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GRADED SUBMODULES WITH PSEUDO IRREDUCIBLE, PSEUDO PRIME AND STRICTLY NON-PRIME COMPONENTS

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Abstract: Let G be a group. Let R be a commutative G-graded ring, M be a graded R-module and N be a graded R-submodule of M. In this paper, we study some cases when R is strongly graded ring and the component N_e of N is strictly non-prime, pseudo prime or pseudo irreducible R_e-submodule. For example, we prove that if R is strongly graded, the components of M are multiplication and N_e is pseudo irreducible, then N_g is pseudo prime for all $g \in G$.

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1. Introduction

A ring R with unity 1 graded by a group G will means that $R = \bigoplus_{g\in G} R_g$

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where R_g is an additive subgroup of R and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is an equality, then the ring is called strongly graded. Clearly, R_e is a subring of R with $1 \in R_e$. An R-module is said to be graded if $M = \bigoplus_{g\in G} M_g$ for a family of subgroups $\{M_g\}_{g\in G}$ of M such that $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Clearly, M_g is an R_e-module for all $g \in G$. In a similar way, we define a strongly graded module. The ring R is strongly graded if and only if every graded R-module is strongly graded. An R-submodule N of a graded R-module M is said to be graded R-submodule of M if $N = \bigoplus_{g\in G}(N \bigcap M_g)$. For more details, we refer the readers to [4], as well as [5].

An R-module is said to be multiplication if for any R-submodule N of M, $N = IM$ for some ideal I of R. One can easily prove that if M is multiplication, then $N = (N : M)M$ for every R-submodule N of M where $(N : M) = \{r \in R : rM \subseteq N\}$ that is an ideal of R. In [1], Abd El-Bast and Smith have shown that N is prime submodule if and only if $(N : M)$ is prime ideal provided that M is multiplication. Let N, K be two R-submodules of multiplication module M. Then $N = IM$ and $K = JM$ for some ideals I, J of R and then the multiplication NK is defined by Ameri in [2] as $NK = (IM)(JM) = IJM$.

In this paper, we study graded submodules with multiplication components for pseudo prime and pseudo irreducible properties. Pseudo irreducible ideals and pseudo prime ideals are defined by McAdam and Swan in [3]. An ideal I of a ring R is said to be pseudo irreducible if one can not write $I = JK$ with $J + K = R$ where $I \neq R$, $J \neq R$ are ideals of R. Also, I is said to be pseudo prime if whenever $J + K = R$ with $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$ where I, J are ideals of R. Similarly, an R-submodule N of a multiplication R-module M is said to be pseudo irreducible if whenever $N = UV$ for some R-submodules U, V of M such that $U + V = M$, then either $U = M$ or $V = M$. Also, N is said to be pseudo prime if whenever U, V are R-submodules of M such that $UV \subseteq N$ and $U + V = M$, then either $U \subseteq N$ or $V \subseteq N$.

Throughout this paper, unless stated otherwise, R is commutative nontrivially graded ring.

2. Results

In this section, we introduce and prove the main results of the paper.

Lemma 2.1. *Let R be a G-graded ring and M be graded R-module. If N is a graded R-module of M, then $R_gN_h \subseteq N_{gh}$ for all $g, h \in G$.*

Proof. Let $g, h \in G$. Then $R_gN_h \subseteq R_gM_h \subseteq M_{gh}$. On the other hand,

$R_gN_h \subseteq RN = N$ and hence, $R_gN_h \subseteq N \bigcap M_{gh} = N_{gh}$. □

Theorem 2.2. *Let R be a strongly G-graded ring and M be a graded R-module. Suppose that M_e is faithful multiplication R_e-module and N is a graded R-submodule of M. If N_e is pseudo irreducible R_e-submodule of M_e, then $(N_g : M_g)$ is pseudo irreducible ideal of R_e for all $g \in G$.*

Proof. Let $g \in G$ and suppose that $(N_g : M_g) = AB$ where A and B are two ideals of R_e satisfy $A + B = R_e$. Then $ABM_e = ABR_{g^{-1}}M_g = R_{g^{-1}}ABM_g \subseteq R_{g^{-1}}N_g \subseteq N_e$, i.e., $AB \subseteq (N_e : M_e)$. Let $r \in (N_e : M_e)$. Then $rM_g = rR_gM_e = R_grM_e \subseteq R_gN_e \subseteq N_g$, i.e., $r \in (N_g : M_g) = AB$. Hence, $AB = (N_e : M_e)$. Since M_e is multiplication, $N_e = (N_e : M_e)M_e = ABM_e = (AM_e)(BM_e)$ with $M_e = R_eM_e = (A + B)M_e = AM_e + BM_e$. Since N_e is pseudo irreducible, either $AM_e = M_e$ or $BM_e = M_e$ and since M_e is faithful, either $A = R_e$ or $B = R_e$. □

