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GENERALIZED HAMMING WEIGHTS OF DUALS OF ALGEBRAIC-GEOMETRIC CODES

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Abstract: Here we extends a work of A. Couvreur on the Hamming distance of the dual of an evaluation code to its generalized Hamming weights. We prove the following result.

Fix integers $r \geq 2$, $m > 0$ and $e \geq 1$. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $\deg(Z) \leq 3m + r - 3$. If $r > 2$ assume that Z spans $\mathbb{P}^r$ and that the sum of the degrees of the non-reduced connected components of Z is at most $2m + 1$. We have $h^1(\mathcal{I}_Z(m)) \geq e$ if and only if there is $W \subseteq Z$ as one of the schemes in the following list:

(a) $\deg(W) = m + 1 + e$ and W is contained in a line;

(b) $\deg(W) = 2m + 1 + e$ and W is contained in a reduced plane conic;

(c) $r \geq 3$, $e \geq 2$, and there are an integer $f \in \{1, \dots, e-1\}$ and lines L_1, L_2, such that $L_1 \cap L_2 = \emptyset$, $\deg(L_1 \cap Z) = m + 1 + f$ and $\deg(L_2 \cap Z) = m + 1 + e - f$.

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1. Introduction

Fix a prime p and a p-power q. We recall that an affine $[n,k]$-code $\mathcal{C}$ over $\mathbb{F}_q$ is often given in the following way. Fix positive integers n, r, distinct points $P_1, \dots, P_n$ of the affine space $\mathbb{F}_q^m$ and a k-dimensional linear subspace W of $\mathbb{F}_q[t_1, \dots, t_m]$ such that no $f \in W \setminus \{0\}$ vanishes at all points $P_1, \dots, P_n$. Fix a basis $f_1, \dots, f_k$ of W. The $k \times n$ matrix $(f_i(P_j))$ gives an injective linear map $\mathbb{F}^k \to \mathbb{F}_q^n$, i.e., this matrix is the generator matrix of an $[n,k]$-code $\mathcal{C}$ ([11]). he dual code $\mathcal{C}^\vee$ is the $[n,(n-k)]$-code whose words are the elements of $\mathbb{F}_q^n$ orthogonal to the words of $\mathcal{C}$ with respect to the canonical inner product, i.e., $(a_1, \dots, a_n) \in \mathbb{F}_q^n$ is a word of $\mathcal{C}^\vee$ if and only if $a_1b_1 + \cdots + a_nb_n = 0$ for all words $(b_1, \dots, b_n)$ of $\mathcal{C}$ (i.e. the generator matrix of $\mathcal{C}^\vee$ is the parity check matrix of $\mathcal{C}$, and conversely). We usually consider the following projective set-up, leaving to the interested reader the task to translate the projective language into the affine language. For any field K let $K[x_0, \dots, x_r]_m$ denote the set of homogeneous polynomials over K with degree m. The K-vector space $K[x_0, \dots, x_r]$ has dimension $\binom{m+r}{r}$. Fix a finite set $S \subseteq \mathbb{P}^r(\mathbb{F}_q)$ and a k-dimensional linear subspace of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ defined over $\mathbb{F}_q$. We fix homogeneous coordinates $x_0, \dots, x_r$ of $\mathbb{P}^r$, so that we may identify V with a k-dimensional linear subspace of $\mathbb{F}_q[x_0, \dots, x_r]$. For any $P \in S$ we fix $P' \in \mathbb{F}_q^{r+1} \setminus \{0\}$ whose equivalence class modulo scalars induces P. We use these choices to identify S with a subset of $\mathbb{F}_q^{r+1} \setminus \{0\}$ (also called S). With these choices every $f \in V$ gives a map $S \to \mathbb{F}_q$. Therefore V induces a linear map $V \to \mathbb{F}_q^S$. If this map is injective, then we get an $[n,k]$-code $\mathcal{C}$ over $\mathbb{F}_q$. If we identify V with a linear subspace of $\mathbb{F}_q^S$, then we denote with $V^\perp$ its orthogonal linear subspace, i.e., the linear subspace of $\mathbb{F}_q^S$ inducing $\mathcal{C}^\vee$. A. Couvreur proved that quite often it is easier to compute the minimum distance of $\mathcal{C}^\vee$ and classify all the codewords of $\mathcal{C}^\vee$ with small weights (not only the ones with minimum distance) than to solve the corresponding problems for $\mathcal{C}$ ([7], [5]). As clear from [7] in evaluation codes arising from $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ very simple objects (lines, small degree plane curves, finite sets which are complete intersections) are often useful. We extend [7] to the higher distances of a code (a.k.a. higher support weights ([10], [12]) (see Lemma 1). In [2] we translated the issue directly on plane curve giving the associated Goppa code. Lemma 1 is more flexible. By Lemma 1 to get the extension it is sufficient to prove the case $e \geq 2$ of the following result.

