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BASIC PROPERTIES OF LINEAR RELATIVE n-WIDTHS

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Abstract: The paper examines the basic properties of linear relative widths as well as the different ways of finding their estimates.

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1. Introduction

The theory of shape-preserving approximation is one of the rapidly developing areas of the approximation theory. Results of this theory are actively used in computational mathematics, the theory of functions, computer-aided geometric design. This theory studies approximation properties of the different methods for approximation of functions preserving its shape properties (monotonicity, convexity). The most significant results were gathered in [1], [2]. Note that if a function f has some shape properties, it usually means that element f belongs to a cone (a convex set, closed under nonnegative scalar multiplication).

Definitions of relative linear widths were introduced in papers [3], [4], [5]. In this paper we are going to examine basic properties of this quantities. The paper organized into five sections. In section 2 one can find two different definitions of linear relative width, in the senses of Konovalov and Korovkin. In Section 3 one can find the list of basic properties of Konovalov linear relative widths. The

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main question of this section how different the properties of Konovalov linear relative widths from linear width are. Section 4 describes basic properties of Korovkin linear relative width. Both in Section 3 and Section 4 we investigate different ways of finding the lower estimations of linear relative widths.

2. Definitions of Linear Relative Width

A nonempty subset V of a linear space is said to be a cone if it satisfies the following properties:

1. V t is closed with respect to taking sums of its elements: if $f, g \in V$, then $f + g \in V$;
2. V is a conic set: if $f \in V$, $\lambda \geqslant 0$, then $\lambda f \in V$.

First we remind the definition of relative width given by Konovalov in [6].

Definition 1. Let X be a normed linear space, V be a cone of X, $A \cap V$ be a nonempty subset of X. Relative n-width of $A \cap V$ in X with the constraint V is defined by

$$d_n(A \cap V, V)_X = \inf_{X_n} \sup_{f \in A \cap V} \inf_{g \in X_n \cap V} \|f - g\|_X,$$

the left-most infimum is taken over all affine subsets X_n of dimension $\leq n$, such that $X_n \cap V \neq \varnothing$.

The definition of relative n-width was first introduced in 1984 by Konovalov [6]. Though he considered a problem not connected with preserving shapes, the concept of relative n-width arises in the theory of shape-preserving approximation naturally. Of course, it is impossible to obtain $d_n(A \cap V, V)_X$ and determine optimal subspaces X_n (if they exist) for all A, V, X. Nevertheless, some estimations of relative shape-preserving n-widths have been obtained in papers [7], [8], [9].

Let $L : X \to X$ be a linear operator and V be a cone in X, $V \neq \varnothing$. We will say that the operator L is shape-preserving operator relative to the cone V, if $L(V) \subset V$.

Let A be a subset of X and $L : X \to X$. The value

$$e(A, L) := \sup_{f \in A} \|f - Lf\|_X = \sup_{f \in A} \|(I - L)f\|_X$$

is the error of approximation of identity operator I by the operator L on the set A.

One might consider the problem of finding (if exists) a linear operator of finite rank n, which gives the minimal error of approximation of identity operator on some set over all finite rank n linear operators L preserving the shape in the sense V. It leads us naturally to the notion of *linear* relative n-width.

The definition of linear relative n-width was introduced in the papers [3], [4], [5]. The paper [3] finds estimates of linear relative n-widths for linear operators preserving an intersection of cones of p-monotonicity functions.

Definition 2. A linear operator $L : X \to X$ is called the operator with finite rank n, if the dimension of linear subspace $L(X)$ is equals to n, $\dim\{L(X)\} = n$.

Two definitions of linear relative widths are presented below. The first one is based on Konovalov's ideas [6].

Definition 3. Let X be a linear normed space. Let V be a cone in X. Konovalov linear n-width of a set $A \cap V \subset X$ in X with constraint V is defined by [5]

$$\delta_n(A \cap V, V)_X := \inf_{L_n(V) \subset V} e(A \cap V, L_n)_X,$$

where infimum is taken over all linear continuous operators L_n such that $L_n : X \to X$ is of finite rank n and $L_n(V) \subset V$.

To determine the negative impact of the property of shape-preserving on the order of linear approximation, the following definition based on ideas of Korovkin was introduced in [3].

