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SOME PROPERTIES OF APPROXIMATION OPERATORS

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Abstract: In this paper, we show that a family of upper approximation operators has a join and a family of lower approximation operators has a meet in complete residuated lattices. We investigate relations between their families.

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1. Introduction

Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Pawlak [9,10] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. By using the concepts of rough approximation operators, information systems and decision rules are investigated in complete residuated lattices [1,3-8,11]. Zhang [12,13] introduced the fuzzy complete lattice which is defined by join and meet on fuzzy partially ordered sets. It is an important mathematical tool for algebraic structure [4-6].

In this paper, by using the concepts of lower and upper approximation operators defined by She and Wang [11], we show that a family of upper approximation operators has a join and a family of lower approximation operators

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has a meet in complete residuated lattices. We investigate relations between their families.

2. Preliminaries

Definition 2.1. [1,2] A structure $(L,\vee,\wedge,\odot,\to,\perp,\top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L,\vee,\wedge,\perp,\top)$ is a complete lattice where $\perp$ is the bottom element and $\top$ is the top element;

(L2) $(L,\odot,\top)$ is a commutative monoid;

(L3) for all $x,y,z\in X$, $x\le y\to z$ iff $x\odot y\le z$.

For $\alpha\in L, A\in L^X, \mathcal{J}:L^X\to L^X$, we denote $(\alpha\to A),(\alpha\odot A),\alpha_X,\top_x,\top_x^*\in L^X,(A\odot\mathcal{H}),(A\to\mathcal{J}):L^X\to L^X$ as $(\alpha\to A)(x)=\alpha\to A(x)$, $(\alpha\odot A)(x)=\alpha\odot A(x)$, $\alpha_X(x)=\alpha$, $(A\odot\mathcal{H})(B)(x)=A(x)\odot\mathcal{H}(B)(x)$, $(A\to\mathcal{J})(B)(x)=A(x)\to\mathcal{J}(B)(x)$ and

$$\top_x(y)=\begin{cases}\top, & \text{if } y=x,\\ \perp, & \text{otherwise,}\end{cases}\quad \top_x^*(y)=\begin{cases}\perp, & \text{if } y=x,\\ \top, & \text{otherwise.}\end{cases}$$

In this paper, we assume $(L,\wedge,\vee,\odot,\to,^*\top,\perp)$ is a complete residuated lattice with a negation;i.e. $x^{**}=x$.

Lemma 2.2[1,2] For each $x,y,z,x_i,y_i\in L$, the following properties hold.

(1) If $y\le z$, then $x\odot y\le x\odot z$.

(2) If $y\le z$, then $x\to y\le x\to z$ and $z\to x\le y\to x$.

(3) $x\to y=\top$ iff $x\le y$.

(4) $x\to\top=\top$ and $\top\to x=x$.

(5) $x\odot y\le x\wedge y$.

(6) $x\odot(\bigvee_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\odot y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\odot y=\bigvee_{i\in\Gamma}(x_i\odot y)$.

(7) $x\to(\bigwedge_{i\in\Gamma}y_i)=\bigwedge_{i\in\Gamma}(x\to y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\to y=\bigwedge_{i\in\Gamma}(x_i\to y)$.

(8) $\bigvee_{i\in\Gamma}x_i\to\bigvee_{i\in\Gamma}y_i\ge\bigwedge_{i\in\Gamma}(x_i\to y_i)$ and $\bigwedge_{i\in\Gamma}x_i\to\bigwedge_{i\in\Gamma}y_i\ge\bigwedge_{i\in\Gamma}(x_i\to y_i)$.

(9) $(x\to y)\odot x\le y$ and $(y\to z)\odot(x\to y)\le(x\to z)$.

(10) $x\to y\le(y\to z)\to(x\to z)$ and $x\to y\le(z\to x)\to(z\to y)$.

(11) $\bigwedge_{i\in\Gamma}x_i^*=(\bigvee_{i\in\Gamma}x_i)^*$ and $\bigvee_{i\in\Gamma}x_i^*=(\bigwedge_{i\in\Gamma}x_i)^*$.

(12) $(x\odot y)\to z=x\to(y\to z)=y\to(x\to z)$ and $(x\odot y)^*=x\to y^*$.

(13) $x^*\to y^*=y\to x$ and $(x\to y)^*=x\odot y^*$.

(14) $y\to z\le x\odot y\to x\odot z$.

(15) $x\to y\odot z\ge(x\to y)\odot z$.

Definition 2.3. [4,12,13] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) reflexive if $e_X(x,x) = \top$ for all $x \in X$,

(E2) transitive if $e_X(x,y) \odot e_X(y,z) \leq e_X(x,z)$, for all $x,y,z \in X$,

(E3) if $e_X(x,y) = e_X(y,x) = \top$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preordered set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set.

Example 2.4.(1) We define a function $e_L : L \times L \to L$ as $e_L(x,y) = x \to y$. Then (L, e_L) is a fuzzy partially order set.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A,B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy partially order set from Lemma 2.2 (9).

(3) We define a function $e^x_{L^X} : L^X \times L^X \to L$ as $e^x_{L^X}(A,B) = A(x) \to B(x)$ for each $x \in X$. Then $(L^X, e^x_{L^X})$ is a fuzzy preordered set for each $x \in X$. If $e^x_{L^X}(A,B) = e^x_{L^X}(B,A) = \top$ for all $x \in X$, then $A = B$.

