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ON THE X-RANK OF A POINTS OF THE TANGENT DEVELOPABLE OF A CURVE IN A PROJECTIVE SPACE

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Abstract: Let $X \subset \mathbb{P}^n$ be a smooth curve. For any $P \in \mathbb{P}^n$ the X-rank of P is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denote the linear span. Let $\tau(X) \subset \mathbb{P}^n$ be the tangent developable of X. We compute upper bounds for the X-rank of all $P \in \tau(X)$ or of the general $P \in \tau(X)$, mainly if X is a canonically embedded curve. To do that we define some invariants for the pair $(X, \mathcal{O}_X(1))$ and compute them if X is canonically embedded and either X is a smooth plane curve or it has general moduli.

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1. Introduction

Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ the X*-rank* $r_X(P)$ of P is the minimal cardinality of a subset $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span ([16], [8], [15]). Now assume that X is a smooth curve and that $n \geq 3$. Let $\tau(X) \subset \mathbb{P}^n$ be the tangent developable of X, i.e. the unions of all tangent lines T_OX.Quite often the maximal among the

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integers $r_X(P)$ is achieved by a point of $\tau(X) \setminus X$ (e.g. this is the case when X is a rational normal curve by a theorem of Sylvester's ([16, Theorem 4.1], [8, Theorem 23], but it appears quite often elsewhere ([4])). In this note we give some criteria to compute upper bounds for points of $\tau(X) \setminus X$ when X is a smooth curve, usually when X is the canonical model of a non-hyperelliptic smooth curve. There are also some easy and well-known lower bound and sometimes we get in this way a good picture). As a byproduct of these definitions we prove the following result.

Corollary 1. *Let $C \subset \mathbb{P}^2$, $d \geq 4$, be a smooth pane curve of degree d. Let $X \subset \mathbb{P}^{g-1}$, $g = (d-1)(d-2)/2$, be the canonical model of C and $\phi : C \to X$ the canonical map Then $r_X(P) \leq 2d-5$ for all $P \in \tau(X)$ and $r_X(P) = d-2$ for a general $P \in \tau(X)$ and $r_X(P) = d-3$ for some $P \in \tau(X)$. If $d \geq 5$ and $O \in C$ is such that the tangent line to C at O is neither a flex nor a multitangent, then $r_X(P) = d-3$ for at least one and at most $d-2$ points of $T_{\phi(O)}(X) \setminus \{\phi(O)\}$ and $r_X(P) = d-2$ for all other points of $T_{\phi(O)}(X) \setminus \{\phi(O)\}$. We have $r_X(P) \geq 2d-6$ for at least one $P \in \tau(X)$.*

In section 2 we study the lowest integers $r_X(P)$, $P \in \tau(X) \setminus X$. In section 3 we study $r_X(P) \in \tau(X)$ (see Proposition 6 for curves with general moduli, Propositions 2, 3 and 5 for an arbitrary curve and Theorem 1 for a result giving, with a little effort, Corollary 1).

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. The Lowest Ranks of Points of $\tau(X) \setminus X$

Notation 1. Let $C \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. Let $\beta(C)$ be the maximal integer such that every zero-dimensional subscheme of C with degree at most $\beta(C)$ is linearly independent.

Let X be a smooth curve of genus $g \geq 3$. The *gonality* $\mathrm{gon}(X)$ of X is the minimal degree of a line bundle L on X with $h^0(X, L) \geq 2$. Let $\mathrm{Gon}(X)$ denote the set of all line bundles L on X with degree $\mathrm{gon}(X)$ and $h^0(X, L) \geq 2$. By the definition of gonality we have $\mathrm{Gon}(X) \neq \emptyset$. For each $L \in \mathrm{Gon}(X)$ we have $h^0(X, L) = 2$ and X is base point free. The set $\mathrm{Gon}(X)$ is a closed subset of $\mathrm{Pic}^{\mathrm{gon}(X)}(X)$ and $\dim(\mathrm{Gon}(X)) \leq 1$ ([11]). If X is not hyperelliptic and $C \subset \mathbb{P}^{g-1}$ is the canonical model of X, then $\beta(C) = \mathrm{gon}(X) - 1$.

Fix $L \in \mathrm{Gon}(X)$. Since L is spanned and $h^0(X, L) = 2$, the complete linear system $|L|$ induces a degree $\mathrm{gon}(X)$ morphism $h_L : X \to \mathbb{P}^1$. Let $R(L)$ be the set of all $O \in X$ such that $h^0(X, L(-2O)) > 0$. Since L has no base points, we have

