z\in X}(A(z) \to \mathcal{K}_\tau(\top_z)(x) \to \bigwedge_{w\in X}(A(w) \to \mathcal{K}_\tau(\top_w)(y)) \\ &= \textstyle\bigwedge_{A\in L^X}\bigwedge_{z\in X}((A(z) \to \mathcal{K}_\tau(\top_z)(x)) \to (A(z) \to \mathcal{K}_\tau(\top_z)(y)) \\ &\geq \textstyle\bigwedge_{x\in X}(\mathcal{K}_\tau(\top_z)(x) \to \mathcal{K}_\tau(\top_z)(y)). \end{aligned}$$

(10) Since $A = \bigvee_{x\in X}(A(x)\odot \top_x)$, by (4) and (5), $\mathcal{K}_\tau(A)(y) = \bigwedge_{x\in X}(A(x) \to \mathcal{K}_\tau(\top_x)(y)) = \bigwedge_{x\in X}(\mathcal{K}^*_\tau(\top_x)(y) \to A^*(x))$. Put $e_{K_\tau}(x,y) = \mathcal{K}^*_\tau(\top_x)(y)$. Then

$$\mathcal{K}_\tau(A)(y) = \bigwedge_{x\in X}(e_{K_\tau}(x,y) \to A^*(x)).$$

$$e_{K_\tau}(x,x) = \mathcal{K}^*_\tau(\top_x)(x) \geq \top_x(x) = \top$$

$$\begin{aligned} &\textstyle\bigvee_{y\in X}(e_{K_\tau}(x,y) \odot e_{K_\tau}(y,z)) \leq e_{K_\tau}(x,z) \\ &\text{iff } \textstyle\bigvee_{y\in X}(\mathcal{K}^*_\tau(\top_x)(y) \odot \mathcal{K}^*_\tau(\top_y)(z)) \leq \mathcal{K}^*_\tau(\top_x)(z) \\ &\text{iff } \textstyle\bigwedge_{y\in X}(\mathcal{K}^*_\tau(\top_x)(y) \to \mathcal{K}_\tau(\top_y)(z)) \geq \mathcal{K}_\tau(\top_x)(z) \\ &\text{iff } \mathcal{K}_\tau(\textstyle\bigvee_{y\in X}(\mathcal{K}^*_\tau(\top_x)(y) \odot \top_y))(z) \geq \mathcal{K}_\tau(\top_x)(z) \\ &\text{iff } \mathcal{K}_\tau(\mathcal{K}^*_\tau(\top_x))(z) \geq \mathcal{K}_\tau(\top_x)(z). \end{aligned}$$

Hence e_{K_τ} is a fuzzy preorder.

By (9),

$$\begin{aligned} e_\tau(x,y) &= \textstyle\bigwedge_{z\in X}(\mathcal{K}_\tau(\top_z)(x) \to \mathcal{K}_\tau(\top_z)(y) \\ &= \textstyle\bigwedge_{z\in X}(\mathcal{K}^*_\tau(\top_z)(y) \to \mathcal{K}^*_\tau(\top_z)(x) \\ &= \textstyle\bigwedge_{z\in X}(e_{K_\tau}(z,y) \to e_{K_\tau}(z,x)) \\ &\leq e_{K_\tau}(y,y) \to e_{K_\tau}(y,x) = e_{K_\tau}(y,x). \end{aligned}$$

Since e_{K_τ} is a fuzzy preorder, $e_{K_\tau}(z,y) \odot e_{K_\tau}(y,x) \leq e_{K_\tau}(z,x)$. Thus

$$e_{K_\tau}(y,x) \leq \bigwedge_{z\in X}(e_{K_\tau}(z,y) \to e_{K_\tau}(z,x)) = e_\tau(x,y).$$

Hence $e_{K_\tau}(y,x) = e_\tau(y,x)$.

Theorem 3.2. Let τ be an Alexandrov topology on X. Define $\mathcal{M}_\tau : L^X \to L^X$ as follows:

$$\mathcal{M}_\tau(A) = \bigwedge_{B\in\tau} (e_{L^X}(A^*, B) \to B).$$

Then the following properties hold.

