THE MODIFIED VARIATIONAL ITERATION METHOD
FOR SOLVING THE IMPENETRABLE AGAR
MODEL PROBLEM

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Abstract: In this paper, we expand the scope of application of the modified
variational iteration method (MVIM) to solve the Impenetrable Agar Model
Problem. This model represents the propagation of a bacterial exo-enzyme
within an impenetrable host material which simulates the manifestations bac-
teria do on tissue wounds in an infection. The numerical results show that the
MVIM is convenient, efficient, and easy to use for the model problem.

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1. Introduction

In 1978 Inokuti et al. [1] proposed a general Lagrange multiplier method to
solve nonlinear problems. Ji-huan He modified the method of Inokuti and
proposed the VIM [2- 6]. This method has been applied to solve a large class of
linear and nonlinear problems, and proved by many researchers to be reliable,
efficient, and convenient for a variety of scientific applications [7-19].

On the other hand, the VIM has also been applied to solve fractional dif-
ferential equations. Soltani and Shirzadi [20] proposed a new modification of the VIM; the given examples show that the method provides great freedom in choosing linear operators for various nonlinear equations which leads to identify the Lagrange multipliers effectively. Geng [21], proposed a modification of the VIM for solving Riccati differential equations, where he compared his modification with the standard VIM.

This work uses the modified method (MVIM) to solve the propagation of the bacteria through wounds and the manipulations it performs to degrade wound tissue. The given example shows that the modified method gives a more accurate approximation in a large region and overcome the restriction of the application area of the VIM. Complete review of He’s VIM can be found in references 5 and 22 [5,22].

This paper is organized as follows: In Section 2 we introduce the VIM and its modification, in Section 3 we discuss the Impenetrable Agar Model Problem, where in Section 4 we implement and analyze the method for the solution of the model problem using the MVIM, and finally we conclude the paper in Section 5.

2. The Variational Iteration Method and the Modified Variational Iteration Method

In this section we introduce the He’s VIM [2-5].

Consider the differential equation

$$Lu + Nu = g(x),$$

where $L$ is a linear operator, $N$ a nonlinear operator and $g(x)$ is the inhomogeneous term. J. He proposed the VIM by constructing a correction functional for equation 1 as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)(Lu_n(t) + N\ddot{u}_n(t) - g(t))dt,$$

where $\lambda$ is a Lagrange’s multiplier and it can be constant or a function, and $\ddot{u}_n$ as a restricted variation, we can assume that the correction functional is stationary, that is $\partial \ddot{u}_n = 0$. In this method we can construct a correction functional form using the general Lagrange multiplier which can be identified by the variational theory [4]. Then the iterative approximation can start with initial approximation $u_0$ which can be obtained from the initial conditions $u(0)$,
u'(0), and u''(0). The exact solution may be obtained by taking the limit as
n \to \infty, that is u(x) = \lim_{n \to \infty} u_n(x).

In 2010 Soltani and Shirzadi [20], presented a very simple and effective mod-
ification of the VIM which is based on introducing an arbitrary linear operator
L_1[u(x)] as follows:

1. Rewrite equation 1 in the form:
   \[ L[u(x)] + L_1[u(x)] - L_1[u(x)] + N[u(x)] = g(x), \]
   where \( L_1[u(x)] \) is an arbitrary linear operator of \( u(x) \).

2. Construct the correction functional
   \[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)(Lu_n(t) + L_1[u_n(x)] - L_1[\tilde{u}_n(x)] + N\tilde{u}_n(t) - g(t))dt, \]
   where \( \tilde{u}_n \) is considered as a restriction variation.

According to Theorem 1 of Section 2 in [20], a sufficient condition for con-
vergence of the MVIM is strictly contraction of \( A \), where \( u_{n+1}(x) = A[u_n] \),
which follows that the sequence \( u_{n+1}(x) = A[u_n] \), converges to the solution of
equation 1.