Theorem 1. *Fix integers $r \geq 2$, $m > 0$ and $e \geq 1$. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $\deg(Z) \leq \max\{3m + \rho - 3, 3m + e - 2\}$ where ρ*

is the dimension of the linear span of Z. If $\rho > 2$ assume that the sum of the degrees of the non-reduced connected components of Z is at most $2m+1$. We have $h^1(\mathcal{I}_Z(m)) \geq e$ if and only if there is $W \subseteq Z$ as one of the schemes in the following list:

(a) $\deg(W) = m + 1 + e$ *and W is contained in a line;*

(b) $\deg(W) = 2m + 1 + e$ *and W is contained in a reduced plane conic;*

(c) *$r \geq 3$, $e \geq 2$, and there are an integer $f \in \{1, \dots, e-1\}$ and lines L_1, L_2, such that $L_1 \cap L_2 = \emptyset$, $\deg(L_1 \cap Z) = m + 1 + f$ and $\deg(L_2 \cap Z) = m + 1 + e - f$.*

Case (c) is the one arising only in the set-up of generalized Hamming weight, because if W is as in case (c), then there is $W' \subsetneq W$ with $h^1(\mathcal{I}_{W'}(m)) > 0$, and it is W', not W, that count for the minimum distance.

Remark 1. Assume that Z is defined over a perfect field K and call $\overline{K}$ the algebraic closure of K. It is easy to check that we may find W with the additional condition that W and the curve (a line, or a conic or a disjoint union of two lines in cases (a), (b) or (c)) are defined over a finite extension K' of K with $\deg([K' : K]) \leq 2$. To get $K' = K$ in cases (a) and (b) it is sufficient that one of the points of W_{red} is defined over K. In case 2 if the conic E is either smooth or a double line, then it is defined over K. If E is a reduced, but a reducible conic to get that W and E are defined over K it is sufficient that two points of W_{red} are defined over K. For the application to coding theory we have $K = \mathbb{F}_q$ and, as in [7] and [5], many of the connected components of Z are reduced and defined over K.

If Z is just a finite set, then the proofs are easier and (at least if $e = 1$) (see [1]). Even weaker assumption on Z may be used (as in [3]). Basically in the proof of Theorem 1 we need in some way to handle the case $D = L$ (or in the case of Lemma 3 the case $D' = D$).

2. The Proof and a Result in $\mathbb{P}^2$

Lemma 1. *Fix $S \subseteq \mathbb{P}^r(\mathbb{F}_q)$ and a linear subspace $V \subseteq \mathbb{F}_q[x_0, \dots, x_r]_m$ and call $\mathcal{C}$ the evaluation code induced by (V, S). Set $n := \sharp(S)$ and $k := \dim(V)$. Assume $V(-S) = \{0\}$, i.e., assume that $\mathcal{C}$ is an $[n, k]$-code.*

(a) Fix $B \subseteq S$ and an integer $e > 0$. There is an e-dimensional linear subspace of $\mathcal{C}^\vee = (V^\perp, S)$ with support contained in B if and only if $i(V, B) \geq e$.

(b) Fix an integer $h \in \{1, \dots, k-1\}$; $d_h(\mathcal{C}^\vee)$ is the minimal cardinality of a set $A \subseteq S$ such that $i(V, A) = h$; all these sets A determine the h-dimensional linear subspaces of $V^\perp$ with minimal support.