Definition 2.5.[4,12,13] Let (X, e_X) be a fuzzy preordered set and $A \in L^X$.

(1) A point x_0 is called a join of A, denoted by $x_0 = \sqcup A$, if it satisfies

(J1) $A(x) \leq e_X(x, x_0)$,

(J2) $\bigwedge_{x \in X}(A(x) \to e_X(x,y)) \leq e_X(x_0, y)$.

A point x_1 is called a meet of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \leq e_X(x_1, x)$,

(M2) $\bigwedge_{x \in X}(A(x) \to e_X(y,x)) \leq e_X(y, x_1)$.

Remark 2.6. Let (X, e_X) be a fuzzy preordered set and $A \in L^X$.

(1) x_0 is a join of A iff $\bigwedge_{x \in X}(A(x) \to e_X(x,y)) = e_X(x_0, y)$.

(2) x_1 is a meet of A iff $\bigwedge_{x \in X}(A(x) \to e_X(y,x)) = e_X(y, x_1)$.

(3) If (X, e_X) be a fuzzy partially ordered set and x_0 is a join of A, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.

Remark 2.7. Let $(L^X, e^x_{L^X})$ be a fuzzy preordered set.

(1) If $e^x_{L^X}(A,B) = e^x_{L^X}(C,B)$ for all $B \in L^X$ and $x \in X$, for $B = \top^*_x$, $A^*(x) = C^*(x)$. Hence $A = C$.

(2) If $e^x_{L^X}(A,B) = e^x_{L^X}(A,C)$ for all $B \in L^X$ and $x \in X$, for $A = \top_x$, $B(x) = C(x)$. Hence $B = C$.

3. Some Properties of Approximation Operators

Definition 3.1. [11] A map $\mathcal{H} : L^X \to L^X$ is called an *upper approximation operator* iff it satisfies the following conditions

(H1) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ for all $A \in L^X$ and $\alpha \in L$.

(H2) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$ for all $A_i \in L^X$.

The set $U(X)$ is a family of upper approximation operators on X.

Definition 3.2. [11] A map $\mathcal{J} : L^X \to L^X$ is called a *lower approximation operator* iff it satisfies the following conditions

(J1) $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$, for all $A \in L^X$ and $\alpha \in L$,

(J2) $\mathcal{J}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{J}(A_i)$, for all $A_i \in L^X$,

The set $M(X)$ is a family of lower approximation operators on X.

Theorem 3.3. For each $x \in X$, we define $e^x_{U(X)} : U(X) \times U(X) \to L$ as follows:

$$\begin{aligned} e^x_{U(X)}(\mathcal{H}_1, \mathcal{H}_2) &= \bigwedge_{A\in L^X} e^x_{L^X}(\mathcal{H}_1(A), \mathcal{H}_2(A)) \\ &= \bigwedge_{A\in L^X} (\mathcal{H}_1(A)(x) \to \mathcal{H}_2(A)(x)). \end{aligned}$$

Then the following properties hold.

(1) For each $x \in X$, $(U(X), e^x_{U(X)})$ is a fuzzy preordered set. If $e^x_{U(X)}(\mathcal{H}_1, \mathcal{H}_2) = \top$ and $e^x_{U(X)}(\mathcal{H}_2, \mathcal{H}_1) = \top$ for all $x \in X$, then $\mathcal{H}_1 = \mathcal{H}_2$.

(2)

$$e^x_{U(X)}(\mathcal{H}_1, \mathcal{H}_2) = \bigwedge_{y\in X} (\mathcal{H}_1(\top_y)(x) \to \mathcal{H}_2(\top_y)(x)).$$

(3) For $\Phi : U(X) \to L^X$, there exists $\sqcup\Phi \in U(X)$.

Proof (1) (reflexive) $e^x_{U(X)}(\mathcal{H}, \mathcal{H}) = \top$.

(transitive) $e^x_{U(X)}(\mathcal{H}_1, \mathcal{H}_2) \odot e^x_{U(X)}(\mathcal{H}_2, \mathcal{H}_3) \le e^x_{U(X)}(\mathcal{H}_1, \mathcal{H}_3)$ from:

$$\begin{aligned} & e^x_{U(X)}(\mathcal{H}_1, \mathcal{H}_2) \odot e^x_{U(X)}(\mathcal{H}_2, \mathcal{H}_3) \\ &= \bigwedge_{A\in L^X} (\mathcal{H}_1(A)(x) \to \mathcal{H}_2(A)(x)) \odot \bigwedge_{A\in L^X} (\mathcal{H}_2(A)(x) \to \mathcal{H}_3(A)(x)) \\ &\le (\mathcal{H}_1(A)(x) \to \mathcal{H}_2(A)(x)) \odot (\mathcal{H}_2(A)(x) \to \mathcal{H}_3(A)(x)) \\ &\le (\mathcal{H}_1(A)(x) \to \mathcal{H}_3(A)(x)). \end{aligned}$$

If $e^x_{U(X)}(\mathcal{H}_1, \mathcal{H}_2) = \top$ and $e^x_{U(X)}(\mathcal{H}_2, \mathcal{H}_1) = \top$ for all $x \in X$, then $\mathcal{H}_1(A)(x) = \mathcal{H}_2(A)(x)$ for all $A \in L^X$ and $x \in X$. Hence $\mathcal{H}_1 = \mathcal{H}_2$.