Definition 4. Let X be a linear normed space. Let V be a cone in X. Korovkin linear n-width of a set $A \subset X$ in the space X with constraint V is defined by

$$\delta_n(A, V)_X := \inf_{L_n(V) \subset V} e(A, L_n)_X,$$

where infimum is taken over all linear continuous operators L_n such that $L_n : X \to X$ is of finite rank n and $L_n(V) \subset V$.

Estimation of linear relative n-widths is of interest in the theory of shape-preserving approximation as, knowing the value of relative linear n-width, we can judge how good or bad (in terms of optimality) this or that finite-dimensional method preserving the shape in the sense V is.

3. Basic Properties of Konovalov Linear Relative n-Widths

Let X be a linear normed space, $A \subset X$, $A \neq \varnothing$. Let $\delta_n(A)_X$ denote the linear n-width of the set A in X,

$$\delta_n(A)_X := \inf_{L_n} \quad e(A, L_n)_X,$$

where infimum is taken over all linear continuous operators $L_n : X \to X$ with finite rank n.

The well-known basic properties of linear widths are listed below (see, for example [10]).

Proposition 1. *Let X be a linear normed space, $A \subset X$, $A \neq \varnothing$. Then*

1. $\delta_n(A)_X = \delta_n(\overline{A})_X$, *where $\overline{A}$ denotes the closure of A.*
2. *Foe every $\alpha \in \mathbb{R}$, $\delta_n(\alpha A)_X = |\alpha|\delta_n(A)_X$.*
3. *If $b(A) := \{\alpha f : \ f \in A, \ |\alpha| \leqslant 1\}$ denotes the balanced hull of A, then*
$$\delta_n(b(A))_X = \delta_n(A)_X.$$
4. $\delta_n(coA)_X = \delta_n(A)_X$.
5. $\delta_n(A)_X \geqslant \delta_{n+1}(A)_X$, $n = 0, 1, \ldots$.

As it will be shown in this section, some of the properties of linear widths mentioned in Proposition 1 do not hold for Konovalov relative linear widths.

1. There exist A, V, X, α, such that $\delta_n(\alpha A \cap V, V)_X \neq \alpha\delta_n(A \cap V, V)_X$ (Example 1).
2. There exist A, V, X, such that $\delta_n(b(A) \cap V, V)_X \neq \delta_n(A \cap V, V)_X$, (Example 2).
3. The inequality $\delta_n(\mathrm{co}A \cap V, V)_X \neq \delta_n(A \cap V, V)_X$ holds for some choice of X, A, V (Example 3).

First we will show that the analogue of property 2 of Proposition 1 does not hold in the case of Konovalov linear widths.

Example 1. Let us consider $X = R^2$ with Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$, $x = (x_1, x_2)$. Let $A = \{(x_1, x_2) \in R^2 : \ x_1 + x_2 \leqslant 1, \ x_1, x_2 \geqslant 0\}$, $V = V_1 = V = \{(x_1, x_2) \in R^2 : \ x_1, x_2 \geqslant 0\}$. Then $\delta_1(A \cap V, V, V)_X = \frac{1}{\sqrt{2}}$. We have $-2A \cap V = \{(0, 0)\}$ and $\delta_1(-2A \cap V, V, V)_X = 0$.

Example 2. Consider $X = R^2$ with Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$, $x = (x_1, x_2)$. Let $A = \{(x_1, x_2) \in R^2 : \ x_1^2 + x_2^2 \leqslant 1, \ x_1 + x_2 \geqslant 0\}$, $V = V = \{(x_1, x_2) \in R^2 : \ x_1 \leqslant 0, x_2 \leqslant 0\}$. Then $b(A) = \{(x_1, x_2) \in R^2 : \ x_1^2 + x_2^2 \leqslant 1\}$, $A \cap V = \{(0,0)\}$, $b(A) \cap V = \{(x_1, x_2) \in R^2 : \ x_1 \leqslant 0, x_2 \leqslant 0, x_1^2 + x_2^2 \leqslant 1\}$, and $\delta_1(b(A) \cap V, V, V)_X = 2 - \sqrt{2}$, $\delta_1(A \cap V, V, V)_X = 0$.