Proof. Fix $B \subseteq S$. We write $S = B \sqcup (S \setminus B)$ and identify $\mathbb{F}_q^S = \{S \to \mathbb{F}_q\}$ with $\mathbb{F}_q^B \times \mathbb{F}_q^{S\setminus B}$. The linear projection of $\mathbb{F}_q^S$ onto its factor $\mathbb{F}_q^B$ and the inclusion $V \hookrightarrow \mathbb{F}_q^S$ induces an inclusion $V/V(-B) \hookrightarrow \mathbb{F}_q^B$. Fix $f \in \mathbb{F}_q^S$ with support on B. Since f has support on B, we have $\sum_{P\in S} f(P)g(P) = \sum_{P\in B} f(P)g(P)$ for all $g \in \mathbb{F}_q^S$. We have $i(V, B) = 0$ if and only if the evaluations of V at the points of B are linearly independent. In general $i(V, B)$ is the number of independent linear relations among the evaluations of V at the points of B. Hence $i(V, B)$ is the dimension of the linear subspace of $\mathcal{C}^\vee$ formed by the words with support on B. Hence we get part (a). Part (b) follows from part (a). □

Remark 2. Let X be any projective scheme and D any effective Cartier divisor of X. For any closed subscheme Z of X let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. If Z is a finite set, then $\mathrm{Res}_D(Z) = Z \setminus Z \cap D$. For every $L \in \mathrm{Pic}(X)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0 \qquad (1)$$

From (1) we get

$$h^i(X, \mathcal{I}_Z \otimes L) \leq h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z\cap D,D} \otimes (L|D))$$

for every integer $i \geq 0$.

Lemma 2. *Let A, B be zero-dimensional subschemes of $\mathbb{P}^r$ such that $A \subseteq B$. If $h^1(\mathbb{P}^r, \mathcal{I}_B(m)) = 0$, then $h^1(\mathbb{P}^r, \mathcal{I}_A(m)) = 0$.*

Proof. Since B is a zero-dimensional scheme, then $h^1(B, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on B. Therefore $h^1(B, \mathcal{I}_{A,B}(m)) = 0$. Hence the restriction map $H^0(B, \mathcal{O}_B(m)) \to H^0(A, \mathcal{O}_A(m))$ is surjective. □

In $\mathbb{P}^2$ by far the strongest tool is the following result due to Ph. Ellia and Ch. Peskine ([8], Corollaire 2) (see [8], Remarques (i) at page 116 for a use of it).

Proposition 1. *Fix an integer $m \geq 3$ and a zero-dimensional scheme $Z \subset \mathbb{P}^2$ such that either $\deg(Z) \leq 10$ (case $m = 3$) or $\deg(Z) \leq 4m - 4$ (case $m \geq 4$). We have $h^1(\mathcal{I}_Z(m)) > 0$ if and only if there is a scheme $W \subseteq Z$ such that one of the following cases occurs:*

(a) $\deg(W) = m + 2$ *and* W *is contained in a line;*

(b) $\deg(W) = 2m + 2$ *and* W *is contained in a conic;*

(c) $\deg(W) = 3m$ *and* W *is the complete intersection of a curve of degree* 3 *and a curve of degree* m*;*

(d) $\deg(W) = 3m + 1$ *and* W *is contained in a degree* 3 *curve;*

(e) $\deg(W) = 4m - 4$ *and* W *is the complete intersection of a degree* 4 *curve and a degree* $m - 1$ *curve.*

Proof. Let T be the degree ≤ 4 curve listed in cases (a), (b), (c), (d), (e). In cases (a), (b), (c), (d), (e) we have $h^1(\mathcal{I}_W(m)) > 0$ either because $\deg(W) > h^0(T, \mathcal{O}_T(m))$ (cases (a), (b), (d)) or because W is a complete intersection, say $W = T \cap T_1$, and we may use the Koszul complex induced by the equations of the curves T and T_1. Hence the " if " part follows from Lemma 2.