(2) For $A = \bigvee_{y\in X} (A(y) \odot \top_y)$, since $\mathcal{H} \in U(X)$, we have

$$\mathcal{H}_i(A)(x) = \mathcal{H}_i(\bigvee_{y\in X} (A(y) \odot \top_y))(x) = \bigvee_{y\in X} (A(y) \odot \mathcal{H}_i(\top_y)(x)).$$

$$\begin{aligned}
&e^x_{U(X)}(\mathcal{H}_1,\mathcal{H}_2) = \bigwedge_{A\in L^X} e^x_{L^X}(\mathcal{H}_1(A),\mathcal{H}_2(A))\\
&= \bigwedge_{A\in L^X}(\mathcal{H}_1(A)(x) \to \mathcal{H}_2(A)(x))\\
&\le (\mathcal{H}_1(\top_y)(x) \to \mathcal{H}_2(\top_y)(x))
\end{aligned}$$

By Lemma 2.2(8,14), we have

$$\begin{aligned}
&e^x_{U(X)}(\mathcal{H}_1,\mathcal{H}_2) = \bigwedge_{A\in L^X}(\mathcal{H}_1(A)(x) \to \mathcal{H}_2(A)(x))\\
&= \bigwedge_{A\in L^X}(\bigvee_{y\in X}(A(y)\odot\mathcal{H}_1(\top_y)(x)) \to \bigvee_{z\in X}(A(z)\odot\mathcal{H}_2(\top_z)(x)))\\
&\ge \bigwedge_{A\in L^X}\bigwedge_{y\in X}((A(y)\odot\mathcal{H}_1(\top_y)(x)) \to (A(y)\odot\mathcal{H}_2(\top_y)(x)))\\
&\ge \bigwedge_{x\in X}(\mathcal{H}_1(\top_y)(x) \to \mathcal{H}_2(\top_y)(x))
\end{aligned}$$

Thus, $e^x_{U(X)}(\mathcal{H}_1,\mathcal{H}_2) = \bigwedge_{x\in X}(\mathcal{H}_1(\top_y)(x) \to \mathcal{H}_2(\top_y)(x))$.

(3) For $\Phi : U(X) \to L^X$, we define $P : L^X \to L^X$

$$P(A) = \bigvee_{\mathcal{H}\in U(X)} (\Phi(\mathcal{H})\odot\mathcal{H}(A)).$$

$$\begin{aligned}
P(\bigvee_{i\in\Gamma} A_i) &= \bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})\odot\mathcal{H}(\bigvee_{i\in\Gamma} A_i))\\
&= \bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})\odot\bigvee_{i\in\Gamma}\mathcal{H}(A_i))\\
&= \bigvee_{i\in\Gamma}\bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})\odot\mathcal{H}(A_i))\\
&= \bigvee_{i\in\Gamma} P(A_i),\\
P(\alpha\odot A) &= \bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})\odot\mathcal{H}(\alpha\odot A))\\
&= \bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})\odot(\alpha\odot\mathcal{H}(A)))\\
&= \bigwedge_{\mathcal{H}\in U(X)}(\alpha\odot(\Phi(\mathcal{H})\odot\mathcal{H}(A)))\\
&= \alpha\odot P(A),
\end{aligned}$$

Hence $P \in U(X)$.

$$\begin{aligned}
&e^x_{U(X)}(P,\mathcal{H}_1) = \bigwedge_{A\in L^X} e^x_{L^X}(P(A),\mathcal{H}_1(A))\\
&= \bigwedge_{A\in L^X}(\bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})(x)\odot\mathcal{H}(A)(x)) \to \mathcal{H}_1(A)(x))\\
&= \bigwedge_{A\in L^X}\bigwedge_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})(x) \to (\mathcal{H}(A)(x) \to \mathcal{H}_1(A)(x)))\\
&= \bigwedge_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})(x) \to \bigwedge_{A\in L^X} e^x_{L^X}(\mathcal{H}(A),\mathcal{H}_1(A)))\\
&= \bigwedge_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})(x) \to e^x_{U(X)}(\mathcal{H},\mathcal{H}_1)) = e^x_{U(X)}(\sqcup\Phi,\mathcal{H}_1).
\end{aligned}$$

Hence $\sqcup\Phi(A)(x) = P(A)(x) = \bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})(x)\odot\mathcal{H}(A)(x))$ for all $x \in X$. Thus, $\sqcup\Phi = \bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})\odot\mathcal{H}) \in U(X)$.

Theorem 3.4. For $\Phi : U(X) \to L^X$, there exists $\sqcup\Phi \in U(X)$ iff for all $\mathcal{H}_i \in U(X)$, $\bigvee_{i\in\Gamma}\mathcal{H}_i \in U(X)$ and $B_i\odot\mathcal{H}_i \in U(X)$.

