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HYBRID ONE STEP BLOCK METHOD FOR THE SOLUTION OF THIRD ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we develop one point, five hybrid points, block method using the method of collocation and interpolation of the power series approximate solution to develop a linear multistep method with a constant step-length which was implemented in block method. The basic properties of the derived method was investigated and found to be constant, zero stable and convergent. The region of absolute stability was equally investigated. The derived method was tested on some numerical examples.

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1. Introduction

Since the advent of computer, numerical methods are important tool for approximate solutions of differential equations. This paper considers the numerical solution to third order initial value problems of the form.

$$y''' = f\left(x, y, y', y''\right),\ y^{(k)}\left(x_0\right) = y_0^{(k)}, k = 0, 1, 2 \tag{1}$$

where x_0 is the initial point, $y_0^{(k)}$are the solutions at the initial point, f is assumed to be continuos within the interval of integration and satisfied the existence and uniqueness condition given in Awoyemi *et al.* [1].

Direct methods for the solution of higher order ordinary differential equation has been established in literature to be better than the method of reduction in terms of approximation, time of execution and cost of implementation. (Adesanya *et al.* [2] James *et al.* [3], Kayode and Obaruah [4], Jator [5]).

Method of collocation and interpolation of power series approximate solution to generate a continuous linear multi-step method has been discussed by many authors, among them are: Awoyemi and Idowu [6], Majid *et al.* [7], Olabode and Yusuf [8], Adesanya *et al.* [9]. These authors developed method which is implemented in predictor-corrector method or block method. Block method has advantage over predictor-corrector method for being cost effective and give better approximations.

Hybrid method has the advantage of reducing the step number of a method and still remains zero stable. This method while retaining certain characteristics of the continuous linear multistep method share with the RungeKutta method the property of utilizing data at other points other than the step points (Anake *et al.* [10]).

In this paper, we proposed an hybrid one step, five hybrid points method for the solution of third order initial value problems which is implemented in block method. The paper is arranged as follows: Section two considers the algorithms involved in deriving the numerical scheme. Section three discussed the analysis of the basic properties of the method which include the consistency, zero stability and region of absolute stability of the method. Section four discussed the numerical experiment where our method is tested on some numerical examples and the results are adequately discussed. Section five gives conclusion and necessary recommendations.

2. Methodology

We consider a power series approximate solution in the form

$$y\left(x\right)=\sum_{j=0}^{r+s-1}a_jx^j \tag{2}$$

where r and s are the number of interpolation and collocation points respectively.

Substituting the third derivatives of (2) into (1) gives

$$f\left(x,y,y',y''\right)=\sum_{j=0}^{r+s-1}j\left(j-1\right)\left(j-2\right)a_jx^{j-3} \tag{3}$$

Interpolating (2) at $x_{n+r},r=0,\frac{1}{6},\frac{1}{3}$ and collocating (3) at $x_{n+s},s=0\left(\frac{1}{6}\right)1$, gives system of non-linear equation of the form

$$y_{n+r}=\sum_{i=0}^{9}a_ix_{n+r}^i,r=0,\frac{1}{6},\frac{1}{3} \tag{4}$$

and

$$f_{n+s}=\sum_{i=3}^{9}j\left(j-1\right)\left(j-2\right)a_ix_{n+s}^i,s=0\left(\frac{1}{6}\right)1 \tag{5}$$

Solving (4) and (5) for the unknown constant $a_j's$ using Gaussian elimination method, substituting the solution into (2) and implementing in block method give a continuous block method in the form.

$$y^{(m)}\left(t\right)=\sum_{j=0}^{2}\frac{(jh)^{(m)}}{m!}y_n^{(m)}+h^3\left(\sum_{j=0}^{1}\sigma_j\left(t\right)f_{n+j}+\sigma_k\left(t\right)f_{n+k}\right),k=\frac{1}{6}\left(\frac{1}{6}\right)1 \tag{6}$$

where

$$\sigma_0=\frac{1}{8400}\left(1080t^9-5670t^8+12600t^7-15435t^6+11368t^5-5145t^4-1400t^3\right)$$

$$\sigma_{\frac{1}{6}}=-\frac{1}{350}\left(270t^9-1350t^8+2790t^7-3045t^6+1827t^5-525t^4\right)$$

$$\sigma_{\frac{1}{3}}=\frac{1}{560}\left(1080t^9-5130t^8+9864t^7-9681t^6+4914t^5-1050t^4\right)$$