Now assume $h^1(\mathcal{I}_Z(m)) > 0$. Fix any subscheme $W \subseteq Z$ such that $h^1(\mathcal{I}_W(m)) = 1$ and $h^1(\mathcal{I}_{Z'}(m)) = 0$ for every $Z' \subsetneq W$ (to prove the existence of W it is essential to work over an algebraically closed base field). Therefore $h^1(\mathcal{I}_W(t)) = 0$ for every $t > m$. Set $d := \deg(W)$. First assume $d \le 3m$ and $d \ge 9$. In the set-up of [8]. Corollaire 2, take $\tau := m$ and $s := 3$. We have $\tau \ge (3-3) + d/3$. By [8], Corollaire 2, we get that either W is the complete intersection of a cubic curve and a degree m curve (and hence $W = Z$ and $\deg(Z) = 3m$) or there is $t \in \{1, 2\}$ and a degree t curve $F \subset \mathbb{P}^2$ such that $\deg(F \cap W) \ge tm + t(3-t)$. Now assume $d \le 8$. Adapt the case of a reduced set done, for instance, in [9], p. 715.

Now assume $d \ge 3m + 1$ and $d \ge 16$. Since $d \ge 16$, we may apply the case $s = 4$, $\tau = m$ of [8], Corollaire 2 (we may apply it, because $d \le 4m - 4$, i.e., $\tau = m \ge 1 + d/4 = s - 3 + d/s$). We get that either W is the complete intersection of a degree 4 curve and a degree $m - 1$ curve (case (e)) or there is $t \in \{1, 2, 3\}$ and $W_t \subseteq W$ with W_t contained in a degree t curve and either $\deg(W_t) = tm + 2$ (case $t \in \{1, 2\}$) or $t = 3$ and $3m \le \deg(W_3) \le 3m + 1$. If $t = 1$ (resp. $t = 2$), then we are in case (a) (resp. (b)). If $t = 3$ and $\deg(W_3) = 3m + 1$, then we are in case (d). Hence we may assume $\deg(W_3) = 3m$. Let E be a degree 3 curve containing W_3. To prove that we are in case (c) (taking W_3 as the subscheme of Z) it is sufficient to prove that W_3 is the complete intersection of E and a degree m plane curve. Since $W \cap E \supseteq W_3$ either we are in case (d) or $W \cap E = W_3$. Hence we may assume $W \cap E = W_3$ (as schemes). Since $\deg(\mathrm{Res}_E(W)) = \deg(W) - \deg(W_3) \le m - 4$, we have $h^1(\mathcal{I}_{\mathrm{Res}_E(W)}(m-1)) = 0$.

Hence Remark 2 gives $h^1(\mathcal{I}_{W_3}(m)) > 0$. Apply the case $d \le 3m$ just done to W_3.

Now assume $d \le 15$, and $3m+1 \le d \le 4m-4$. Hence $m = 3$ and $10 \le d \le 11$. Since $\deg(Z) \le 10$ if $m = 3$, it is sufficient to do the case $m = 3$ and $d = 10$. Since $\deg(W) = 10 = h^0(\mathcal{O}_{\mathbb{P}^2}(3))$, we have $h^0(\mathcal{I}_W(3)) = h^1(\mathcal{I}_W(3))$. Hence we are in case(d). □

Lemma 3. *Fix integers $r \ge 2$ and $m > 1$. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $\deg(Z) \le \max\{3m+1, 3m+r-2\}$. If $r > 2$ assume that Z spans $\mathbb{P}^r$ and that the union of the non-reduced connected components of Z has degree $\le 2m+1$. We have $h^1(\mathcal{I}_Z(m)) > 0$ if and only if there is a scheme $W \subseteq Z$ as one of the schemes in the following list:*

- *(a)* $\deg(W) = m+2$ *and W is contained in a line;*
- *(b)* $\deg(W) = 2m+2$ *and W is contained in a plane conic;*
- *(c)* $\deg(W) = 3m$ *and W is the complete intersection of a degree 3 plane curve and a degree m hypersurface;*
- *(d)* $\deg(W) = 3m+1$ *and W is contained in a plane cubic.*

Proof. The " if " part follows from Lemma 2 and the cohomology of low degree plane curves. Now assume $h^1(\mathcal{I}_Z(m)) > 0$. If $m = 1$, then $\deg(Z) \le r+1$. Since Z spans $\mathbb{P}^r$ and any degree r subscheme of $\mathbb{P}^r$ is contained in a hyperplane, we have $\deg(Z) = r+1$. Since $h^0(\mathcal{I}_Z(1)) = 0$ and $\deg(Z) = r+1$, we have $h^1(\mathcal{I}_Z(1)) = 0$. Hence Proposition 3 is true if $m = 1$. We use induction on r. For fixed r we use induction on m, starting the induction with the case $m = 1$ just done. In the case $r = 2$ use Proposition 1.

