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ON PERMANENTS OF SOME TRIDIAGONAL MATRICES CONNECTED WITH FIBONACCI NUMBERS

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Abstract: In this paper we study such sequences of special tridiagonal matrices, which permanents are equal a Fibonacci number. Firstly we summarize the previous results and then we derive some new special cases of square tridiagonal matrices, whose permanents are equal to some Fibonacci number.

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1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. F_n is called the n^{th} Fibonacci number. The Fibonacci numbers are well-known for possessing many amazing properties (see e. g. [7] or [12]). Let α and β be the roots of the characteristic equation $x^2 - x - 1 = 0$ (thus $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$), then the Binet formula for the

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Fibonacci numbers has the form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

The recurrence relation for the Fibonacci numbers can be used to extend the sequence backward

$$F_{-1} = F_1 - F_0 = 1, F_{-2} = F_0 - F_{-1} = -1, F_{-3} = F_{-1} - F_{-2} = 2, \dots,$$

thus

$$F_{-n} = (-1)^{n+1} F_n$$

for any positive integer n. Square matrix $A = (a_{jk})$ of the order n, where $a_{jk} = 0$ for $|k - j| > 1$, is called tridiagonal matrix. Let $B = (b_{jk})$ be any square matrix of the order n. The permanent of matrix M is defined by the following way

$$\operatorname{per} M = \sum_{\sigma \in S_n} \prod_{j=1}^{n} m_{j\sigma(j)},$$

where the summation extends over all permutations σ of the symmetric group S_n. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive (see [11]). Permanents have many applications in physics, chemistry and electrical engineering. Probably the first result on the connection between the permanent of a tridiagonal matrix and the Fibonacci numbers can be extracted from a more general case, which is due to Minc [10], but this result was exactly given by King and Parker [6] (G. Y. Lee and S. G. Lee [8] later rediscovered this result). They derived that for permanent of sequence of tridiagonal matrices $\{A(n), n = 1, 2, 3, \dots\}$, where n is order of these matrices, with entries $a_{jk} = 1$ for $|k - j| \le 1$ and $a_{jk} = 0$ otherwise, thus matrix in the form

$$A(n) = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}, \tag{1}$$

the following holds

$$\operatorname{per} A(n) = F_{n+1}.$$

Kiliç and Taşci [4] studied permanents of tridiagonal matrices and they derived some relationships between permanents of tridiagonal matrices and Fibonacci and Lucas numbers. They found, using *contraction method* of a square matrix (Brualdi and Gibson [1] introduced this method), that for permanent of the matrix

$$B(n) = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & -1 & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}, \tag{2}$$

the following holds

$$\text{per } B(n) = F_{n+1}.$$

Using a general recurrence relation (see Lemma 1 below) for sequence of tridiagonal matrices $\{C(n), n = 1, 2, 3, \dots\}$ they derived that for the tridiagonal matrix with entries $c_{j,j} = -1$, $c_{j,j+1} = c_{j+1,j} = -1$ for $1 \leq j \leq n$, thus the matrix in the form

$$C(n) = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \ddots & \vdots \\ 0 & 1 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix},$$

the following holds

$$\text{per } C(n) = F_{-(n+1)}.$$

The reader can found many interesting results on connection permanents with Fibonacci and Lucas numbers and their various generalizations in [5, 9, 13].

2. Preliminary Results

We will use the following lemma from [4], which can be easily proved by Laplace expansion for permanents.

Lemma 1. *(Lemma 3 of [4]) Let $\{D(n), n = 1, 2, \dots\}$ be the sequence of matrices of type $n \times n$ in the following form*

$$D(n) = \begin{pmatrix} d_{1,1} & d_{1,2} & 0 & \cdots & 0 \\ d_{2,1} & d_{2,2} & d_{2,3} & \ddots & \vdots \\ 0 & d_{3,2} & d_{3,3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & d_{n-1,n} \\ 0 & \cdots & 0 & d_{n,n-1} & d_{n,n} \end{pmatrix}.$$

Then the successive permanents of sequence $D(n)$ are given by recursive formula

$$\begin{aligned} \text{per } D(1) &= d_{1,1}; \\ \text{per } D(2) &= d_{1,1}d_{2,2} + d_{1,2}d_{2,1}; \\ \text{per } D(n) &= d_{n,n}\text{per } D(n-1) + d_{n-1,n}d_{n,n-1}\text{per } D(n-2). \end{aligned} \tag{3}$$

The results in [10] and [4] can be generalized by the following way.