$$\sigma_{\frac{1}{2}} = -\frac{1}{105}\left(270t^9 - 1215t^8 + 2178t^7 - 1953t^6 + 889t^5 - 193t^4\right)$$

$$\sigma_{\frac{2}{3}} = \frac{1}{560}\left(1080t^9 - 4590t^8 + 7704t^7 - 6447t^6 + 2772t^5 - 525t^4\right)$$

$$\sigma_{\frac{5}{6}} = -\frac{1}{350}\left(270t^9 - 1080t^8 + 1710t^7 - 1365t^6 + 567t^5 - 105t^4\right)$$

$$\sigma_1 = \frac{1}{8400}\left(1080t^9 - 4050t^8 + 6120t^7 + 4725t^6 + 1918t^5 - 350t^4\right)$$

$t = \frac{x - x_n}{h}$

Evaluating (5) at $t = \frac{1}{6}\left(\frac{1}{6}\right)1$, gives a discrete block formula of the form

$$A^{(0)}Y_m^{(i)} = \sum_i^2 h^{(i)} e_i y_n^{(i)} + h^3 d_i f\left(y_n\right) + h^3 b_i f\left(Y_m\right) \tag{7}$$

where

$A^{(0)} = 6 \times 6$ identity matrix

when $i = 0$,

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{72} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{18} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{9} \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{72} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \; d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{343801}{783820800} \\ 0 & 0 & 0 & 0 & 0 & \frac{6887}{3061800} \\ 0 & 0 & 0 & 0 & 0 & \frac{1959}{358400} \\ 0 & 0 & 0 & 0 & 0 & \frac{3863}{382725} \\ 0 & 0 & 0 & 0 & 0 & \frac{505625}{31352832} \\ 0 & 0 & 0 & 0 & 0 & \frac{33}{1400} \end{bmatrix}$$

$$b_0 = \begin{bmatrix} \frac{6031}{9331200} & -\frac{32981}{52254720} & \frac{5177}{9797760} & \frac{-15107}{52254720} & \frac{5947}{65318400} & \frac{-9809}{783820800} \\ \frac{1499}{255150} & \frac{-233}{58320} & \frac{52}{15309} & \frac{-379}{204120} & \frac{149}{255150} & \frac{-491}{6123600} \\ \frac{1599}{89600} & \frac{-537}{71680} & \frac{1}{120} & \frac{-327}{71680} & \frac{129}{89600} & \frac{-71}{358400} \\ \frac{4664}{127575} & \frac{-226}{25515} & \frac{272}{15309} & \frac{-31}{3645} & \frac{344}{127575} & \frac{-142}{382725} \\ \frac{162125}{2612736} & \frac{-85625}{10450944} & \frac{66875}{1959552} & \frac{-119375}{10450944} & \frac{1625}{373248} & \frac{-18625}{31352832} \\ \frac{33}{350} & \frac{-3}{560} & \frac{2}{35} & \frac{-3}{280} & \frac{3}{350} & \frac{-1}{1200} \end{bmatrix}$$

when $i = 1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{4354560} \\ 0 & 0 & 0 & 0 & 0 & \frac{1027}{68040} \\ 0 & 0 & 0 & 0 & 0 & \frac{253}{10752} \\ 0 & 0 & 0 & 0 & 0 & \frac{272}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225}{870912} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{840} \end{bmatrix},$$

$$b_1 = \begin{bmatrix} \frac{275}{20736} & \frac{-5717}{483840} & \frac{10621}{1088640} & \frac{-7703}{1451520} & \frac{403}{241920} & \frac{-199}{870912} \\ \frac{97}{1890} & \frac{-2}{81} & \frac{197}{8505} & \frac{-97}{7560} & \frac{23}{5670} & \frac{-19}{34020} \\ \frac{165}{1792} & \frac{-267}{17920} & \frac{5}{128} & \frac{-363}{17920} & \frac{57}{8960} & \frac{-47}{53760} \\ \frac{376}{2835} & \frac{-2}{945} & \frac{656}{8505} & \frac{-2}{81} & \frac{8}{945} & \frac{-2}{1701} \\ \frac{8375}{48384} & \frac{3125}{290304} & \frac{25625}{217728} & \frac{-625}{96768} & \frac{275}{20736} & \frac{-1375}{870912} \\ \frac{3}{14} & \frac{3}{140} & \frac{17}{105} & \frac{3}{280} & \frac{3}{70} & 0 \end{bmatrix}$$