3. The Impenetrable Agar Model Problem

In 2003 King et al [23], studied a model in which a bacterial colony is unable to
penetrate the ‘tissue’ (agar substrate), and must obtain nutrients by producing
exo-proteases to break down the milk protein contained in an agar matrix.
They were trying to simulate the propagation of the bacteria (\emph{P. aeruginosa})
through wounds and the manipulations it performs to degrade wound tissue.
This model is a simplified model for the propagation of the bacteria due to the
fact that it does not involve the complications occurs within the tissue. Hence,
their case provides useful insights and validation of the introduced modeling
approach they presented, and the simplified model lends itself to a detailed
mathematical analysis. In their experiment, an impenetrable matrix of agar
infused with dissolved milk powder was constructed to provide a simplified
system in which \emph{P. aeruginosa} was unable to move into the substrate but
produces exo-proteases in order to degrade milk proteins into utilizable peptides
for its growth.
The simplest model they used is
\[ \frac{\partial c}{\partial t} = D \nabla^2 c - \lambda c - \alpha c m \]
\[ \frac{\partial m}{\partial t} = -\beta c m, \]
where the concentration of the most active bacterial enzyme is denoted by \( c \), and that of milk by \( m \). While the constants \( \alpha, \beta, \lambda \) and \( D \) respectively denote the rates of bacterial enzyme, milk used up in the reaction, the natural inactivation rate of the exo-enzyme, and the rate of diffusion of the exo-enzyme within the agar. Hence, the milk is thus assumed to be immobile until it is degraded by the enzyme to produce diffusible peptides.

They considered a semi-infinite region of substrate, with \( z \) defined as positive in the downward direction (into the agar) with a disc-shaped colony of bacteria deposited on the surface. Hence the initial data becomes
\[ c = 0, \ m = m_i \ at \ t = 0, \ z > 0, \]
where \( m_i \) is the initial concentration of milk protein, and boundary conditions
\[ c = c_b \ on \ z = 0, \ x^2 + y^2 < l^2 \ and \ \frac{\partial c}{\partial z} = 0 \ on \ z = 0, \ x^2 + y^2 > l^2, \]
which correspond to the surface bacterial layer suddenly producing enzymes at a constant level \( c_b \) on the boundary and then continuing to hold that level. This sudden switch is an exaggerated version of the quorum sensing-induced jump in virulence determinant production that does occur in \( P.aeruginosa \).

Therefore, the dimensionless model will be
\[ \frac{\partial \hat{c}}{\partial \hat{t}} = \nabla^2 \hat{c} - k \hat{c} - \frac{\gamma}{\varepsilon^2} \hat{c} \hat{m} \]
\[ \frac{\partial \hat{m}}{\partial \hat{t}} = -\frac{1}{\varepsilon^2} \hat{c} \hat{m} \]
where,
\[ \hat{m} = \frac{m}{m_i}, \ \hat{c} = \frac{c}{c_b}, \ \hat{\ell} = \frac{D}{l^2}, \ x = \frac{l}{\ell} \]
\[ \hat{x} = \frac{1}{l} x \]

immediately dropping the hats will give
\[ \frac{\partial c}{\partial t} = \nabla^2 c - kc - \frac{\gamma}{\varepsilon^2} cm \] (3)
THE MODIFIED VARIATIONAL ITERATION METHOD... 449

\[
\frac{\partial m}{\partial t} = -\frac{1}{\varepsilon^2}cm
\]  

(4)

with initial condition
\(c = 0, \ m = 1\) at \(t = 0, \ z > 0\)

and boundary conditions
\(c = 1\) on \(z = 0, \ x^2 + y^2 < 1\) and \(\frac{\partial c}{\partial z} = 0\) on \(z = 0, \ x^2 + y^2 > 1\).

The dimensionless parameters are
\[\varepsilon = \sqrt{\frac{D}{\beta c b l^2}}, \ \gamma = \frac{\alpha c m}{\beta c b}, \text{ and } k = \frac{\lambda l^2}{D}.
\]

4. The Solution Method and Analysis

In this section, we implement and analyze the MVIM for solving the model problem in equations 3 and 4 when \(x^2 + y^2 > 1\).