Proof ($\Rightarrow$) Put $\Phi_i : U(X) \to L^X$ as $\Phi_i(\mathcal{H}_i) = B_i$ and $\Phi_i(\mathcal{H}) = 0$, otherwise. By Theorem 3.3 (3), we have

$$(\sqcup \Phi_i)(A) = \bigvee_{\mathcal{H} \in U(X)} (\Phi_i(\mathcal{H}) \odot \mathcal{H}(A)) = B_i \odot \mathcal{H}_i(A)$$

Hence $\sqcup \Phi_i = B_i \odot \mathcal{H}_i \in U(X)$.

Let $\{\mathcal{H}_i \in U(X) \mid i \in \Gamma\}$ be given. Define $\Phi : U(X) \to L^X$ as $\Phi(\mathcal{H}_i) = \top_X$ for $i \in \Gamma$ and $\Phi(\mathcal{H}) = 0$ otherwise. By Theorem 3.3 (3),

$$\sqcup \Phi(A) = \bigvee_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H}) \odot \mathcal{H}(A)) = \bigvee_{i \in \Gamma} \mathcal{H}_i(A).$$

Hence $\sqcup \Phi = \bigvee_{i \in \Gamma} \mathcal{H}_i \in U(X)$.

($\Leftarrow$) For $\Phi : U(X) \to L^X$, $\sqcup \Phi = \bigvee_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H}) \odot \mathcal{H}) \in U(X)$ and

$$\begin{aligned}
& e^x_{U(X)}(\sqcup \Phi, \mathcal{H}_1) = \bigwedge_{A \in L^X} e^x_{L^X}(\sqcup \Phi(A), \mathcal{H}_1(A)) \\
& = \bigwedge_{A \in L^X} (\sqcup \Phi(A)(x) \to \mathcal{H}_1(A)(x)) \\
& = \bigwedge_{A \in L^X} (\bigvee_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H})(x) \odot \mathcal{H}(A)(x)) \to \mathcal{H}_1(A)(x)) \\
& = \bigwedge_{A \in L^X} \bigwedge_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H})(x) \to (\mathcal{H}(A)(x) \to \mathcal{H}_1(A)(x))) \\
& = \bigwedge_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H})(x) \to \bigwedge_{A \in L^X} (\mathcal{H}(A)(x) \to \mathcal{H}_1(A)(x))) \\
& = \bigwedge_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H})(x) \to e^x_{U(X)}(\mathcal{H}(A), \mathcal{H}_1(A)))
\end{aligned}$$

Then $\sqcup \Phi(A)(x) = \bigvee_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H})(x) \odot \mathcal{H}(A)(x))$ for all $x \in X$. Hence $\sqcup \Phi = \bigvee_{\mathcal{H} \in U(X)} (\Phi(\mathcal{H}) \odot \mathcal{H}) \in U(X)$.

Example 3.5. (1) Let $R \in L^{X \times X}$ be a fuzzy relation. Define $\mathcal{H} : L^X \to L^X$ as follows

$$\mathcal{H}(A)(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)).$$

Since $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ and $\mathcal{H}(\bigvee_{i \in \Gamma} A_i) = \bigvee_{i \in \Gamma} \mathcal{H}(A_i)$, $\mathcal{H}$ is an upper approximation operator; i.e. $\mathcal{H} \in U(X)$.

(2) Let $\{R_i \in L^{X \times X} \mid i \in \Gamma\}$ be a family of fuzzy relations. Then $\mathcal{H}_i$ is an upper approximation operator such that

$$\mathcal{H}_i(A)(y) = \bigvee_{x \in X} (A(x) \odot R_i(x, y)).$$

By Theorem 3.4, $\bigvee_{i \in \Gamma} \mathcal{H}_i \in U(X)$. For each $B_i \in L^X$, $B_i \odot \mathcal{H}_i \in U(X)$ such that

$$\begin{aligned}
(B_i \odot \mathcal{H}_i)(A)(y) & = B_i(y) \odot \bigvee_{x \in X} (A(x) \odot R_i(x, y)) \\
& = \bigvee_{x \in X} (B_i(y) \odot A(x) \odot R_i(x, y)).
\end{aligned}$$

Example 3.6. Let $(L=[0,1],\odot,\rightarrow,^*)$ be a complete residuated lattice with a negation defined by

$$x\odot y=(x+y-1)\vee 0,\ \ x\rightarrow y=(1-x+y)\wedge 1,\ x^*=1-x.$$

Let $X=\{x,y,z\}$ be a set.