Example 3. Consider $X = R^2$ with Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$, $x = (x_1, x_2)$. Let $A = \{(x_1, x_2) \in R^2 : \ |x_1| + |x_2| \leqslant 1, \ x_1, x_2 \geqslant 0\} \cup \{(x_1, x_2) \in R^2 : \ |x_1| + |x_2| \leqslant 1, \ x_1, x_2 \leqslant 0\}$, $V = V = \{(x_1, x_2) \in R^2 : \ x_1 + x_2 \geqslant 0, x_1 \geqslant 0, x_2 \leqslant 0\}$. Then $\mathrm{co}A = \{(x_1, x_2) \in R^2 : \ |x_1| + |x_2| \leqslant 1\}$, $A \cap V = \{(x_1, x_2) \in R^2 : \ 0 \leqslant x_1 \leqslant 1, x_2 = 0\}$, $\mathrm{co}A \cap V = \{(x_1, x_2) \in R^2 : \ x_1 + x_2 \geqslant 0, x_1 \geqslant 0, x_2 \leqslant 0, \ x_1 - x_2 \leqslant 1\}$, and $\delta_1(\mathrm{co}A \cap V, V, V)_X = 1/(2\sqrt{2})$, $\delta_1(A \cap V, V, V)_X = 0$.

We can establish the following properties of $\delta_n(A \cap V, V)_X$.

Theorem 1. *Let X be a linear normed space, $A \subset X$, $A \neq \varnothing$, $V \subset X$ is a non-empty cone. Then*

1. $\delta_n(A \cap V, V)_X = \delta_n(\overline{A \cap V}, V)_X$, *where $\overline{A \cap V}$ denotes the closure of $A \cap V$.*

2. *For every $\alpha \geqslant 0$, $\delta_n(\alpha A \cap V, V)_X = \alpha \delta_n(A \cap V, V)_X$.*

3. *If $b_+(A) := \{\alpha f : \ f \in A, \ \alpha \in [0,1]\}$, then*

$$\delta_n(b_+(A) \cap V, V)_X = \delta_n(A \cap V, V)_X.$$

4. $\delta_n(A \cap V, V)_X \geqslant \delta_{n+1}(A \cap V, V)_X$, $n = 0, 1, \ldots$.

Proof. It follows from the definition of cone that $\alpha V \in V$ for every $\alpha \geqslant 0$. Then 2) follows from equalities $\alpha A \cap V = \alpha A \cap \alpha V = \alpha(A \cap V)$ and definition 3.

Let us prove 3). Since $b_+(A) \supset A$, we have

$$\delta_n(b_+(A), V)_X \geqslant \delta_n(A, V)_X.$$

On the other hand, for every $f \in b_+(A)$, there exist $f^* \in A$ and $\alpha \in [0,1]$, such that $f = \alpha f^*$. We can write $b_+(A) = \cup_{\alpha \in [0,1]} b_\alpha(A)$, where $b_\alpha(A) := \{\alpha f : \ f \in A\}$. We have

$$\delta_n(b_+(A) \cap V, V)_X = \inf_{L_n(V) \subset V} \ \sup_{f \in b_+(A) \cap V} \|f - L_n f\|_X =$$

$$= \inf_{L_n(V) \subset V} \ \sup_{\alpha \in [0,1]} \ \sup_{f \in b_\alpha(A) \cap V} \|f - L_n f\|_X =$$

$$= \inf_{L_n(V)\subset V} \sup_{\alpha\in[0,1]} \sup_{f\in\alpha A\cap V} \|f - L_n f\|_X =$$
$$= \inf_{L_n(V)\subset V} \sup_{\alpha\in[0,1]} \sup_{f\in\alpha(A\cap V)} \|f - L_n f\|_X \leqslant$$
$$\leqslant \inf_{L_n(V)\subset V} \sup_{f\in A\cap V} \|f - L_n f\|_X = \delta_n(A\cap V, V)_X.$$

Properties 1 and 4 easily follow from the definition of Konovalov linear relative width 3 .

□

The following example shows that in the proposition 1) of Theorem 1 we can not write $\overline{A}\cap V$ (instead of $\overline{A\cap V}$ as it is in Proposition 1).