Consider the following correction functional for \(c\) and \(m\) respectively:

\[
\delta c_{n+1}(t) = \delta c_n(t) + \delta \int_0^t \lambda_1(\tau)(-\frac{\partial^2 c_n}{\partial \tau^2} + \frac{\partial c_n}{\partial \tau} - \frac{\partial^2 \tilde{c}_n}{\partial z^2} + \kappa \tilde{c}_n + \frac{\gamma}{\varepsilon^2} \tilde{c}_n \tilde{m}_n) d\tau, \quad (5)
\]

\[
\delta m_{n+1}(t) = \delta m_n(t) + \delta \int_0^t \lambda_2(\tau)\left(\frac{\partial m_n}{\partial \tau} + \frac{1}{\varepsilon^2} \tilde{c}_n \tilde{m}_n\right) d\tau, \quad (6)
\]

which leads to the following
\(\delta c_{n+1}(t) = \delta c_n(t) + \delta \int_0^t \lambda_1(\tau)(-\frac{\partial^2 c_n}{\partial \tau^2}) d\tau, \) and

\(\delta m_{n+1}(t) = \delta m_n(t) + \delta \int_0^t \lambda_2(\tau)\frac{\partial m_n}{\partial \tau} d\tau.\) Since \(\delta \tilde{c}_n = 0, \) and \(\delta \tilde{m}_n = 0.\)

The extremum conditions of \(c_{n+1}\) and \(m_{n+1}\) requires that \(\delta c_{n+1}(t) = 0,\) and \(\delta m_{n+1}(t) = 0.\) Doing this will lead to the stationary conditions for \(\lambda_1\) and \(\lambda_2\)

\[
1 + \lambda_1' |_{\tau=t} = 0,
\lambda_1 |_{\tau=t} = 0,
\lambda_1'' = 0.
\]  

(7)

\[
1 + \lambda_2' |_{\tau=t} = 0,
\lambda_2' = 0.
\]  

(8)

The solutions of 7 and 8 implies that \(\lambda_1 = t - \tau,\) and \(\lambda_2 = -1.\)

Substituting this value of the Lagrange multiplier into the functional 5 and 6 gives the iteration formulae:
\[ \begin{align*}
c_{n+1}(t) &= c_n(t) + \int_0^t (t - \tau) \left( \frac{\partial c_n}{\partial \tau} - \frac{\partial^2 c_n}{\partial \tau^2} - \frac{\partial^2 c_n}{\partial z^2} + \kappa c_n + \frac{\gamma}{\varepsilon^2} c_n m_n \right) d\tau \\
m_{n+1}(t) &= m_n(t) - \int_0^t \left( \frac{\partial m_n}{\partial \tau} + \frac{1}{\varepsilon^2} c_n m_n \right) d\tau.
\end{align*} \] (9) (10)

By choosing the initial approximation \( c_0(t, z) = t \cosh[z] \), and \( m_0(t, z) = 1 \), which satisfies the initial condition of the original problem \( c = 0, m = 1 \) at \( t = 0, z > 0 \), and the boundary condition \( \frac{\partial c}{\partial z} = 0 \) at \( z = 0 \).

Applying the iteration formula, will give the following first approximation

\[ c_1(t, z) = \frac{1}{6} t \left( 6 + 3t + \left( -1 + \frac{b}{c^2} + k \right) t^2 \right) \cosh[z] \]

and

\[ m_1(t, z) = 1 - \frac{t^2 \cosh[z]}{2 c^2}. \]