Now assume $r > 2$. Let $H \subset \mathbb{P}^r$ be a hyperplane such that $\deg(Z \cap H)$ is maximal. Since Z spans $\mathbb{P}^r$, we have $\deg(Z \cap H) < \deg(Z)$ and hence $\deg(Z \cap H) \le 3m + \dim(H) - 2$. We have $h^1(H, \mathcal{I}_{Z\cap H}(m)) = h^1(\mathbb{P}^r, \mathcal{I}_{Z\cap H}(m))$. Hence if $h^1(H, \mathcal{I}_{Z\cap H}(m)) > 0$, then we may take $W \subseteq Z \cap H$ by the inductive assumption on r. Hence we may assume $h^1(H, \mathcal{I}_{Z\cap H}(m)) = 0$. Remark 2 implies $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) > 0$. By the inductive assumption on m there is $W' \subseteq \mathrm{Res}_H(Z)$ satisfying one of the assumptions (a), (b), (c) or (d) of Proposition 3 with respect to the integer $m' := m-1$. In particular there is a plane $M \subset \mathbb{P}^r$ such that $W' \subset M$ and $\deg(W') \ge m+1$, with strict inequality, unless W' is contained in a line. Every zero-dimensional scheme with degree at most r is contained in a hyperplane. Since Z is not contained in a hyperplane, there is a hyperplane $U \subset \mathbb{P}^r$ such that $W' \subset U$ and either

$\deg(Z\cap U)-\deg(W')\ge r-2$ (case W' non collinear) or $\deg(Z\cap U)\ge m+r-1$ (case W' collinear). We took H so that $\deg(Z\cap H)\ge \deg(Z\cap U)$. We have $\deg(Z)=\deg(Z\cap H)+\deg(\mathrm{Res}_H(Z))$ (first part of Remark 2). Since $\deg(Z)<2m+2+2(r-2)$, we get that W' is collinear. Call D the line spanned by W' and set $W'':=D\cap Z$. If $W''\neq W'$, then $\deg(W'')\ge m+2$ and hence we are in case (a). Therefore we may assume $W=W'$. If $h^1(\mathcal{I}_{Z\cap U}(m))>0$, then we may take as W a subscheme of $Z\cap U$. Therefore we may assume $h^1(\mathcal{I}_{Z\cap U}(m))=0$. Therefore $h^1(\mathcal{I}_{\mathrm{Res}_U(Z)}(m-1))>0$. Hence there is $A\subseteq \mathrm{Res}_U(Z)$ as in one of the cases (a), (b), (c), (d) for the integer $m-1$. As in the case with H we get that A is collinear, $\deg(A)=m+1$ and $A=Z\cap D'$ with D' a line. First assume $D=D'$. Let E be the union of all non-reduced connected components A of Z with $A_{red}\in D$, but with $A\nsubseteq D$. Let F be the union of the connected components of Z contained in D (among them there are all the reduced connected components of Z contained in D). We have $\mathrm{Res}_U(Z)=\mathrm{Res}_U(Z\setminus F)$ and hence $\mathrm{Res}_U(Z)\cap D=\mathrm{Res}_U(E)\cap D$. Since $\mathrm{Res}_U(E)\subseteq E$, we have $\deg(D\cap \mathrm{Res}_U(E)\cap D)\le \deg(E\cap D)$. Therefore $m+1\le \deg(E\cap D)$. Since $2m+1\ge \deg(E)=\deg(E\cap D)+\deg(\mathrm{Res}_U(E))\ge \deg(E\cap D)+m+1$, we get a contradiction. Therefore $D\neq D'$. If $D\cap D'\neq\emptyset$, then we are in case (b). Hence we may assume $D\cap D'=\emptyset$.