(1) $e_{L^X}(A, B) \le e_{L^X}(\mathcal{M}_\tau(B), \mathcal{M}_\tau(A))$, for all $A, B \in L^X$.

(2) $A^* \le \mathcal{M}_\tau(A)$ for all $A \in L^X$.

(3) $\mathcal{M}_\tau(\mathcal{M}^*_\tau(A)) = \mathcal{M}_\tau(A)$ for all $A \in L^X$.

(4) $\mathcal{M}_\tau(\alpha \to A) = \alpha \odot \mathcal{M}_\tau(A)$ for all $\alpha \in L, A \in L^X$.

(5) $\mathcal{M}_\tau(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i)$ for all $A_i \in L^X$.

(6) $\mathcal{M}_\tau(\sqcap\Phi) = \sqcup\mathcal{M}^{\to}_\tau(\Phi)$ for each $\Phi : L^X \to L$ where $\mathcal{M}^{\to}_\tau : L^{L^X} \to L^{L^X}$ defined as $\mathcal{M}^{\to}_\tau(\Phi)(B) = \bigvee_{\mathcal{M}_\tau(A)=B}(\Phi(A))$.

(7) $\mathcal{M}_\tau(A) = \bigwedge\{B \in L^X \mid A^* \le B, B \in \tau\}$.

(8) Define $\tau_{\mathcal{M}_\tau} = \{A \mid A = \mathcal{M}_\tau(A^*)\}$. Then $\tau = \tau_{\mathcal{M}_\tau} = \{\mathcal{M}_\tau(A^*) \mid A \in L^X\}$.

(9) Define $e_\tau : X \times X \to L$ as follows

$$e_\tau(x, y) = \bigwedge_{A\in\tau} (A(x) \to A(y)).$$

Then e_τ is a fuzzy preorder such that

$$e_\tau(x, y) = \bigwedge_{A\in\tau} (\mathcal{M}_\tau(\top^*_z)(x) \to \mathcal{M}_\tau(\top^*_z)(y)).$$

Moreover, $e_{\tau^*}(x, y) = e_\tau(y, x) = e^{-1}_\tau(x, y)$.

(10) $(\mathcal{M}_\tau(A^*))^* = \mathcal{K}_{\tau^*}(A)$ for all $A \in L^X$.

(11) There exists a fuzzy preorder $e_{M_\tau} : X \times X \to L$ such that

$$\mathcal{M}_\tau(A)(y) = \bigvee_{x\in X} (e_{M_\tau}(x, y) \odot A^*(x)),$$

$$\mathcal{K}_{\tau^*}(A)(y) = \bigwedge_{x\in X} (e_{M_\tau}(x, y) \to A^*(x)).$$

Moreover, $e_{M_\tau}(x, y) = e_{K_\tau}(y, x) = e_\tau(x, y) = e_{M_{\tau^*}}(y, x) = e_{K^*_\tau}(x, y) = e_{\tau^*}(y, x)$ for all $x, y \in X$.

Proof. (1) By Lemma 2.2 (6,8), we have

$$\begin{aligned}
&e_{L^X}(\mathcal{M}_\tau(B), \mathcal{M}_\tau(A))\\
&= \textstyle\bigwedge_{x\in X}(\bigwedge_{C\in\tau}(e_{L^X}(B^*, C) \to C(x)) \to \bigwedge_{D\in\tau}(e_{L^X}(A^*, D) \to D(x))\\
&\ge \textstyle\bigwedge_{x\in X} \bigwedge_{C\in\tau}((e_{L^X}(B^*, C) \to C(x)) \to (e_{L^X}(A^*, C) \to C(x))\\
&\ge \textstyle\bigwedge_{C\in\tau}((e_{L^X}(A^*, C) \to (e_{L^X}(B^*, C))\\
&\ge e_{L^X}(B^*, A^*) = e_{L^X}(A, B).
\end{aligned}$$

(2) Since $e_{L^X}(A^*, B) \odot A^* \le B$ iff $A^* \le e_{L^X}(A^*, B) \to B$, then $A^* \le \mathcal{M}_\tau(A)$.