Theorem 2.3. *Let R be a strongly G-graded ring and M be a graded R-module. Suppose that M_e is multiplication R_e-module and N is a graded R-submodule of M. If N_e is pseudo prime R_e-submodule of M_e, then $(N_g : M_g)$ is pseudo prime ideal of R_e for all $g \in G$.*

Proof. Let $g \in G$ and suppose that I, J are two ideals of R_e such that $IJ \subseteq (N_g : M_g)$ with $I + J = R_e$. Then $R_{g^{-1}}IM_g$ and $R_{g^{-1}}JM_g$ are R_e-submodules of M_e such that $R_{g^{-1}}IM_g + R_{g^{-1}}JM_g = R_{g^{-1}}(I + J)M_g = R_{g^{-1}}R_eM_g = M_e$ and $(R_{g^{-1}}IM_g)(R_{g^{-1}}JM_g) = R_{g^{-1}}IJM_g \subseteq R_{g^{-1}}N_g \subseteq N_e$. Since N_e is pseudo prime, either $R_{g^{-1}}IM_g \subseteq N_e$ or $R_{g^{-1}}JM_g \subseteq N_e$ and then either $IM_g \subseteq R_gN_e \subseteq N_g$ or $JM_g \subseteq R_gN_e \subseteq N_g$, i.e., $I \subseteq (N_g : M_g)$ or $J \subseteq (N_g : M_g)$. □

Theorem 2.4. *Let R be a strongly G-graded ring and M be a graded R-module such that all components of M are multiplication R_e-modules. Suppose that N is a graded R-submodule of M.Then N_e is pseudo prime R_e-submodule of M_e if and only if N_g is pseudo prime R_e-submodule of M_g for all $g \in G$.*

Proof. Suppose that N_e is pseudo prime R_e-submodule of M_e. Let $g \in G$ and assume that U, V are two R_e-submodules of M_g such that $UV \subseteq N_g$ and $U + V = M_g$. Then $R_{g^{-1}}U$ and $R_{g^{-1}}V$ are R_e-submodules of M_e such that $(R_{g^{-1}}U)(R_{g^{-1}}V) = R_{g^{-1}}UV \subseteq R_{g^{-1}}N_g \subseteq N_e$ and $R_{g^{-1}}U + R_{g^{-1}}V = R_{g^{-1}}(U + V) = R_{g^{-1}}M_g = M_e$. Since N_e is pseudo prime, either $R_{g^{-1}}U \subseteq N_e$ or $R_{g^{-1}}V \subseteq N_e$ and then either $U \subseteq R_gN_e \subseteq N_g$ or $V \subseteq R_gN_e \subseteq N_g$. The converse is obvious. □

Theorem 2.5. *Let R be a strongly G-graded ring and M be a graded R-module such that all components of M are multiplication R_e-modules. Suppose that N is a graded R-submodule of M. If N_e is pseudo irreducible R_e-submodule of M_e, then N_g is pseudo prime R_e-submodule of M_g for all $g \in G$.*

Proof. By Theorem 2.4, it is enough to prove that N_e is pseudo prime. Let U, V be two R_e-submodules of M_e such that $UV \subseteq N_e$ and $U+V = M_e$. Then $U + N_e$ and $V + N_e$ are R_e-submodules of M_e such that $(U + N_e)(V + N_e) = UV+UN_e+VN_e+N_e^2 \subseteq N_e$ and $(U+N_e)+(V+N_e) = (U+V)+N_e = M_e+N_e = M_e$. Since N_e is pseudo irreducible, either $U + N_e = M_e$ or $V + N_e = M_e$. If $U+N_e = M_e$, then $U+N_e = U+V$ and then $V \subseteq N_e$. Similarly, if $V+N_e = M_e$, then $U \subseteq N_e$. □

An R-submodule N of an R-module M is said to be strictly non-prime if there exist $m \notin N$ and $r \notin (N : M)$ such that $rm \in N$ and $rM + Rm = M$. An ideal of a ring R is said to be strictly non-prime if there exist $a, b \notin I$ such that $ab \in I$ and $\langle a\rangle + \langle b\rangle = R$.