Lemma 2. *Let $x \neq 0$ be any complex number. Let $\{E(n), n = 1, 2, 3, \dots\}$ be a sequence of tridiagonal matrices in the form*

$$e_{jk} = \begin{cases} x, & j = k+1\ ; \\ \frac{1}{x}, & j = k-1; \\ 0, & \textit{otherwise} \end{cases}$$

i. e.

$$E(n) = \begin{pmatrix} 1 & \frac{1}{x} & 0 & \cdots & 0 \\ x & 1 & \ddots & \ddots & \vdots \\ 0 & x & \ddots & \frac{1}{x} & 0 \\ \vdots & \ddots & x & 1 & \frac{1}{x} \\ 0 & \cdots & 0 & x & 1 \end{pmatrix}.$$

Then

$$\text{per } E(n) = F_{n+1}$$

holds for $n \geq 1$.

Proof. We use the induction on n. The assertion holds for $n = 1$ and $n = 2$ as

$$\text{per } E(1) = 1 \text{ and per } E(2) = \text{ per } \begin{pmatrix} 1 & \frac{1}{x} \\ x & 1 \end{pmatrix} = 2.$$

Suppose that the assertion holds for every k, $3 \leq k \leq n$. Then we have to show that the assertion is true for $n+1$. We use recurrence (3)

$$\begin{aligned} \text{per } E(n+1) &= e_{n+1,n+1}\text{per } E(n) + e_{n,n+1}e_{n+1,n}\text{per } E(n-1) \\ &= 1 \cdot \text{per } E(n) + \frac{1}{x}x\text{per } E(n-1) \\ &= \text{per } E(n) + \text{per } E(n-1) = F_{n+1} + F_n = F_{n+2}. \end{aligned}$$

□

For example setting $x = i = \sqrt{-1}$ we obtain the following relation, which is analogous with the result for the determinant in [3],

$$\text{per} \begin{pmatrix} 1 & -i & 0 & \cdots & 0 \\ i & 1 & \ddots & \ddots & \vdots \\ 0 & i & \ddots & -i & 0 \\ \vdots & \ddots & i & 1 & -i \\ 0 & \cdots & 0 & i & 1 \end{pmatrix} = F_{n+1}.$$

3. Main Results

In this major section we introduce permanents of new special families of matrices, which are connected with Fibonacci numbers.

Theorem 3. *Let $\{D(n) = (d_{jk})_{1 \leq j,k \leq n}, n = 1, 2, 3, \dots\}$ be a sequence of tridiagonal matrices in the form*

$$d_{jk} = \begin{cases} 1, & (j = k \neq 2) \text{ or } (j = k \pm 1 \text{ and } j + k > 4); \\ 2, & j = k = 2; \\ 0, & \text{otherwise} \end{cases}$$

i. e.

$$D(n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 1 & \ddots & \vdots \\ 0 & 1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$\text{per } D(n) = F_{n+1}$$

holds for $n \geq 1$.

Proof. We use induction on n. The assertion holds for $n = 1$ and $n = 2$ as

$$\text{per } D(1) = 1 \text{ and per } D(2) = 2.$$

Now we assume that the assertion holds for every k, $3 \leq k \leq n$, then for $n + 1$ by (3) we obtain

$$\begin{aligned} \text{per } D(n+1) &= d_{n+1,n+1}\text{per } D(n) + d_{n,n+1}d_{n+1,n}\text{per } D(n-1) \\ &= \text{per } D(n) + \text{per } D(n-1) = F_{n+1} + F_n = F_{n+2}. \end{aligned}$$

□

Now we show matrix, that permanents are equal negatively subscripted Fibonnaci sequence.

Theorem 4. *Let* $\{H(n) = (h_{jk})_{1 \leq j,k \leq n}, n = 1, 2, 3, \ldots\}$ *be a sequence of tridiagonal matrices in the form*

$$h_{jk} = \begin{cases} i, & j = k; \\ -1, & j = k \pm 1; \\ 0, & \textit{otherwise} \end{cases}$$

i. e.

$$H(n) = \begin{pmatrix} i & -1 & 0 & \cdots & 0 \\ -1 & i & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \ddots & -1 & i & -1 \\ 0 & \cdots & 0 & -1 & i \end{pmatrix},$$

where $i = \sqrt{-1}$. *Then the following holds*

$$\text{per } H(n) = F_{-(n+1)} \tag{4}$$

for $n \geq 1$.