(3) Since $\mathcal{M}_\tau(A) \in \tau$, then $\mathcal{M}_\tau(\mathcal{M}^*_\tau(A)) \le e_{L^X}(\mathcal{M}_\tau(A), \mathcal{M}_\tau(A)) \to \mathcal{M}_\tau(A) = \mathcal{M}_\tau(A)$. By (2), $\mathcal{M}_\tau(\mathcal{M}^*_\tau(A)) = \mathcal{M}_\tau(A)$.

(4) Since $\alpha \odot A^* \le \alpha \odot \mathcal{M}_\tau(A)$ and $\alpha \odot \mathcal{M}_\tau(A) \in \tau$,

$$\begin{aligned} \mathcal{M}_\tau(\alpha \to A) &\le e_{L^X}(\alpha \odot A^*, \alpha \odot \mathcal{M}_\tau(A)) \to \alpha \odot \mathcal{M}_\tau(A) \\ &= \alpha \odot \mathcal{M}_\tau(A). \end{aligned}$$

$$\begin{aligned} &\mathcal{M}_\tau(\alpha \to A) = \textstyle\bigwedge_{B\in\tau}(e_{L^X}(\alpha \odot A^*, B) \to B) \\ &= \textstyle\bigwedge_{B\in\tau}((\alpha \to e_{L^X}(A^*, B)) \to B) \\ &\ge \textstyle\bigwedge_{B\in\tau}(\alpha \odot (e_{L^X}(A^*, B) \to B)) \text{ (by Lemmma 2.2(13))} \\ &\ge \alpha \odot \textstyle\bigwedge_{B\in\tau}(e_{L^X}(A^*, B) \to B) \\ &= \alpha \odot \mathcal{M}_\tau(A). \end{aligned}$$

Thus, $\mathcal{M}_\tau(\alpha \to A) = \alpha \odot \mathcal{M}_\tau(A)$.

(5) By (1), since $\mathcal{M}_\tau(B) \le \mathcal{M}_\tau(A)$ for $A \le B$, $\bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i) \le \mathcal{M}_\tau(\bigwedge_{i\in\Gamma} A_i)$. Since $\bigvee_{i\in\Gamma} A^*_i \le \bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i)$ and $\bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i) \in \tau$, we have

$$\begin{aligned} \mathcal{M}_\tau(\textstyle\bigwedge_{i\in\Gamma} A_i) &\le e_{L^X}(\textstyle\bigvee_{i\in\Gamma} A^*_i, \bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i)) \to \bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i) \\ &= \textstyle\bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i). \end{aligned}$$

Hence $\mathcal{M}_\tau(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathcal{M}_\tau(A_i)$.

(6) For each $\Phi : L^X \to L$, put $P = \sqcap \mathcal{M}^{\to}_\tau(\Phi)$. Since $\mathcal{M}^{\to}_\tau(\Phi) : L^X \to L$ is a map, we have $\sqcup \mathcal{M}^{\to}_\tau(\Phi) = \bigvee_{C\in\tau}(\mathcal{M}^{\to}_\tau(\Phi)(C) \odot C)$ and $P = \sqcup \mathcal{M}^{\to}_\tau(\Phi) = \mathcal{M}_\tau(\sqcup\Phi)$ from:

$$\begin{aligned} e_{L^X}(P, B) &= \textstyle\bigwedge_{C\in L^X}(\mathcal{M}^{\to}_\tau(\Phi)(C) \to e_{L^X}(C, B)) \\ &= \textstyle\bigwedge_{C\in L^X} e_{L^X}(\mathcal{M}^{\to}_\tau(\Phi)(C) \odot C, B) \\ &= e_{L^X}(\textstyle\bigvee_{C\in L^X}(\mathcal{M}^{\to}_\tau(\Phi)(C) \odot C), B) \\ &= e_{L^X}(\textstyle\bigvee_{A\in L^X}(\Phi(A) \odot \mathcal{M}_\tau(A)), B) \\ &= e_{L^X}(\mathcal{M}_\tau(\textstyle\bigwedge_{A\in L^X}(\Phi(A) \to A)), B) \text{ (by (4) and (5))} \\ &= e_{L^X}(\mathcal{M}_\tau(\sqcap\Phi), B) \end{aligned}$$

(7) Put $M(A) = \bigwedge\{B \in L^X \mid A^* \le B, B \in \tau\}$. Since $A^* \le M(A)$ and $M(A) \in \tau$, we have

$\mathcal{M}_\tau(A) = \bigwedge_{B\in\tau}(e_{L^X}(A^*, B) \to B) \le e_{L^X}(A^*, M(A)) \to M(A) = M(A)$.