Theorem 2.6. *Let R be a strongly G-graded ring and M be a graded R-module. Suppose that M_e is multiplication R_e-module and R_e is P.I.D. If N is a graded R-submodule of M such that N_e is strictly non-prime R_e-submodule of M_e, then $(N_g : M_g)$ is strictly non-prime ideal of R_e for all $g \in G$.*

Proof. Since N_e is strictly non-prime, there exist $m \notin N_e$ and $r \notin (N_e : M_e)$ such that $rm \in N_e$ and $rM_e + R_em = M_e$. Since M_e is multiplication, $R_em = IM_e$ for some ideal I of R_e and since R_e is P.I.D, $I = \langle a\rangle$ for some $a \in R_e$. Now, let $g \in G$. Then if $a \in (N_g : M_g)$, then $I \subseteq (N_g : M_g)$ and then $IM_g \subseteq N_g$. So, $IM_e = IR_{g^{-1}}M_g = R_{g^{-1}}IM_g \subseteq R_{g^{-1}}N_g \subseteq N_e$ and then $R_em \subseteq N_e$ a contradiction since $m \notin N_e$. So, $a \notin (N_g : M_g)$. Also, if $r \in (N_g : M_g)$, then $rM_g \subseteq N_g$ and then $rM_e = rR_{g^{-1}}M_g = R_{g^{-1}}rM_g \subseteq R_{g^{-1}}N_g \subseteq N_e$, i.e., $r \in (N_e : M_e)$ a contradiction. So, $r \notin (N_g : M_g)$. Now, $raM_g = raR_gM_e = rR_gaM_e \subseteq rR_gIM_e = rR_gR_em = R_grm \subseteq R_gN_e \subseteq N_g$, i.e., $ra \in (N_g : M_g)$. Also, $M_e = rM_e + R_em = rM_e + IM_e = rM_e + \langle a\rangle M_e$ and then $\langle r\rangle + \langle a\rangle = R_e$. Hence, $(N_g : M_g)$ is strictly non-prime ideal. □

Theorem 2.7. *Let R be a strongly G-graded, M be a graded R-module and N be a graded R-submodule of M. Then N_e is strictly non-prime R_e-submodule of M_e if and only if N_g is strictly non-prime R_e-submodule of M_g for all $g \in G$.*

Proof. Since N_e is strictly non-prime, there exist $m \notin N_e$ and $r \notin (N_e : M_e)$ such that $rm \in N_e$ and $rM_e + R_em = M_e$. Let $g \in G$. Then if $r \in (N_g : M_g)$, then $rM_g \subseteq N_g$ and then $rM_e = rR_{g^{-1}}M_g = R_{g^{-1}}rM_g \subseteq R_{g^{-1}}N_g \subseteq N_e$, i.e., $r \in (N_e : M_e)$ a contradiction. So, $r \notin (N_g : M_g)$. Suppose $R_g m \subseteq N_g$. Then $m = 1.m \in R_e m = R_{g^{-1}}R_g m \subseteq R_{g^{-1}}N_g \subseteq N_e$ a contradiction. So, $R_g m \nsubseteq N_g$. Now, $rR_g m = R_g rm \subseteq R_g N_e \subseteq N_g$ and also, $M_g = R_g M_e = R_g(rM_e + R_e m) \subseteq R_g rM_e + R_g R_e m = rM_g + R_e(R_g m) \subseteq R_e M_g + R_e(R_g m) = M_g + R_e(R_g m) \subseteq M_g + R_e(R_g M_e) = M_g + M_g \subseteq M_g$ and hence, $rM_g + R_e(R_g m) = M_g$. Therefore, N_g is strictly non-prime. The converse is obvious. □

References

[1] Z. Abd El-Bast, P.F. Smith, Multiplication modules, *Comm. Algebra*, **16**, No. 4 (1988), 755-779.

[2] R. Ameri, On the prime submodules of multiplication modules, *Inter. J. of Mathematics and Mathematical Sci.*, **27** (2003), 1715-1724.

[3] S. McAdam, R.G. Swan, Unique comaximal factorization, *J. Algebra*, **276**, No. 1 (2004), 180-192.

[4] C. Nastasescu, F. Van Oystaeyen, Graded ring theory, *Mathematical Library*, **28**, North Holland, Amesterdam (1982).

[5] M. Refai, Various types of strongly graded rings, *Abhath Al-Yarmouk Journal (Pure Sciences and Engineering Series)*, **4**, No. 2 (1995), 9-19.