when $i = 2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087}{362880} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139}{22680} \\ 0 & 0 & 0 & 0 & 0 & \frac{137}{2688} \\ 0 & 0 & 0 & 0 & 0 & \frac{143}{2835} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715}{72576} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{840} \end{bmatrix}$$

$$b_2 = \begin{bmatrix} \frac{2713}{15120} & \frac{-15487}{120960} & \frac{293}{2835} & \frac{-6737}{120960} & \frac{263}{15120} & \frac{-863}{362880} \\ \frac{47}{189} & \frac{11}{7560} & \frac{166}{2835} & \frac{-269}{7560} & \frac{11}{945} & \frac{-37}{22680} \\ \frac{27}{112} & \frac{387}{4480} & \frac{17}{105} & \frac{-243}{4480} & \frac{9}{560} & \frac{-29}{13440} \\ \frac{232}{945} & \frac{64}{945} & \frac{752}{2835} & \frac{29}{945} & \frac{8}{945} & \frac{-4}{3835} \\ \frac{725}{3024} & \frac{2125}{24192} & \frac{125}{567} & \frac{3875}{24192} & \frac{235}{3024} & \frac{-275}{72576} \\ \frac{9}{35} & \frac{9}{280} & \frac{34}{105} & \frac{9}{280} & \frac{9}{35} & \frac{41}{840} \end{bmatrix}$$

3. Analysis of the Basic Properties of the Corrector

3.1. Order of the Method

Let the linear operator $\triangle\{y(x):h\}$ associated with the block method be defined as

$$\triangle\{y(x):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{2} h^{(i)}e_i y_n^{(i)} + h^3 d_i f(y_n) + h^3 b_i f(Y_m) \tag{8}$$

expanding (8) in Taylor's series gives

$$\triangle\{y(x):h\} = C_0 y(x) + C_1 hy(x) + ... + C_p h^p y(x) + C_{p+1}h^{p+1}y^{p+1}(x)$$

Definition 1. *Order: The linear operator and associated block method (8) are said to be of order p if* $C_0 = C_1 = ...C_{p+1} = 0$ *and* $C_{p+2} \neq 0$, C_{p+2} *is called the error constant and implies that the local truncation error is given by*

$$t_{n+k} = C_{p+2}h^{p+2}y^{p+2}(x) + 0(h^{p+3})$$

The order of our method is 8 with error constant

$$[3.649(-11), 2.321(-10), 5.981(-10), 1.0827(-09), 1.7375(-09), 2.5516(-09)]^T.$$

3.2. Zero Stability

A block method is said to be zero stable if as h→0 the root $r_j, j = 1(1)k$ of the first characteristics polynomial $\rho(r) = 0$ that is

$$\rho(r) = \left|\sum A^0 R^{k-1}\right| = 0$$

satisfying $\mid R \mid \leq 1$, for those root with $\mid R \mid \leq 1$ must be simple.

For our method

$$\rho(r) = \left| R\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right|$$

$R = 0, 0, 0.0, 0, 1$

hence our method is zero stable

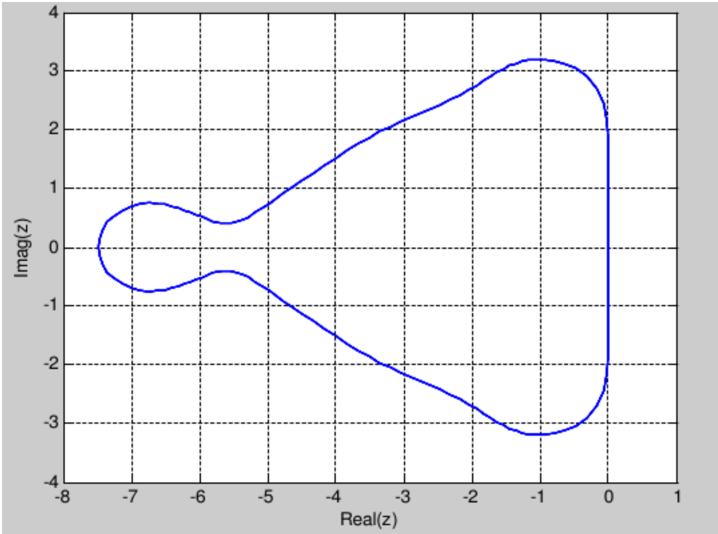

Figure 1: The stability region

3.3. Consistency

A block method is said to be consistent if it has order greater than one.

3.4. Convergence

A block method is said to be convergent if and only if it is consistent and zero stable.