Since $A^* \le \mathcal{M}_\tau(A)$ and $\mathcal{M}_\tau(A) \in \tau$, we have $M(A) \le \mathcal{M}_\tau(A)$. Hence $\mathcal{M}_\tau = M$.

(8) We have $\tau = \tau_{\mathcal{M}_\tau}$ from $A \in \tau$ iff $\mathcal{M}_\tau(A^*) = A$ iff $A \in \tau_{\mathcal{M}_\tau}$. Put $\tau = \{\mathcal{M}_\tau(A^*) \mid A \in L^X\}$. Let $A \in \tau$. Then $A = \mathcal{M}_\tau(A^*)$. Let $\mathcal{M}_\tau(A^*) \in \tau_1$. Then $\mathcal{M}_\tau(\mathcal{M}^*_\tau(A^*)) = \mathcal{M}_\tau(A^*) \in \tau_{\mathcal{M}_\tau} = \tau$. Hence $\tau = \tau_1$.

(9) By (8), since $\mathcal{M}_\tau(\top_z^*) \in \tau$,

$$e_\tau(x,y) = \bigwedge_{A\in\tau}(A(x) \to A(y)) \leq \bigwedge_{x\in X}(\mathcal{M}_\tau(\top_z^*)(x) \to \mathcal{M}_\tau(\top_z^*)(y)).$$

Since $A^* = \bigwedge_{z\in X}(A(z) \to \top_z^*)$, by (4) and (5), $\mathcal{M}_\tau(A^*)(x) = \bigvee_{z\in X}(A(z) \odot \mathcal{M}_\tau(\top_z^*)(x))$.

$$\begin{aligned} e_\tau(x,y) &= \bigwedge_{A\in\tau}(A(x) \to A(y)) = \bigwedge_{A\in L^X}(\mathcal{M}_\tau(A^*)(x) \to \mathcal{M}_\tau(A^*)(y)) \\ &= \bigwedge_{A\in L^X}(\bigwedge_{z\in X}(A(z) \odot \mathcal{M}_\tau(\top_z^*)(x) \to \bigwedge_{w\in X}(A(w) \odot \mathcal{M}_\tau(\top_w^*)(y)) \\ &= \bigwedge_{A\in L^X}\bigwedge_{z\in X}((A(z) \odot \mathcal{M}_\tau(\top_z^*)(x)) \to (A(z) \odot \mathcal{M}_\tau(\top_z^*)(y)) \\ &\geq \bigwedge_{x\in X}(\mathcal{M}_\tau(\top_z^*)(x) \to \mathcal{M}_\tau(\top_z^*)(y)). \end{aligned}$$

(10)

$$\begin{aligned} (\mathcal{M}_\tau(A^*))^* &= (\bigwedge_{B\in\tau}(e_{L^X}(A,B) \to B))^* \\ &= \bigvee_{B\in\tau}(e_{L^X}(B^*,A^*) \odot B^*) \\ &= \bigvee_{B^*\in\tau^*}(e_{L^X}(B^*,A^*) \odot B^*) \\ &= \mathcal{K}_{\tau^*}(A). \end{aligned}$$

(11) Since $A = \bigwedge_{z\in X}(A^*(z) \to \top_z^*)$, by (4) and (5),

$$\mathcal{M}_\tau(A)(x) = \mathcal{M}_\tau(\bigwedge_{z\in X}(A^*(z) \to \top_z^*))(x) = \bigvee_{z\in X}(A^*(z) \odot \mathcal{M}_\tau(\top_z^*)(x)).$$