First assume $r=3$. Take a smooth quadric Q containing the two disjoint lines D and D', say as lines of type $(1,0)$. Since $\deg(\mathrm{Res}_Q(Z))\le m-1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(m-2))=0$. Hence Remark 2 gives $h^1(Q,\mathcal{I}_{Z\cap Q}(m))>0$. We have $h^1(Q,\mathcal{I}_{Z\cap Q}(m))=h^1(\mathcal{I}_{Z\cap Q}(m))$. See $D\cup D'$ as a Cartier divisor of Q. Since $\deg(Z\cap D)=\deg(Z\cap D')=m+1$, we have $h^i(Q,\mathcal{I}_{Z\cap(D\cup D')}(m))=0$, $i=0,1$. Hence $h^1(Q,\mathcal{I}_{Z\cap Q}(m))=h^1(Q,\mathcal{I}_{\mathrm{Res}_{D\cup D'}(Z\cap Q)}(m-2,m))$. Therefore $h^1(Q,\mathcal{I}_{\mathrm{Res}_{D\cup D'}(Z\cap Q)}(m-2,m))>0$ and so $h^1(Q,\mathcal{I}_{\mathrm{Res}_{D\cup D'}(Z\cap Q)}(m-2,m-2))>0$. Hence $\deg(\mathrm{Res}_{D\cup D'}(Z\cap Q))\ge m$. Hence $\deg(Z)\ge 3m+2$, contradiction.

Now assume $r>3$. Let H' be any hyperplane containing $D\cup D'$ and (among these hyperplanes) with maximal $\deg(Z\cap H')$. Since Z spans $\mathbb{P}^r$, we have $\deg(Z\cap H')\ge 2m+r-3$. Hence $\deg(\mathrm{Res}_{H'}(Z))\le m$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_{H'}(Z))}(m-1))=0$. Hence Remark 2 gives $h^1(\mathcal{I}_{Z\cap H'}(m))\ge e$. Apply the inductive assumption on r to the scheme $Z\cap H'$. □

Proof of Theorem 1. First assume $\deg(Z)\le 3m+\rho-2$. With no loss of generality we may assume $r=\rho$. Since the case $r=2$ is true by Proposition 1, we may assume $r>2$ and use induction on r. Since the case $e=1$ is true by Proposition 1 we may assume $e\ge 2$ and use induction on e. Since $h^1(\mathcal{I}_Z(m))\ge e\ge e-1$, there is $W\subseteq Z$ with W one of the schemes in the following list:

(i) $\deg(W) = m + e$ and W is contained in a line L;

(ii) $\deg(W) = 2m + e$ and W is contained in a plane conic;

(iii) $e \geq 3$, $\deg(W) = 2m+e+1$, and there are an integer $f \in \{1, \dots, e-2\}$ and lines L_1, L_2, such that $L_1 \cap L_2 = \emptyset$, $W \subset L_1 \cup L_2$, $\deg(L_1 \cap W) = m+1+f$ and $\deg(L_2 \cap W) = m + e - f$.