Example 4. Let us consider $X = R^3$ with Euclidean norm

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad x = (x_1, x_2, x_3).$$

Let $A = \{(x_1, x_2, x_3) \in R^3 : \ |x_1| + |x_2| + |x_3| \leqslant 1,\ x_1, x_2 \geqslant 0,\ x_3 > 0\} \cup \{(0,0,0)\}$, $V = V = \{(x_1, x_2, x_3) \in R^3 :\ x_1, x_2 \geqslant 0, x_3 = 0\}$. Then $\overline{A}\cap V = \{(x_1, x_2, x_3) \in R^3 :\ x_1, x_2 \geqslant 0, x_1 + x_2 \leqslant 1,\ x_3 = 0\}$, $A \cap V = \{(0,0,0)\}$, and $\delta_1(\overline{A}\cap V, V)_X = 1/\sqrt{2}$, $\delta_1(A\cap V, V)_X = 0$.

Note that the property 2) of Theorem 1 is not true if V is an arbitrary set (not necessary a cone). The next example show that $\delta_n(\alpha A\cap V, V)_X \neq |\alpha|\delta_n(A\cap V, V)_X$ X, A, V and $\alpha\in\mathbb{R}$.

Example 5. Let us consider $X = R^2$ with Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$, $x = (x_1, x_2)$. Let $A = V = V = \{(x_1, x_2) \in R^2 :\ x_1 + x_2 \leqslant 1,\ x_1, x_2 \geqslant 0\}$. Then $\delta_1(A\cap V, V)_X = \delta_1(\alpha A\cap V, V)_X = 1/(2\sqrt{2})$ for all $\alpha \geqslant 1$.

The next theorem points out several ways of finding lower estimations of Konovalov linear relative widths.

Theorem 2. *If $A \subset X \subset Y$, where X is the subspace of the normed linear space Y, then*

$$\delta_n(A\cap V, V)_X \geqslant \delta_n(A\cap V, V)_Y. \tag{1}$$

Proof. Every continuous linear operator L_n of finite rank n may be written in the form $L_n f = \sum_{i=1}^{n} l_i(f) f_i$, where $f_i \in X$ $l_i \in X'$. The analogous property holds for continuous linear operators of finite rank n, defined in Y. Since X is a subspace of Y, it follows from HahnBanach theorem that there exists a continuous operator $\tilde{L}_n$ on Y, for which $\tilde{L}_n = L_n$ for all $f \in X$ and for all $f \in A$. □

Using on example [10, p. 10] it is possible to achieve a strong inequality in 2 for particular choices of X, A, V.

Since linear cone-preserving approximation is not better than best cone-preserving approximation, we get the following proposition.

Theorem 3. *Let X be a linear normed space and $A \subset X$, $V \subset X$. Then*

$$\delta_n(A \cap V, V)_X \geqslant d_n(A \cap V, V)_X. \tag{2}$$

Proof. Denote $X_n = \text{range } L_n$. Then

$$\delta_n(A \cap V, V)_X = \inf_{L_n(V) \subset V} \sup_{f \in A \cap V} \|f - L_n f\|_X \geqslant$$
$$\geqslant \sup_{f \in A \cap V} \inf_{g \in X_n \cap V} \|f - g\|_X \geqslant d_n(A \cap V, V)_X.$$

□

It follows from 3 follows that one of the methods for estimation of Konovalov linear relative width $\delta_n(A \cap V, V)_X$ from below is estimation of the corresponding relative n-width $d_n(A \cap V, V)_X$.

The following example 6 shows that $\delta_n(A \cap V, V)_X > d_n(A \cap V, V)_X$ for some choice of A, V in $X = L^2_{2\pi}(R)$, where $L^2_{2\pi}(R)$ denotes the Banach space of all (equivalence classes of) functions $f : R \to R$ that are Lebesgue integrable to the 2-th power over $[-\pi, \pi]$ and that satisfy $f(x + 2\pi) = f(x)$ for a.e. $x \in R$. The space $L^2_{2\pi}(R)$ is endowed with the norm

$$\|f\|_2 = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f^2(x) dx \right)^{\frac{1}{2}}.$$

Example 6. Let $X = L^2_{2\pi}(R)$, $P := \text{span}\{1, \cos x, \sin x\}$, $A = \{f \in P : \|f\|_2 \leqslant 1\}$ and $V = V = \{f \in L^2_{2\pi}(R) : \ f \geqslant 0\}$. Then every linear operator L_n of finite rank n, such that $L_n(V) \subset V$, may be written in the form

$$L_n f = \sum_{i=1}^{n} \langle a_i, f \rangle g_i,$$

where $a_i, g_i \in L^2_{2\pi}(R)$, $i = 1, \ldots, n$, are a.e. non-negative functions.