Put $e_{M_\tau}(x,y) = \mathcal{M}_\tau(\top_x^*)(y)$. Then

$$\mathcal{M}_\tau(A)(y) = \bigvee_{x\in X}(e_{M_\tau}(x,y) \odot A^*(x)).$$

$$e_{M_\tau}(x,x) = \mathcal{M}_\tau(\top_x^*)(x) \geq \top_x(x) = \top$$

$$\begin{aligned} &\bigvee_{y\in X}(e_{M_\tau}(x,y) \odot e_{M_\tau}(y,z)) \leq e_{M_\tau}(x,z) \\ \text{iff } &\bigvee_{y\in X}(\mathcal{M}_\tau(\top_x^*)(y) \odot \mathcal{M}_\tau(\top_y^*)(z)) \leq \mathcal{M}_\tau(\top_x^*)(z) \\ \text{iff } &\mathcal{M}_\tau(\bigwedge_{y\in X}(\mathcal{M}_\tau(\top_x^*)(y) \to \top_y^*))(z) \leq \mathcal{M}_\tau(\top_x^*)(z) \\ \text{iff } &\mathcal{M}_\tau(\mathcal{M}_\tau^*(\top_x^*))(z) \leq \mathcal{M}_\tau(\top_x^*)(z) \end{aligned}$$

Hence e_{M_τ} is a fuzzy preorder. Since $e_{M_\tau}(x,y) = \mathcal{M}_\tau(\top_x^*)(y) = \mathcal{K}_{\tau^*}^*(\top_x)(y)$, by Theorem 3.1(10),

$$\mathcal{K}_{\tau^*}(A)(y) = \bigwedge_{x\in X}(e_{M_\tau}(x,y) \to A^*(x)).$$

Moreover, by Theorem 3.1(10), $e_{M_\tau}(x,y) = e_{K_\tau}(y,x) = e_\tau(x,y) = e_{M_{\tau^*}}(y,x) = e_{K_\tau^*}(x,y) = e_{\tau^*}(y,x)$ for all $x,y \in X$.

Example 3.3. Let $(L = [0,1], \odot, \rightarrow, ^*)$ be a complete residuated lattice with a strong negation defined by

$$x \odot y = (x + y - 1) \vee 0, \quad x \rightarrow y = (1 - x + y) \wedge 1, \; x^* = 1 - x.$$

Let $X = \{x, y, z\}$ be a set and $B \in L^X$ as follows:

$$B(x) = 0.5, B(y) = 0.2, B(z) = 0.9.$$

Define $e_B(x, y) = B(x) \rightarrow B(y)$ such that

$$e_B = \begin{pmatrix} 1 & 0.7 & 1 \\ 1 & 1 & 1 \\ 0.6 & 0.3 & 1 \end{pmatrix}.$$

(1) We define

$$\tau = \{e_B(A)(y) = \bigvee_{y \in X} (e_B(x, y) \odot A(x)) \mid A \in L^X\}.$$

(T1) For $\perp_X \in L^X$, $e_B(\perp_X) = \perp_X \in \tau$. For $\top_X \in L^X$, $e_B(\top_X) = \top_X \in \tau$.

(T2) For $e_B(A_i) \in \tau$ for each $i \in \Gamma$, $\bigvee_{i \in \Gamma} e_B(A_i) = e_B(\bigvee_{i \in \Gamma} A_i) \in \tau$. Moreover, since $e_B(A)(x) \geq e_B(x, x) \odot A(x) = A(x)$ and $e_B(e_X(A)) = e_X(A)$,

$$\bigwedge_{i \in \Gamma} e_B(A_i) \leq e_B(\bigwedge_{i \in \Gamma} e_B(A_i)) \leq \bigwedge_{i \in \Gamma} e_B(e_B(A_i)) = \bigwedge_{i \in \Gamma} e_B(A_i).$$

Hence $\bigwedge_{i \in \Gamma} e_B(A_i) = e_B(\bigwedge_{i \in \Gamma} A_i) \in \tau$.

(T3) For $e_B(A) \in \tau$, $\alpha \odot e_B(A) = e_B(\alpha \odot A) \in \tau$.

(T4) Since $\alpha \odot e_B(\alpha \rightarrow e_B(A)) \leq e_B(e_B(A)) = e_B(A)$, we have

$$\alpha \rightarrow e_B(A) \leq e_B(\alpha \rightarrow e_B(A)) \leq \alpha \rightarrow e_B(A)$$

Hence, for $e_B(A) \in \tau$, $\alpha \rightarrow e_B(A) = e_B(\alpha \rightarrow e_B(A)) \in \tau$. Hence τ is an Alexandrov topology on X.