(1) Define $R_1,R_2\in L^{X\times X}$ as follows

$$R_1=\begin{pmatrix}0.5 & 0.9 & 0.1\\ 0.7 & 0.8 & 0.5\\ 0.9 & 0.6 & 0.7\end{pmatrix},\ R_2=\begin{pmatrix}0.2 & 0.6 & 0.7\\ 0.8 & 1 & 0.3\\ 0.5 & 0.3 & 0.4\end{pmatrix}.$$

Since $\mathcal{H}_i(A)(y)=\bigvee_{x\in X}(A(x)\odot R_i(x,y))$ for $i=1,2$,

$$\begin{array}{l} e^y_{U(X)}(\mathcal{H}_1,\mathcal{H}_2)=\bigwedge_{A\in L^X} e^y_{L^X}(\mathcal{H}_1(A),\mathcal{H}_2(A))\\ =\bigwedge_{A\in L^X}(\mathcal{H}_1(A)(y)\rightarrow\mathcal{H}_2(A)(y))\\ \leq(\mathcal{H}_1(1_x)(y)\rightarrow\mathcal{H}_2(1_x)(y))\\ =R_1(x,y)\rightarrow R_2(x,y).\end{array}$$

Hence $e^y_{U(X)}(\mathcal{H}_1,\mathcal{H}_2)\leq\bigwedge_{x\in X}(R_1(x,y)\rightarrow R_2(x,y))$. By Lemma 2.2(8,14),

$$\begin{array}{l} e^y_{U(X)}(\mathcal{H}_1,\mathcal{H}_2)=\bigwedge_{A\in L^X}(\mathcal{H}_1(A)(y)\rightarrow\mathcal{H}_2(A)(y))\\ =\bigwedge_{A\in L^X}(\bigvee_{x\in X}(A(x)\odot R_1(x,y))\rightarrow\bigvee_{w\in X}(A(w)\odot R_2(w,y))\\ \geq\bigwedge_{A\in L^X}\bigwedge_{x\in X}((A(x)\odot R_1(x,y))\rightarrow(A(x)\odot R_2(x,y))\\ \geq\bigwedge_{x\in X}(R_1(x,y)\rightarrow R_2(x,y))\end{array}$$

Thus $e^y_{U(X)}(\mathcal{H}_1,\mathcal{H}_2)=\bigwedge_{x\in X}(R_1(x,y)\rightarrow R_2(x,y))=0.7$. Moreover,

$$\begin{array}{l} e^x_{U(X)}(\mathcal{H}_1,\mathcal{H}_2)=0.6,\ \ e^z_{U(X)}(\mathcal{H}_1,\mathcal{H}_2)=0.7\\ e^x_{U(X)}(\mathcal{H}_2,\mathcal{H}_1)=0.7,\ \ e^y_{U(X)}(\mathcal{H}_2,\mathcal{H}_1)=0.8,\\ e^z_{U(X)}(\mathcal{H}_2,\mathcal{H}_1)=0.4.\end{array}$$

Theorem 3.7. We define $e^x_{M(X)}:M(X)\times M(X)\rightarrow L$ as follows:

$$\begin{array}{ll} e^x_{M(X)}(\mathcal{J}_1,\mathcal{J}_2) & =\bigwedge_{A\in L^X} e^x_{L^X}(\mathcal{J}_1(A),\mathcal{J}_2(A))\\ & =\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x)\rightarrow\mathcal{J}_2(A)(x)).\end{array}$$

Then the following properties hold.

(1) $(M(X), e^x_{M(X)})$ is a fuzzy preordered set. If $e^x_{M(X)}(\mathcal{J}_1,\mathcal{J}_2)=\top$ and $e^x_{M(X)}(\mathcal{J}_2,\mathcal{J}_1)=\top$ for all $x\in X$, then $\mathcal{J}_1=\mathcal{J}_2$.

(2) $e^x_{M(X)}(\mathcal{J}_1,\mathcal{J}_2)=\bigwedge_{y\in X}(\mathcal{J}_1(\top^*_y)(x)\to \mathcal{J}_2(\top^*_y)(x))$.

(3) For $\Psi: M(X)\to L^X$, there exists $\sqcap\Psi\in M(X)$.

Proof (1) It is similarly proved as Theorem 3.1.

(2) For $A=\bigwedge_{y\in X}(A^*(y)\to \top^*_y)$, we have

$$\mathcal{J}_i(A)(x)=\mathcal{J}_i(\bigwedge_{y\in X}(A^*(y)\to\top^*_y))(x)=\bigwedge_{y\in X}(A^*(y)\to\mathcal{J}_i(\top^*_y)(x)).$$

$$\begin{array}{l} e^x_{M(X)}(\mathcal{J}_1,\mathcal{J}_2)=\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x)\to\mathcal{J}_2(A)(x))\\ \le\bigwedge_{y\in X}(\mathcal{J}_1(\top^*_y)(x)\to\mathcal{J}_2(\top^*_y)(x)).\end{array}$$

By Lemma 2.2(8,10), we have

$$\begin{array}{l} e^x_{M(X)}(\mathcal{J}_1,\mathcal{J}_2)=\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x)\to\mathcal{J}_2(A)(x))\\ =\bigwedge_{A\in L^X}(\bigwedge_{y\in X}(A^*(y)\to\mathcal{J}_1(\top^*_y)(x))\to\bigwedge_{y\in X}(A^*(y)\to\mathcal{J}_2(\top^*_y)(x)))\\ \ge\bigwedge_{A\in L^X}\bigwedge_{y\in X}((A^*(y)\to\mathcal{J}_1(\top^*_y)(x))\to(A^*(y)\to\mathcal{J}_2(\top^*_y)(x)))\\ \ge\bigwedge_{y\in X}(\mathcal{J}_1(\top^*_y)(x)\to\mathcal{J}_2(\top^*_y)(x)).\end{array}$$

Hence $e^x_{M(X)}(\mathcal{J}_1,\mathcal{J}_2)=\bigwedge_{y\in X}(\mathcal{J}_1(\top^*_y)(x)\to\mathcal{J}_2(\top^*_y)(x))$.