It is obvious that $d_n(A \cap V, V)_{L^2_{2\pi}(R)} = 0$.

Our aim is to show that the set of all linear operators satisfying

1. $\|f - L_n f\|_2 = 0$ for all $f \in P$;

2. $L_n(V) \subset V$

is empty. Denote $h_t(x) = \sin^2 \frac{t-x}{2}$. Then $h_t \in A$ and

$$\|h_t - L_n h\|_2^2 = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left(h_t(x) - \sum_{i=1}^{n} \langle a_i, h_t \rangle g_i(x) \right)^2 dx = 0.$$

Thus

$$h_t = \sum_{i=1}^{n} \langle a_i, h_t \rangle g_i$$

It follows from $h_t(t) = 0$ follows that $\sum_{i=1}^{n} \langle a_i, h_t \rangle g_i(t) = 0$. It is imposable for all $t \in [-\pi, \pi]$ because of $a_i, g_i \geqslant 0$ a.e.

Theorem 4. *Let X be a linear normed space and V be a cone in X. Then*

$$\delta_n(A \cap V, V)_X \geqslant \delta_n(A \cap V)_X, \tag{3}$$

where $\delta_n(A \cap V)_X$ denotes the linear n-width of $A \cap V$ in X.

Proof. We have

$$\delta_n(A \cap V, V)_X = \inf_{L_n(V) \subset V} \sup_{f \in A \cap V} \|f - L_n f\|_X \geqslant \inf_{L_n} \sup_{f \in A \cap V} \|f - L_n f\|_X,$$

where infimum is taken over all continuous linear operators $L_n : X \to X$ with finite rank n. The last quantity is the linear n-width of $A \cap V$ in X. $\square$

Note that if $V = X$ then relative n-width of A in X is equal to n-width of A in X for every A, i.e. $\delta_n(A \cap X, X)_X = \delta_n(A \cap X)_X = \delta_n(A)_X$.

The example 7 shows that there is a choice of X, A, V for which strong inequality holds in (3).

Example 7. Let $X = R^4$, $\|y\|_X = \max_{0 \leqslant i \leqslant 3} |y_i|$, $y = (y_0, y_1, y_2, y_3)$, and let $V = V = R^4_+$ is non-negative octant of R^4. Denote $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$, $x \in R$. For $f \in C(R)$ denote $If = (f(0), f(1), f(2), f(3))$. Let $A = \{If : \; f = \sum_{j=0}^{2} a_j e_j, \; |a_2| \leqslant 1\} \subset R^4$.

Consider the linear operator $L : R^4 \to R^4$ of finite rank 3, defined by

$$Ly = y_0 I e_0 + \frac{1}{6}(9y_1 - 8y_0 - y_3) I e_1 + \frac{1}{6}(2y_0 - 3y_1 + y_3) I e_2, \; y = (y_0, y_1, y_2, y_3). \tag{4}$$

Note that range range$L = \mathrm{span}\{Ie_0, Ie_1, Ie_2\}$ and dimension of range range$L$ are equal to 3.

It follows from equalities $(Ly)_i = y_i$, $i = 0, 1, 3$, that $Ly = y$ for all $y \in A$. As $\|y - Ly\|_X = 0$ for all $y \in A$, we have $\delta_3(A \cap V)_X = 0$.

It is worth noting that operator L defined above does not satisfy the shape-preserving property $L(V) \subset V$. To prove this let us consider vectors $y = (1, 0, 0, 0) \in V$ and $z = Ie_0 - \frac{4}{3}Ie_1 + \frac{1}{3}Ie_2 = (1, 0, -\frac{1}{3}, 0)$. We have $Ly = L(z + (0, 0, \frac{1}{3}, 0)) = Lz + L(0, 0, \frac{1}{3}, 0) = (1, 0, -\frac{1}{3}, 0) + (0, 0, 0, 0) = (1, 0, -\frac{1}{3}, 0) \notin V$.

Moreover, there does not exist a linear operator $L : R^4 \to R^4$ with finite rank 3, such that $Ly = y$ for all $y \in A$ and for which the shape-preserving property $L(V) \subset V$ holds.