(2) Since $B(x) \rightarrow B(y) \leq (\alpha \odot B)(x) \rightarrow (\alpha \odot B)(y)$, $B(x) \rightarrow B(y) \leq (\alpha \rightarrow B)(x) \rightarrow (\alpha \rightarrow B)(y)$, we have $e_\tau(x, y) = e_B(x, y)$. Moreover, $e_{\tau^*}(x, y) = e_{B^*}(x, y) = e_B(y, x)$.

Since

$$\tau = \{(x \odot (1, 0.7, 1)) \vee (y \odot (1, 1, 1)) \vee (z \odot (0.6, 0.3, 1)) \mid (x, y, z) \in L^X\},$$

by Theorem 3.1 (7), we obtain:

$$\begin{aligned}\mathcal{K}_\tau(1_x) &= \bigvee\{A\in\tau \mid A=(x\odot(1,0.7,1))\vee(y\odot(1,1,1))\\ &\quad \vee(z\odot(0.6,0.3,1))\le(0,1,1)\}=(0,0,0.4)\\ \mathcal{K}_\tau(1_y) &= \bigvee\{A\in\tau \mid A=(x\odot(1,0.7,1))\vee(y\odot(1,1,1))\\ &\quad \vee(z\odot(0.6,0.3,1))\le(1,0,1)\}=(0.3,0,0.7)\\ \mathcal{K}_\tau(1_z) &= \bigvee\{A\in\tau \mid A=(x\odot(1,0.7,1))\vee(y\odot(1,1,1))\\ &\quad \vee(z\odot(0.6,0.3,1))\le(1,1,0)\}=(0,0,0)\end{aligned}$$

Put $e_{K_\tau}(x,y)=\mathcal{K}^*_\tau(1_x)(y)$, then

$$e_{K_\tau}(x,y)=\mathcal{K}^*_\tau(1_x)(y)=e_\tau(y,x)=e_B(y,x)=e_{\tau^*}(x,y).$$

(3) By Theorem 3.2 (7), we obtain:

$$\begin{aligned}\mathcal{M}_\tau(1^*_x) &= \bigwedge\{A\in\tau \mid (1,0,0)\le A=(x\odot(1,0.7,1))\vee(y\odot(1,1,1))\\ &\quad \vee(z\odot(0.6,0.3,1))\}=(1,0.7,1)\\ \mathcal{M}_\tau(1^*_y) &= \bigwedge\{A\in\tau \mid (0,1,0)\le A=(x\odot(1,0.7,1))\vee(y\odot(1,1,1))\\ &\quad \vee(z\odot(0.6,0.3,1))\}=(1,1,1)\\ \mathcal{M}_\tau(1^*_z) &= \bigwedge\{A\in\tau \mid (0,0,1)\le A=(x\odot(1,0.7,1))\vee(y\odot(1,1,1))\\ &\quad \vee(z\odot(0.6,0.3,1))\}=(0.6,0.3,1)\end{aligned}$$

Moreover, we have $e_{M_\tau}(y,x)=\mathcal{M}_\tau(1^*_y)(x)=e_{K_\tau}(x,y)=\mathcal{K}^*_\tau(1_x)(y)=e_\tau(y,x)=e_B(y,x)=e_{\tau^*}(x,y)$.

(4) For $D_1=(0.5,0.3,0.7), D_2=(0.4,0.8,0.5),$ we obtain

$$\mathcal{K}_\tau(D_i)(y)=\bigwedge_{x\in X}(e_{K_\tau}(x,y)\to D^*_i(x))=\bigwedge_{x\in X}(e_B(y,x)\to D^*_i(x)),$$

$$\mathcal{M}_\tau(D_i)(y)=\bigvee_{x\in X}(e_{M_\tau}(x,y)\odot D^*_i(x))=\bigvee_{x\in X}(e_B(x,y)\odot D^*_i(x)).$$

$$\mathcal{K}_\tau(D_1)=(0.3,0.3,0.3),\mathcal{K}_\tau(D_2)=(0.5,0.2,0.5),$$

$$\mathcal{M}_\tau(D_1)=(0.7,0.7,0.7),\mathcal{M}_\tau(D_2)=(0.6,0.3,0.6).$$

Let $\Phi: L^X\to L$ as follows

$$\Phi(B)=\begin{cases}0.7, & \text{if } B=B_1,\\ 0.8, & \text{if } B=B_2,\\ 0, & \text{otherwise.}\end{cases}$$