(3) For $\Psi: M(X)\to L^X$, we define $Q: L^X\to L^X$

$$Q(A)(x)=\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to\mathcal{J}(A)(x)).$$

$$\begin{array}{rl} Q(\bigwedge_{i\in\Gamma}A_i)(x) & =\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to\mathcal{J}(\bigwedge_{i\in\Gamma}A_i)(x))\\ & =\bigwedge_{i\in\Gamma}(\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to\mathcal{J}(A_i)(x)))\\ & =\bigwedge_{i\in\Gamma}Q(A_i)(x),\\ Q(\alpha\to A)(x) & =\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to\mathcal{J}(\alpha\to A)(x))\\ & =\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to(\alpha\to\mathcal{J}(A)(x)))\\ & =\bigwedge_{\mathcal{J}\in M(X)}(\alpha\to(\Psi(\mathcal{J})(x)\to\mathcal{J}(A)(x)))\\ & =\alpha\to Q(A)(x).\end{array}$$

Hence $Q\in M(X)$.

$$\begin{array}{l} e^x_{M(X)}(\mathcal{J}_1,Q)=\bigwedge_{A\in L^X}e^x_{L^X}(\mathcal{J}_1(A),Q(A))\\ =\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x)\to\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to\mathcal{J}(A)(x)))\\ =\bigwedge_{A\in L^X}\bigwedge_{\mathcal{J}\in M(X)}(\mathcal{J}_1(A)(x)\to(\Psi(\mathcal{J})(x)\to\mathcal{J}(A)(x)))\\ =\bigwedge_{A\in L^X}\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to(\mathcal{J}_1(A)(x)\to\mathcal{J}(A)(x)))\\ =\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x)\to\mathcal{J}(A)(x)))\\ =\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x)\to e^x_{M(X)}(\mathcal{J}_1,\mathcal{J}))=e^x_{M(X)}(\mathcal{J}_1,\sqcap\Psi).\end{array}$$

Hence $\sqcap\Psi(A)(x) = Q(A)(x) = \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x) \to \mathcal{J}(A)(x))$;i.e. $\sqcap\Psi = \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J}) \to \mathcal{J}) \in M(X)$.

Theorem 3.8. For $\Psi : M(X) \to L^X$, there exists $\sqcap\Psi \in M(X)$ iff for all $\mathcal{J}_i \in M(X)$, $\bigwedge_{i\in\Gamma} \mathcal{J}_i \in M(X)$ and $B_i \to \mathcal{J}_i \in M(X)$.

Proof ($\Rightarrow$) Put $\Psi_i : M(X) \to L^X$ as $\Psi_i(\mathcal{J}_i) = B_i$ and $\Psi_i(\mathcal{J}) = 0$, otherwise. Then

$$(\sqcap\Psi_i)(A) = \bigwedge_{\mathcal{J}\in M(X)} (\Psi_i(\mathcal{J}) \to \mathcal{J}(A)) = B_i \to \mathcal{J}_i(A)$$

Hence $\sqcap\Psi_i = B_i \to \mathcal{J}_i \in M(X)$.

Let $\{\mathcal{J}_i \in M(X) \mid i \in \Gamma\}$ be given. Define $\Psi : L^X \to L$ as $\Psi(\mathcal{J}_i) = \top_X$ for $i \in \Gamma$ and $\Psi(\mathcal{J}) = 0$ otherwise. By Theorem 3.7(3),

$$\sqcap\Psi(A) = \bigwedge_{\mathcal{J}\in M(X)} (\Psi(\mathcal{J}) \to \mathcal{J}(A)) = \bigwedge_{i\in\Gamma} \mathcal{J}_i(A).$$

Hence $\sqcap\Psi = \bigwedge_{i\in\Gamma} \mathcal{J}_i \in M(X)$.

($\Leftarrow$) For $\Psi : M(X) \to L^X$, $\sqcap\Psi = \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J}) \to \mathcal{J}) \in M(X)$ and

$$\begin{aligned}
& e^x_{M(X)}(\mathcal{J}_1, \sqcap\Psi) = \textstyle\bigwedge_{A\in L^X} e^x_{L^X}(\mathcal{J}_1(A), \sqcap\Psi(A)) \\
&= \textstyle\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x) \to \sqcap\Psi(A)(x)) \\
&= \textstyle\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x) \to \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x) \to \mathcal{J}(A)(x))) \\
&= \textstyle\bigwedge_{A\in L^X} \bigwedge_{\mathcal{J}\in M(X)}(\mathcal{J}_1(A)(x) \to (\Psi(\mathcal{J})(x) \to \mathcal{J}(A)(x))) \\
&= \textstyle\bigwedge_{A\in L^X} \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x) \to (\mathcal{J}_1(A)(x) \to \mathcal{J}(A)(x))) \\
&= \textstyle\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x) \to \bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x) \to \mathcal{J}(A)(x))) \\
&= \textstyle\bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J})(x) \to e^x_{M(X)}(\mathcal{J}_1, \mathcal{J}))
\end{aligned}$$

Thus, $\sqcap\Psi = \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J}) \to \mathcal{J}) \in M(X)$

Theorem 3.9. For each $\mathcal{H} \in U(X)$, we define $\mathcal{J}_H(A) = (\mathcal{H}(A^*))^*$ for all $A \in L^X$. We have the following properties.