Suppose that such operator L exists. Then L may be presented in the form $Ly = \sum_{i=0}^{2} a_i(y) Ie_i$. Since $L(Ie_0) = Ie_0$, $L(Ie_1) = Ie_1$, $L(Ie_2) = Ie_2$, we have

$$\begin{cases} a_0 e_0(y_0) + a_1 e_0(y_1) + a_2 e_0(y_3) = e_0(y_2) \\ a_0 e_1(y_0) + a_1 e_1(y_1) + a_2 e_1(y_3) = e_1(y_2) \\ a_0 e_2(y_0) + a_1 e_2(y_1) + a_2 e_2(y_3) = e_2(y_2) \end{cases} \tag{5}$$

Let us consider the vector $y^* = I(y_2^2 e_0 - 2y_2 e_1 + e_2)$. We have

$$y_i^* \quad \begin{cases} = 0, \ i = 2 \\ > 0, \ i = 0, 1, 3. \end{cases}$$

It follows from (5) that $a_0 y_0^* + a_1 y_1^* + a_2 y_2^* = 0$, and therefore one of the coefficients a_0, a_1, a_2 is strongly less than 0, i.e. L is not positive and the property $L(V) \subset V$ does not hold. Thus, $\delta_3(A \cap V, V) \neq 0$.

All inequalities listed in Theorems 2, 3, 4 can not be improved. Easiest way to show this is to set $X = Y$ in (1), $V = X = R^2$ in (2), $V = X$ in (3).

4. Basic Properties of Korovkin Relative Linear n-Widths

Some basic properties of Korovkin relative linear n-widths $\delta_n(A, V)_X$ which follows from the definition are listed below.

Theorem 5. *Let X be a linear normed space, $A \subset X$, $A \neq \varnothing$, $V \subset X$ be a cone in X. Then*

1. *$\delta_n(A, V)_X = \delta_n(\overline{A}, V)_X$, where $\overline{A}$ denotes the closure of A.*

2. *For every $\alpha \in \mathbb{R}$, $\delta_n(\alpha A, V)_X = |\alpha| \delta_n(A, V)_X$.*

3. Let $b(A)$ denotes the balanced hull of A, then

$$\delta_n(b(A), V)_X = \delta_n(A, V)_X.$$

4. $\delta_n(coA, V)_X = \delta_n(A, V)_X$.

5. $\delta_n(A, V)_X \geqslant \delta_{n+1}(A, V)_X$, $n = 0, 1, \ldots$.

Analogues of theorems 2, 3, 4 are holds and can be find below.

Theorem 6. *If $A \subset X \subset Y$, where X is the subspace of the normed linear space Y, then*

$$\delta_n(A, V)_X \geqslant \delta_n(A, V)_Y.$$

Theorem 7. *Let X be a linear normed space and $A \subset X$, $V \subset X$. Then*

$$\delta_n(A, V)_X \geqslant d_n(A, V)_X.$$

Theorem 8. *Let X be a linear normed space and V be a cone in X. Then*

$$\delta_n(A, V)_X \geqslant \delta_n(A)_X, \tag{6}$$

where $\delta_n(A)_X$ is the linear n-width of A in X.

Of course, the value of Konovalov linear relative n-width is not greater than the value of Korovkin linear relative n-width:

$$\delta_n(A, V)_X \geqslant \delta_n(A \cap V, V)_X. \tag{7}$$

If we compare the value of Korovkin linear relative n–width $\delta_n(A, V)_X$ of set A in X with the constraint V to the value of linear n–width $\delta_n(A)_X$ of the set A in X we can evaluate the negative impact of the shape-preserving constraint $L_n(V) \subset V$ on the intrinsic error of approximation by means of the *shape-preserving* linear operators of finite rank n compare to the error of *unconstrained* linear finite-rank approximation on the same set.

It seems that the problem of estimation of Konovalov linear relative n-width $\delta_n(A \cap V, V)_X$ is much harder that the problem of estimation of Korovkin linear relative n-width $\delta_n(A, V)_X$, so it will be difficult to use (7) for getting lower estimations for the value of $\delta_n(A, V)_X$. On the other hand, (7) can potentially help for getting upper estimations for the value of Konovalov linear relative n-width $\delta_n(A \cap V, V)_X$.

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