Proof. We use induction on n. Identity (4) holds for $n = 1$ and $n = 2$ as

$$\text{per } H(1) = -1 \text{ and per } H(2) = 2.$$

We assume the assertion holds for every k, $3 \leq k \leq n$, then for $n+1$ by (3) we obtain

$$\begin{aligned}
\text{per } H(n+1) &= h_{n+1,n+1}\text{per } H(n) + h_{n,n+1}h_{n+1,n}\text{per } H(n-1) \\
&= -1 \cdot \text{per } H(n) + (-1)(-1)\text{per } H(n-1) \\
&= -\text{per } H(n) + \text{per } H(n-1) \\
&= -F_{-(n+1)} + F_{-n} = F_{-(n+2).}
\end{aligned}$$

□

Cahill et al [2] proved that for the following matrix of type $n \times n$ holds

$$\det \begin{pmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \ddots & -1 & 3 & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{pmatrix} = F_{2(n+1)}.$$

We show that the permanent of similar matrix is equal to the same Fibonacci number.

Theorem 5. *Let* $\{G(n) = (g_{jk})_{1 \leq j,k \leq n}, n = 1, 2, 3, \dots\}$ *be a sequence of tridiagonal matrices in the form*

$$g_{jk} = \begin{cases} 3, & j = k; \\ -1, & j = k-1; \\ 1, & j = k+1; \\ 0, & \textit{otherwise} \end{cases}$$

i. e.

$$G(n) = \begin{pmatrix} 3 & -1 & 0 & \cdots & 0 \\ 1 & 3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \ddots & 1 & 3 & -1 \\ 0 & \cdots & 0 & 1 & 3 \end{pmatrix}.$$

Then the following holds

$$\text{per } G(n) = F_{2(n+1)}. \tag{5}$$

Proof. We use induction on n. Identity (5) holds for $n = 1$ and $n = 2$ as

$$\text{per } G(1) = 3 \text{ and per } G(2) = 8.$$

Now we assume, that (5) holds for every k, $3 \leq k \leq n$, then for $n + 1$ by recurrence (3)

$$\begin{aligned}
\text{per } G(n+1) &= g_{n+1,n+1}\text{per } G(n) + g_{n,n+1}g_{n+1,n}\text{per } G(n-1) \\
&= 3 \text{ per } G(n) - \text{per } G(n-1) \\
&= 3F_{2(n+1)} - F_{2n} = 2F_{2n+2} + (F_{2n+1} + F_{2n}) - F_{2n} \\
&= F_{2n+2} + (F_{2n+2} + F_{2n+1}) = F_{2n+2} + F_{2n+3} \\
&= F_{2n+4}.
\end{aligned}$$

□

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References

[1] R. A. Brualdi, P. M. Gibson, Convex polyhedra of doubly stochastic matrices. I. Applications of permanent function, *J. Combin. Theory* (*A*) **22,** (1977), 194-230.

[2] N. D. Cahill, J. R. D´Errico, D. A. Narayan, J. Y. Narayan, Fibonacci determinants, *College Math. J.*, **33**, No. 3 (2002), 221-225. http://dx.doi.org/10.2307/1559033

[3] N. D. Cahill, J. R. D'Errico, J. P. Spence, Complex factorizations of the Fibonacci and Lucas numbers, *Fib. Quart.*, **41**, No. 1 (2003), 13-19.

[4] E. Kiliç, D. Taşci, On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers, *Rocky Mountain J. Math.*, **37**, No. 6 (2007), 1953-1969. http://dx.doi.org/10.1216/rmjm/1199649832

[5] K. Kaygisiz, A. Şahin, Determinant and Permanent of Hessenberg Matrix and Fibonacci Type Numbers, Gen. Math. Notes, **9**, No. 2 (2012), 32-41

[6] B. W. King, F. D. Parker, Fibonacci matrix and the permanent function, *Fib. Quart.*, **7**, No. 5 (1969), 539-544.

[7] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[8] G. Y. Lee, S. G. Lee, A note on generalized Fibonacci numbers, *Fib. Quart.*, **33**, No. 3 (1995), 273-278.

[9] F. Lu, Z. Jiang, The sum and product of Fibonacci numbers and Lucas numbers, Pell numbers and Pell-Lucas numbers representation by matrix method, *WSEAS TRANSACTIONS on MATHEMATICS*, **12,** No. 4 (2013), 449-458.

[10] H. Minc, Permanents of (0,1)-circulants, *Canad. Math. Bul.* **7**, (1964), 253-263.

[11] H. Minc, *Encyclopedia of Mathematics and Its Applications, Permanents*, Vol. 6, Addison-Wesley Publishing Company, London, 1978.

[12] S. Vajda, Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications, Ellis Horwood, Chichester, 1989.

[13] F. Yilmaz, D. Bozkurt, On the Fibonacci and Lucas numbers, their sums and permanents of one type of Hessenberg Matrices, arXiv:1302.0668 [math.NT] (2013).