(1) $\mathcal{J}_H \in M(X)$.

(2) $e^x_{M(X)}(\mathcal{J}_{H_1}, \mathcal{J}_{H_2}) = e^x_{U(X)}(\mathcal{H}_2, \mathcal{H}_1)$.

(3) For $\Psi : M(X) \to L^X$, there exist $\sqcap\Psi \in M(X)$ with $\Psi(\mathcal{J}_H) = \Phi(\mathcal{H}^*)$ such that

$$\sqcap\Psi(A)(x) = \bigwedge_{\mathcal{J}_H\in M(X)} (\Psi(\mathcal{J}_H)(x) \to \mathcal{J}_H(A)(x)).$$

Proof (1) We have $\mathcal{J}_H \in M(X)$ from:

$$\begin{array}{ll}\mathcal{J}_H(\bigwedge_{i\in\Gamma} A_i) & = (\mathcal{H}(\bigvee_{i\in\Gamma} A_i^*))^* = (\bigvee_{i\in\Gamma} \mathcal{H}(A_i^*))^* = \bigwedge_{i\in\Gamma} \mathcal{J}_H(A_i) \\ \mathcal{J}_H(\alpha \to A) & = (\mathcal{H}(\alpha \odot A^*))^* = (\alpha \odot \mathcal{H}(A^*))^* = \alpha \to \mathcal{J}_H(A).\end{array}$$

(2)

$$\begin{array}{l} e^x_{M(X)}(\mathcal{J}_{H_1}, \mathcal{J}_{H_2}) = \bigwedge_{A\in L^X} e^x_{L^X}(\mathcal{J}_{H_1}(A), \mathcal{J}_{H_2}(A)) \\ = \bigwedge_{A\in L^X}(\mathcal{J}_{H_1}(A)(x) \to \mathcal{J}_{H_2}(A)(x)) \\ = \bigwedge_{A\in L^X}(\mathcal{H}_1^*(A^*)(x) \to \mathcal{H}_2^*(A^*)(x)) \\ = \bigwedge_{A\in L^X}(\mathcal{H}_2(A^*)(x) \to \mathcal{H}_1(A^*)(x)) \\ = e^x_{U(X)}(\mathcal{H}_2, \mathcal{H}_1).\end{array}$$

$$\begin{array}{l} e^x_{M(X)}(\mathcal{J}_1, \sqcap\Psi) = \bigwedge_{A\in L^X} e^x_{L^X}(\mathcal{J}_1(A), \sqcap\Psi(A)) \\ = \bigwedge_{A\in L^X}(\mathcal{J}_1(A)(x) \to \sqcap\Psi(A)(x)) \\ = \bigwedge_{A^*\in L^X}((\sqcap\Psi(A^*))^*(x) \to \mathcal{J}_1^*(A^*)(x)) \\ = \bigwedge_{A^*\in L^X}((\sqcap\Psi(A^*))^*(x) \to \mathcal{H}_1(A)(x))\end{array}$$

(3) Put $\Psi : M(X) \to L^X$ as $\Psi(\mathcal{J}_H) = \Phi(\mathcal{H})$. Then

$$\begin{array}{l} e^x_{U(X)}(\sqcup\Phi, \mathcal{H}_1) = \bigwedge_{A\in L^X} e^x_{L^X}(\sqcup\Phi(A), \mathcal{H}_1(A)) \\ = \bigwedge_{A\in L^X}(\sqcup\Phi(A)(x) \to \mathcal{H}_1(A)(x)) \\ = \bigwedge_{A\in L^X}(\mathcal{H}_1^*(A)(x) \to (\sqcup\Phi(A))^*(x)) \\ = \bigwedge_{A\in L^X}(\mathcal{H}_1^*(A)(x) \to (\bigvee_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})(x) \odot \mathcal{H}(A)(x)))^*) \\ = \bigwedge_{A\in L^X}(\mathcal{H}_1^*(A)(x) \to \bigwedge_{\mathcal{H}\in U(X)}(\Phi(\mathcal{H})(x) \to \mathcal{H}^*(A)(x))) \\ = \bigwedge_{A\in L^X}(\mathcal{J}_{H_1}(A^*)(x) \to \bigwedge_{\mathcal{J}_H\in M(X)}(\Psi(\mathcal{J}_H)(x) \to \mathcal{J}_H(A^*)(x))) \\ = e^x_{L^X}(\mathcal{J}_{H_1}(A), \bigwedge_{\mathcal{J}_H\in M(X)}(\Psi(\mathcal{J}_H)(x) \to \mathcal{J}_H(A)) \\ = \bigwedge_{\mathcal{H}^*\in M(X)}(\Phi(\mathcal{H})(x) \to \bigwedge_{A\in L^X}(\mathcal{H}_1^*(A)(x) \to \mathcal{H}^*(A)(x))) \\ = \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J}_H)(x) \to \bigwedge_{A\in L^X}(\mathcal{J}_{H_1}(A)(x) \to \mathcal{J}_H(A)(x))) \\ = \bigwedge_{\mathcal{J}\in M(X)}(\Psi(\mathcal{J}_H)(x) \to e^x_{M(X)}(\mathcal{J}_{H_1}, \mathcal{J}_H)\end{array}$$

Then $\sqcap\Psi(A)(x) = \bigwedge_{\mathcal{J}_H\in M(X)}(\Psi(\mathcal{J}_H)(x) \to \mathcal{J}_H(A)(x))$.