From above, it shows clearly that our method is convergent.

3.5. Stability Region of the Method

Definition 2. *A method is said to be absolutely stable if for a given value of* h*, all the roots* z_s *of the characteristics polynomials* $\pi(z, /) = \rho(z) + /(z) = 0$ *satisfied* $|z_z| < 1, s = 1, 2, ..., n$ *where* $/ = \lambda^3 h^3$ *and* $\lambda = \frac{\partial f}{\partial y}$.

We adopted the boundary locus method to find the region of absolute stability of the method. Substituting $y''' = \lambda^3 y$, $y'' = \lambda^2 y$ and $y' = \lambda y$ into the block formula and writing $r = e^{i\theta}$, the stability region is shown in Figure 1.

h	y	Error	EABLM
3	0.120916084328232	9.166724(-06)	-
2	0.120908010329808	1.092725(-06)	-
1	0.120906924474128	6.869709(-09)	-
0.1	0.120906917604419	9.714451(-17)	1.7458(-14)

Table 1: Results of Problem I, Note: EABLM → Error ssee Adesanya et al, [11]

4. Numerical Experiments

We considered three numerical examples to test the efficiency of our method

Example 3. *We consider a special third order initial value problem*

$$y''' = 3\sin x,\ \ y(0) = 1, y'(0) = 0, y''(0) = -2, 0 \leq x \leq 1, h = 0.1$$

Exact solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Source: Adesanya *et al.* [11]

The theoretical solution at $x = 1$ is $y(1) \simeq 0.120906917604419$.The error were obtained at $x = 1$ using our new method at a fixed step-size $h = 3, 2, 1, 0.1$. We compared our result with Adesanya *et al.* [11] where they proposed block method of order six with step-size 0.01. The results are shown in Table I.

Example 4. *We consider a linear third order initial value problem*

$$y''' + y' = 0,\ \ y(0) = 0, y'(0) = 1, y''(0) = 2, h = 0.1$$

Exact solution: $y(x) = 2(1 - \cos x) + \sin x$

Source: Adesanya *et al.* [11]

The theoretical solution at $x = 1$ is $y(1) \simeq 1.760866373071617$.The error were obtained at $x = 1$ using our new method at a fixed step-size $h = 3, 2, 1, 0.1$. We compared our result with Adesanya *et al.* [11] where they proposed block method of order six. The results are shown in Table II..

Example 5. *we consider a linear third order initial value problem*

$$y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$$

$$y(0) = -2, y'(0) = -8, y''(0) = -12$$

Exact solution: $y(x) = -2e^{3x} + e^{-2x} + x^2 - 1$

h	y	Error	EABLM
3	1.760838982583774	2.739049(-05)	-
2	1.760865218172317	1.154899(-06)	-
1	1.760866397013195	2.394158(-08)	-
0.1	1.76088373033251	3.836531(-11)	1.0693(-07)

Table 2: Results of Problem II

h	y	Error	EAPCM
0.1	-40.037079448651731	1.340886(-03)	2.12(–02)
0.05	-40.035831152134719	9.258900(-05)	2.92(-03)
0.025	-40.035744638502806	6.075364(-06)	3.82(-04)
0.0125	-40.035738952091137	3.889526(-07)	4.90(-05)
0.00625	-40.035738587740624	2.460220(-08)	6.20(-06)

Table 3: Results of Problem III, Note: EAPCM $\rightarrow$ Error in Awoyemi and Idowu [11].

Source: Awoyemi and Idowu [6]

The theoretical solution at $x = 1$ is $y(1) \simeq -40.035738563138423$.The error were obtained at $x = 1$ using our new method at a fixed step-size $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$ We compared our result with Awoyemi and Idowu[6] where they proposed a method of order seven implemented in predictor corrector method over an overlapping interval. The results are shown in Table III

4.1. Discussion of Result

We have considered three numerical examples in this paper. Though it is expected that our method should perform better than Adesanya et al. [11] but despite the high step-size used in our method, it still performed better as shown in Tables I and II. Awoyemi and Idowu [6] solved problem III using an implicit order seven method. Table III showed that our method gave better approximation

5. Conclusion

We have developed one step hybrid method for the solution of third order initial value problems in this paper. Our new method is convergent and the stability region was shown to be A-stable. This method is cost effective in term of cost of developing the scheme and the time of execution. It must be noted that the method adopted Taylor series expansion to generate the starting value, hence it is not a self starting block method. The results showed that the method is more accurate than the predictor corrector method and the self starting block method.

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