$$\begin{aligned}&\sqcap\Phi=\textstyle\bigwedge_{A\in L^X}(\Phi(A)\to A)=(\Phi(D_1)\to D_1)\wedge(\Phi(D_2)\to D_2)\\ &=(0.7\to(0.5,0.3,0.7))\wedge(0.8\to(0.4,0.8,0.5))=(0.6,0.6,0.7)\end{aligned}$$

$$\sqcup\Phi = \bigvee_{A\in L^X}(\Phi(A)\odot A) = (\Phi(D_1)\odot D_1)\vee(\Phi(D_2)\odot D_2)$$
$$= (0.7\odot(0.5,0.3,0.7))\vee(0.8\odot(0.4,0.8,0.5)) = (0.2,0.6,0.4)$$

$$\mathcal{K}_\tau(\sqcup\Phi) = (0.6,0.4,0.6).$$
$$\sqcap\mathcal{K}_\tau^{\rightarrow}(\Phi) = \bigwedge_{A\in L^X}(\Phi(A)\rightarrow\mathcal{K}_\tau(A))$$
$$= (\Phi(D_1)\rightarrow\mathcal{K}_\tau(D_1))\wedge(\Phi(D_2)\rightarrow\mathcal{K}_\tau(D_2))$$
$$= (0.7\rightarrow(0.3,0.3,0.3))\wedge(0.8\rightarrow(0.5,0.2,0.5))$$
$$= (0.6,0.4,0.6)$$

we have $\mathcal{K}_\tau(\sqcup\Phi) = \sqcap\mathcal{K}_\tau^{\rightarrow}(\Phi)$.

$$\mathcal{M}_\tau(\sqcap\Phi) = (0.4,0.4,0.4).$$
$$\sqcup\mathcal{M}_\tau^{\rightarrow}(\Phi) = \bigvee_{A\in L^X}(\Phi(A)\odot\mathcal{M}_\tau(A))$$
$$= (\Phi(D_1)\odot\mathcal{M}_\tau(D_1))\vee(\Phi(D_2)\odot\mathcal{M}_\tau(D_2))$$
$$= (0.7\odot(0.7,0.7,0.7))\vee(0.8\odot(0.6,0.3,0.6))$$
$$= (0.4,0.4,0.4)$$

Thus, $\mathcal{M}_\tau(\sqcap\Phi) = \sqcup\mathcal{M}_\tau^{\rightarrow}(\Phi)$.

(5) Since $\tau^* = \{A^* \mid A\in\tau\}$, we have

$$\tau^* = \{\bigwedge_{x\in X}(e_X(x,-)\rightarrow B(x)) \mid B\in L^X\}.$$

$$\begin{aligned}
\mathcal{K}_{\tau^*}(1_x) &= \bigwedge\{A\in\tau^* \mid A = ((1,0.7,1)\rightarrow x)\wedge((1,1,1)\rightarrow y)\\
&\quad \wedge((0.6,0.3,1)\rightarrow z)\le(0,1,1)\} = (0,0.3,0)\\
\mathcal{K}_{\tau^*}(1_y) &= \bigwedge\{A\in\tau^* \mid A = ((1,0.7,1)\rightarrow x)\wedge((1,1,1)\rightarrow y)\\
&\quad \wedge((0.6,0.3,1)\rightarrow z)\le(1,0,1)\} = (0,0,0)\\
\mathcal{K}_{\tau^*}(1_z) &= \bigwedge\{A\in\tau^* \mid A = ((1,0.7,1)\rightarrow x)\wedge((1,1,1)\rightarrow y)\\
&\quad \wedge((0.6,0.3,1)\rightarrow z)\le(1,1,0)\} = (0.4,0.7,0)
\end{aligned}$$

$$\tau_1 = \{e_{B^*}(A)(y) = \bigvee_{y\in X}(e_{B^*}(x,y)\odot A(x)) \mid A\in L^X\}.$$