And the second iteration of "c2" and "m2" can be obtained from 9 and 10;

\[ c_2 = \frac{1}{2520 c^6} t \cosh(z) \left( 21 (b^2 c^2 t^4 + 2bc^4 t^2 (10 + 5t + (-1 + k) t^2)) + c^6 \left( 120 + 60t + 20kt^2 + 10 (-1 + k) t^3 + (-1 + k)^2 t^4 \right) - \right), \]

and

\[ m_2 = 1 - \frac{t^2 \left( bt^2 + c^2 (12 + 4t + (-1 + k) t^2) \right) \cosh(z)}{24c^4} + \frac{t^4 \left( 5bt^2 + c^2 (45 + 18t + 5 (-1 + k) t^2) \right) \cosh^2(z)}{360c^6}. \]

The following two Figures represent the third iteration of the approximate solutions of \( m_3 \) and \( c_3 \) respectively:

From the above we notice that all iterations satisfy the initial and the boundary conditions and we conclude that the third iteration of MVIM is a reasonable approximation for the solution of the impenetrable agar model problem. These figures show that the concentration of milk "\( m \)" decreases as time increases.
Figure 1: Depicts the approximate solution of $m_3$ when $c = 0.5$, $k = 1$ and $b = 1$.

While the concentration of the enzyme increases with time. This demonstrates a reasonable explanation of the degradation of milk in the presence of bacteria as time increases.

5. Solving the Impenetrable Agar Model Problem Using Variational Iteration Algorithm II

In this section a comparison between the results obtained for the Impenetrable Agar Model Problem using algorithm II and that obtained in the previous section. He [11] summarized three variational iteration algorithms. The variational iteration method mentioned in section two is called variational iteration
Figure 2: Depicts the approximate solution of $c_3$ when $c = 0.5$, $k = 1$ and $b = 1$.

algorithm I. The variational iteration algorithm II

$$u_{n+1}(x) = u_0(x) + \int_0^x \lambda(t)(N\dot{u}_n(t) - g(t))dt,$$  \hspace{1cm} (11)

was used to construct a solution for the problem of concern. The algorithm becomes

$$c_{n+1}(t) = c_n(t) + \int_0^t (t - \tau)(-\frac{\partial^2 c_n}{\partial \tau^2} - \frac{\partial^2 c_n}{\partial z^2} + \kappa c_n + \frac{\gamma}{\varepsilon^2}c_nm_n)d\tau$$ \hspace{1cm} (12)

$$m_{n+1}(t) = m_n(t) - \int_0^t (\frac{1}{\varepsilon^2}c_nm_n)d\tau.$$ \hspace{1cm} (13)

The following figures show the comparison between the solutions obtained using algorithm II and that obtained by the modified method. Figure 3 display
the third iteration $c_3$ when $\gamma = \kappa = 1$ with different values for $\varepsilon$. Figure 3(A) represent the solution when $\varepsilon = 0.5$, where $z=0.6$. In (B) $\varepsilon = 0.5$, and $z = 0.3$. In (C) the value for $\varepsilon = 0.8 = z$. While in (D) $\varepsilon = 0.8$, and $z = 0.7$.

Figure 3: The mesh curve represents the approximate solution of $c_3$using algorithm II, while the second curve represent the approximate solution of $c_3$ using the modified method when $\kappa = 1$ and $\gamma = 1$.

Figure 4 shows the approximate solutions of $m_3$ for $\gamma = \kappa = 1$ with different values for $\varepsilon$. Figure 4 (A)represent the solution when $\varepsilon = 0.8$, where $z = 1.0$. In (B)$\varepsilon = 0.7$, and$z = 0.9$. In (C) the value for $\varepsilon = 1.0$, and $z = 1.4$. While in (D) $\varepsilon = 0.8$, and $z = 0.7$.

6. Conclusion

In this paper, we have applied the MVIM for solving the Impenetrable Agar Model Problem. The numerical results show that the method converges and provides powerful tools for the solution of the model problem and gives a reasonable explanation of the degradation of milk in the presence of bacteria as time increases.
Figure 4: The mesh curve represents the approximate solution of $m_3$ using algorithm II, while the second curve represent the approximate solution of $m_3$ using the modified method when $\kappa = 1$ and $\gamma = 1$

References


THE MODIFIED VARIATIONAL ITERATION METHOD... 455