(a) First assume that W is as in cases (i) or (ii) or that $r \geq 4$ and that we are as in case (iii). Let $H \subset \mathbb{P}^r$ be any hyperplane containing $\langle W \rangle$, where $\langle \ \rangle$ denote the linear span, and with maximal $u := \deg(Z \cap H)$. Since Z spans $\mathbb{P}^r$, we have $\deg(Z \cap H) < \deg(Z) \leq 3m + r - 2$. Since Z spans $\mathbb{P}^r$, the maximality property of the integer u gives $\langle Z \cap H \rangle = H$. If $A \subseteq Z \cap H$ is a connected component of $Z \cap H$, which is not reduced, then the connected component B of Z with $B_{red} = A_{red}$ contains A and hence $\deg(A) \leq \deg(B)$. Therefore we may use induction on r (for an arbirary integer e) for the scheme $Z \cap H \subset H$. Since $Z \cap H \supseteq W$, we have $h^1(H, \mathcal{I}_{Z \cap H}(m)) \geq e - 1$. If $h^1(H, \mathcal{I}_{Z \cap H}(m)) \geq e$, then we may use the inductive assumption on r. Hence we may assume that $h^1(H, \mathcal{I}_{Z \cap H}(m)) = e - 1$. By the residual exact sequence (1) we get $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) > 0$. Therefore $\deg(\mathrm{Res}_H(Z)) \geq m + 1$ ([6, Lemma 34]). Since $Z \cap H$ spans H, we have $u \geq r - 1 - \dim(\langle W \rangle) + \deg(W)$. Hence $\deg(Z) \geq m + r - \dim(\langle W \rangle) + \deg(W)$. In case (ii) we get $\deg(Z) \geq m + r - 2 + 2m + e = 3m + r - 2 + e$, a contradiction. In case (iii) we get $\deg(Z) \geq m + r - 3 + 2m + e + 1$, a contradiction. Now assume that we are in case (i). We may assume $W = Z \cap L$ (as schemes), because otherwise $\deg(Z \cap L) \geq m + 1 + e$ and we may take as new W a subscheme of $Z \cap L$ with degree $m + 1 + e$. Since $\deg(Z \cap H) \geq m + e + r - 2$, we have $\deg(\mathrm{Res}_H(Z)) \leq 2m - e - 1$. By [6, Lemma 34] there is a line $D \subset \mathbb{P}^r$ such that $\deg(D \cap \mathrm{Res}_H(Z)) \geq m + 1$.

(a1) First assume $D \cap L \neq \emptyset$ and $D \neq L$. Therefore $\langle D \cup L \rangle$ is a plane. We are in case (b).

(a2) Now assume $D \cap L = \emptyset$. If $D \cap Z \supsetneq D \cap \mathrm{Res}_H(L)$ (or if $\deg(Z \cap D) \geq m + 2$), then we are as in case (c) with $f = e - 1$, $L_1 = L$ and $L_2 = D$. Now assume $\deg(D \cap Z) = m + 1$. Let $M := \langle D \cup L \rangle$ be the 3-dimensional linear space. First assume $r = 3$. Let $Q \subset M$ be any smooth quadric containing $D \cup L$. Since $\deg(Q \cap Z) \geq 2m + 1 + e$, we have $\deg(\mathrm{Res}_Q(Z)) \leq m - 1 - e$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(m-2)) = 0$. Remark 2 gives $h^1(Q, \mathcal{I}_{Z \cap Q}(m)) \geq e$. Call $(1, 0)$ the type of the ruling of Q containing D and L. Since $\deg(\mathrm{Res}_{D \cup L}(Q \cap Z)) \leq m - 1 - e$, we have $h^1(Q, \mathcal{I}_{\mathrm{Res}_{D \cup L}(Q \cap Z)}(m-2, m)) \leq h^1(Q, \mathcal{I}_{\mathrm{Res}_{D \cup L}(Q \cap Z)}(m-

$2, m-2)) = 0$. Therefore Remark 2 gives $h^1(Q, \mathcal{I}_{Z\cap Q}(m)) \le h^1(D, \mathcal{I}_{Z\cap D}(m)) + h^1(L, \mathcal{I}_{L\cap Z}(m)) = 0 + e - 1$, contradicting the inequality $h^1(Q, \mathcal{I}_{Z\cap Q}(m)) \ge e$ just proved. Now assume $r \ge 4$. Let $N \subset \mathbb{P}^r$ be any hyperplane containing M and with maximal $\deg(Z \cap N)$ among all hyperplanes containing M. Since Z is non-degenerate, we have $\deg(Z\cap N) \ge \deg(Z\cap M) + r - 4 \ge 2m + e + r - 3$. By induction on r we get $h^1(N, \mathcal{I}_{N\cap Z}(m)) = e - 1$. Since $\deg(\mathrm{Res}_N(Z)) \le \deg(Z) - \deg(Z \cap (L \cup D)) \le 3m + r - 2 - 2m - e - r + 3 = m + 1 - e$, we have $h^1(\mathcal{I}_{\mathrm{Res}_N(Z)}(m)) = 0$. Remark 2 gives $h^1(\mathcal{I}_Z(m)) \le e - 1$, a contradiction.