Example 3.10. (1) Let $R \in L^{X\times X}$ be a fuzzy relation. Define $\mathcal{J} : L^X \to L^X$ as follows

$$\mathcal{J}(A)(y) = \bigwedge_{x\in X}(R(x, y) \to A(x)).$$

Since $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$ and $\mathcal{J}(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{J}(A_i)$, $\mathcal{J}$ is a lower approximation operator; i.e. $\mathcal{J} \in M(X)$.

(2) Let $\{R_i \in L^{X\times X} \mid i \in \Gamma\}$ be a family of fuzzy relations. Then $\mathcal{J}_i$ is a lower approximation operator such that

$$\mathcal{J}_i(A)(y) = \bigwedge_{x\in X}(R_i(x,y) \to A(x)).$$

By Theorem 3.8, $\bigwedge_{i\in\Gamma}\mathcal{J}_i \in M(X)$. For each $B_i \in L^X$, $B_i \to \mathcal{J}_i \in M(X)$ such that

$$\bigwedge_{i\in\Gamma}\mathcal{J}_i(A)(y) = \bigwedge_{x\in X}((\bigvee_{i\in\Gamma}R_i(x,y)) \to A(x)).$$

$$\begin{aligned}(B_i \to \mathcal{J}_i)(A)(y) = B_i(y) \to \bigwedge_{x\in X}(R_i(x,y) \to A(x))\\ = \bigwedge_{x\in X}(B_i(y)\odot R_i(x,y) \to A(x))\end{aligned}$$

Example 3.11. Let $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with a negation defined by

$$x\odot y = (x+y-1)\vee 0,\ \ x\to y = (1-x+y)\wedge 1,\ x^* = 1-x.$$

Let $X = \{x,y,z\}$ be a set.

(1) Define $R_1, R_2 \in L^{X\times X}$ as follows

$$R_1 = \begin{pmatrix} 0.5 & 0.9 & 0.1\\ 0.7 & 0.8 & 0.5\\ 0.9 & 0.6 & 0.7\end{pmatrix}, R_2 = \begin{pmatrix} 0.2 & 0.6 & 0.7\\ 0.8 & 1 & 0.3\\ 0.5 & 0.3 & 0.4\end{pmatrix}.$$

Since $\mathcal{J}_i(A)(y) = \bigwedge_{x\in X}(R_i(x,y) \to A(x))$ for $i = 1,2$,

$$\begin{aligned}&e^y_{M(X)}(\mathcal{J}_1,\mathcal{J}_2) = \textstyle\bigwedge_{A\in L^X} e^y_{L^X}(\mathcal{J}_1(A),\mathcal{J}_2(A))\\ &= \textstyle\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(y) \to \mathcal{J}_2(A)(y))\\ &\le (\mathcal{J}_1(\top^*_x)(y) \to \mathcal{J}_2(\top^*_x)(y))\\ &= R^*_1(x,y) \to R^*_2(x,y) = R_2(x,y) \to R_1(x,y).\end{aligned}$$

Hence $e^y_{M(X)}(\mathcal{J}_1,\mathcal{J}_2) \le \bigwedge_{x\in X}(R_2(x,y) \to R_1(x,y))$.

$$\begin{aligned}&e^y_{M(X)}(\mathcal{J}_1,\mathcal{J}_2) = \textstyle\bigwedge_{A\in L^X}(\mathcal{J}_1(A)(y) \to \mathcal{J}_2(A)(y))\\ &= \textstyle\bigwedge_{A\in L^X}(\bigwedge_{x\in X}(R_1(x,y) \to A(x)) \to \bigwedge_{w\in X}(R_2(w,y) \to A(w))\\ &\ge \textstyle\bigwedge_{A\in L^X}\bigwedge_{x\in X}(R_1(x,y) \to A(x)) \to (R_2(x,y) \to A(x))\\ &\ge \textstyle\bigwedge_{x\in X}(R_2(x,y) \to R_1(x,y))\end{aligned}$$

Thus $e^y_{M(X)}(\mathcal{J}_1, \mathcal{J}_2) = \bigwedge_{x\in X}(R_2(x, y) \to R_1(x, y)) = 0.8$. Moreover,

$$e^x_{M(X)}(\mathcal{J}_1, \mathcal{J}_2) = 0.7, \quad e^z_{M(X)}(\mathcal{J}_1, \mathcal{J}_2) = 0.4$$
$$e^x_{M(X)}(\mathcal{J}_2, \mathcal{J}_1) = 0.6, \quad e^y_{M(X)}(\mathcal{J}_2, \mathcal{J}_1) = 0.7,$$
$$e^z_{M(X)}(\mathcal{J}_2, \mathcal{J}_1) = 0.7.$$

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