$$\begin{aligned}
\mathcal{K}_{\tau_1}(1_x) &= \bigvee\{A\in\tau_1 \mid A = (x\odot(1,1,0.6))\vee(y\odot(0.7,1,0.3))\\
&\quad \vee(z\odot(1,1,1))\le(0,1,1)\} = (0,0.3,0)\\
\mathcal{K}_{\tau_1}(1_y) &= \bigvee\{A\in\tau_1 \mid A = (x\odot(1,1,0.6))\vee(y\odot(0.7,1,0.3))\\
&\quad \vee(z\odot(1,1,1))\le(1,0,1)\} = (0,0,0)\\
\mathcal{K}_{\tau_1}(1_z) &= \bigvee\{A\in\tau_1 \mid A = (x\odot(1,1,0.6))\vee(y\odot(0.7,1,0.3))\\
&\quad \vee(z\odot(1,1,1))\le(1,1,0)\} = (0.4,0.7,0)
\end{aligned}$$

Hence $\mathcal{K}_{\tau_1}(1_x)(y) = \mathcal{K}_{\tau^*}(1_x)(y)$ for all $x,y\in X$, then $\mathcal{K}_{\tau_1} = \mathcal{K}_{\tau^*}$. Thus,

$$\tau^* = \tau^*_{\mathcal{K}_{\tau^*}} = \tau^*_{\mathcal{K}_{\tau_1}} = \tau_1.$$

For D_1, D_2 and Φ in (3), we obtain

$$\mathcal{K}_{\tau^*}(D_i)(y) = \bigwedge_{x\in X} (e_{K_{\tau^*}}(x,y) \to D_i^*(x))$$

$$\mathcal{M}_{\tau^*}(D_i)(y) = \bigvee_{x\in X} (e_{M_{\tau^*}}(x,y) \odot D_i^*(x)) = \bigvee_{x\in X} (e_{K_{\tau^*}}(y,x) \odot D_i^*(x)).$$

$$\mathcal{K}_{\tau^*}(D_1) = (0.5, 0.7, 0.3), \mathcal{K}_{\tau^*}(D_2) = (0.2, 0.2, 0.2),$$

$$\mathcal{M}_{\tau}(D_1) = (0.5, 0.7, 0.3), \mathcal{M}_{\tau}(D_2) = (0.6, 0.6, 0.5).$$

Since $\sqcup\Phi = (0.2, 0.6, 0.4)$ and

$$\begin{aligned}
&\mathcal{K}_{\tau^*}(\sqcup\Phi) = (0.4, 0.4, 0.4).\\
&\sqcap\mathcal{K}_{\tau^*}^{\to}(\Phi) = \textstyle\bigwedge_{A\in L^X}(\Phi(A) \to \mathcal{K}_{\tau^*}(A))\\
&= (\Phi(D_1) \to \mathcal{K}_{\tau^*}(D_1)) \wedge (\Phi(D_2) \to \mathcal{K}_{\tau^*}(D_2))\\
&= (0.7 \to (0.5, 0.7, 0.3)) \wedge (0.8 \to (0.2, 0.2, 0.2))\\
&= (0.4, 0.4, 0.4)
\end{aligned}$$

we have $\mathcal{K}_{\tau^*}(\sqcup\Phi) = \sqcap\mathcal{K}_{\tau^*}^{\to}(\Phi)$.

Since $\sqcap\Phi = (0.6, 0.6, 0.7)$ and

$$\begin{aligned}
&\mathcal{M}_{\tau^*}(\sqcap\Phi) = (0.4, 0.4, 0.3).\\
&\sqcup\mathcal{M}_{\tau^*}^{\to}(\Phi) = \textstyle\bigvee_{A\in L^X}(\Phi(A) \odot \mathcal{M}_{\tau^*}(A))\\
&= (\Phi(D_1) \odot \mathcal{M}_{\tau^*}(D_1)) \vee (\Phi(D_2) \odot \mathcal{M}_{\tau^*}(D_2))\\
&= (0.7 \odot (0.5, 0.7, 0.3)) \vee (0.8 \odot (0.6, 0.6, 0.5))\\
&= (0.4, 0.4, 0.3),
\end{aligned}$$

then $\mathcal{M}_{\tau^*}(\sqcap\Phi) = \sqcup\mathcal{M}_{\tau^*}^{\to}(\Phi)$.

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