(a3) Now assume $D = L$. Let E be the union of all non-reduced connected components A of Z with $A_{red} \in L$, but with $A \nsubseteq L$. Let F be the union of the connected components of Z contained in L (among them there are all the reduced connected components of Z contained in F. We have $\mathrm{Res}_H(Z) = \mathrm{Res}_H(Z \setminus F)$ and hence $\mathrm{Res}_H(Z) \cap L = \mathrm{Res}_H(E) \cap L$. Since $\mathrm{Res}_H(E) \subseteq E$, we have $\deg(L \cap \mathrm{Res}_H(E) \cap L) \le \deg(E \cap L)$. Therefore $m + 1 \le \deg(E \cap L)$. Since $2m + 1 \ge \deg(E) = \deg(E \cap H) + \deg(\mathrm{Res}_H(E)) \ge \deg(E \cap L) + m + 1$, we get a contradiction.

(b) Now assume $r = 3$, $e \ge 3$, and that W is as in case (iii). If either $W \cap L_1 \ne Z \cap L_1$ or $W \cap L_2 \ne Z \cap L_2$, then we are in case (c) with as integer f either f or $f + 1$. Therefore we may assume $Z \cap (L_1 \cup L_2) = W$. Let Q be any smooth quadric containing $L_1 \cup L_2$. We have $\deg(\mathrm{Res}_Q(Z)) = \deg(Z) - \deg(Z \cap Q) \le \deg(Z) - \deg(Z \cap L_1 \cup L_2) \le m - 1 - e$. Therefore $h^1(\mathcal{I}_{\mathrm{Res}_{Z}(Q)}(m-2)) = 0$ ([6, Lemma 34]). Remark 2 gives $h^1(Q, \mathcal{I}_{Q\cap Z}(m)) = 0$. Set $T := L_1 \cup L_2$ seeing as a divisors of Q, say of type $(2, 0)$. Since $T \cap Z = W$, we have $h^1(T, \mathcal{I}_{T\cap Z}(m)) = e - 1$. We have $\deg(\mathrm{Res}_T(Z \cap Q)) = \deg(Z \cap Q) - \deg(Z \cap T) \le m - 1 - e$ and hence $h^1(Q, \mathcal{I}_{\mathrm{Res}_T(Z\cap Q)}(m-2, m)) \le h^1(Q, \mathcal{I}_{\mathrm{Res}_T(Z\cap Q)}(m-2, m-2)) = 0$. The residual exact sequence of the inclusion $T \subset Q$ gives a contradiction.

(c) Now assume $\deg(Z) \ge 3m + r - 1$, but $\deg(Z) \le 3m + e - 2$. Taking the linear span $\langle Z\rangle$ of Z instead of $\mathbb{P}^r$, we may assume $\langle Z\rangle = \mathbb{P}^r$. Since the case $r = 2$ is true by Proposition 1 we may assume $r > 2$ and use induction on r. Since the case $e = 1$ is true by Lemma 3 we may assume $e \ge 2$ and use induction on e. Take H, W as in the first part of the proof. In case (ii) (resp. case (iii) with $r > 3$) of step (a) we use that $\deg(\mathrm{Res}_H(W)) \le m - 2$ (resp. $m - 3$). Now assume that we are in case (i). We have $\deg(Z) - \deg(Z \cap H) \le 3m + e - 2 - (m + 1 + e) - (r - 2) \le 2(m - 1) + 1$. As in the proof of Theorem 1 quoting [6, Lemma 34] we get a line D with $\deg(D \cap \mathrm{Res}_H(Z)) \ge m + 1$. The assumption on the non-reduced connected components of Z gives $D \ne L$. If $D \cap L \ne \emptyset$, then we get $\deg(Z \cap (D \cup L)) \ge 2m + 1 + e$ and hence we are in case (b). If $D \cap L = \emptyset$, we are in case (c) with $f = e - 1$, $L_1 = L$ and $L_2 = D$. If $r = 3$ and we are in case (iii), then step (b) gives a contradiction, unless

$(L_1 \cup L_2) \cap Z \supsetneq W$, which gives case (c) of Theorem 1. □

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