$h^0(X, L(-2O)) = 1$ (i.e. $h^1(X, L(-2O)) = h^1(X, L) + 1$) for each $O \in R(L)$ and $h^0(X, L(-2O)) = 0$ for each $O \in X \setminus R(L)$. For each $O \in R(L)$ let $D_{L,O}$ be the unique effective divisor such that $D_{L,O} + 2O \in |L|$. Since $g > 0$ we have $R(L) \neq \emptyset$ for each $L \in \mathrm{gon}(X)$. Let $R(L)'$ be the set of all $O \in R(L)$ such that $D_{L,O}$ is reduced (we allow the case in which O appears in $D_{L,O}$, i.e. we allow the case $h^0(X, L(-3O)) = 1$, but of course $h^0(X, L(-4O)) = 0$ if $O \in R(L)'$). Let $R(L)''$ be the set of all $O \in R(L)'$ such that $D_{L,O} - O$ is effective; $R(L)''$ is the set of all $O \in X$ such that $h^0(X(L - 3O)) = 1$, the unique divisor $D'_{L,O}$ of $|L(-3O)|$ is reduced and $O \notin D'_{L,O}$. Set $R(L)_1 := R(L)' \setminus R(L)''$, i.e. let $R(L)_1$ be the set of all $O \in X$ with $2O + S \in |L|$ with S reduced and $O \notin S$. Let $\Delta(X)_1$ be the union of all $O \in R(L)_1$, $L \in \mathrm{Gon}(X)$. For each $O \in \Delta_1(X)$ let $\mathbb{M}(X, O)$ denote the set of all $L \in \mathrm{Gon}(X)$ such that $O \in R(L)_1$. Set $\mu(X, O) := \sharp(\mathbb{M}(X, O))$ with the convention $\mu(X, O) = +\infty$ if $\mathbb{M}(X, O)$ is infinite.

Remark 1. Easy examples (standard cyclic coverings with degree k and large g) show that we may have $R'(L) = \emptyset$ for all $L \in \mathrm{Gon}(X)$. In particular we may have $\Delta(X)_1 = \emptyset$.

Remark 2. Let $C \subset \mathbb{P}^n$ be a smooth curve. Take zero-dimensional schemes $A, B \subset C$. Therefore A and B are effective divisors of C. For each $P \in C$ let a_P (resp. b_P) be the degree of the connected component of A (resp. B). The effective divisor $A+B$ has degree $\deg(A)+\deg(B)$ (each P appars with multiplicity $a_P + b_P$ in $A+B$). The scheme $A \cup B$ is the minimal subscheme of C containing both A and B. Each $P \in C$ appears with multiplicity $\max\{a_P, b_P\}$ in $A \cup B$. Therefore $A + B = A \cup B$ if and only if $A \cap B = \emptyset$.

Proposition 1. *Assume gon*$(X) \geq 6$. *We have* $r_X(P) \geq$ *gon*$(X) - 2$ *for all* $P \in \tau(X) \setminus X$. *Fix* $O \in X$.

(i) If $O \notin \Delta(X)_1$, *then* $r_X(P) \geq$ *gon*$(X) - 1$ *for all* $P \in T_O X \setminus \{O\}$.

(ii) Assume $O \in \Delta(X)_1$. *Then gon*$(X) - 2 \leq r_X(P) \leq$ *gon*$(X) - 1$ *for all* $P \in T_O X \setminus \{O\}$. *Let* $\mathcal{W}$ *be the set of all* $P \in T_O X \setminus \{O\}$ *such that* $r_X(P) =$ *gon*$(X) - 2$. *We have* $\mathcal{W} \neq \emptyset$ *and* $\sharp(\mathcal{W}) \leq \mu(X, O)$.

Proof. We have $\beta(X) = \mathrm{gon}(X) - 1 \geq 5$. Since $\beta(X) \geq 4$, for each $P \in \tau(X)$ there is a unique zero-dimensional scheme $O \in X$ with $P \in T_O X$. Fix $P \in \tau(X) \setminus X$ and take the unique $O \in X$ such that $P \in T_O X$, i.e. such that $P \in \langle 2O \rangle$. Fix $S \subset X$ evincing $r_X(P)$, i.e. fix $S \subset X$ with $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. By [7, Lemma 1] we have $h^1(\mathbb{P}^{g-1}, \mathcal{I}_{2O \cup S}(1)) > 0$, i.e. $h^1(X, \omega_X(-(2O \cup S)) \geq 2$ and hence either $\sharp(S) \geq \mathrm{gon}(X) - 1$ or $\sharp(S) = \mathrm{gon}(X) - 2$ and $O \notin S$ (Remark 2). We get part (i).

Now assume $O \in \Delta(X)_1$. Fix $L \in \mathbb{M}(X, O)$ and let $S \subset X$ be the set such that $2O + S \in |L|$. Since $O \notin S$, we have $2O + S = 2O \cup S$, i.e. $2O + S$ is the minimal zero-dimensional scheme containing both $2O$ and S. Since $2O + S$ is a divisor of a line bundle evincing the gonality of X, Grassmann's formula gives that $\langle 2O \rangle \cap \langle S \rangle$ is a single point, P_L. We get $\mathcal{W} \neq \emptyset$ and the existence of a surjection $\mathbb{M}(X, O) \to \mathcal{W}$. Hence $\sharp(\mathcal{W}) \leq \mu(X, O)$. Fix any $P' \in \mathcal{W}$, say associated to $L \in \Delta(X)_1$, and take the only $S \subset X$ such that $2O + S \in |L|$. We have $O \notin S$ and we saw that $\{P'\} = \langle 2O \rangle \cap \langle S \rangle$. Fix any $P \in T_O X$. Since $P' \neq O$, we have $P \in \langle \{O\} \cup S \rangle$ and hence $r_X(P) \leq \mathrm{gon}(X) - 1$. □

3. The Rank of a General Point of the Tangent Developable

Let C be a smooth and connected projective curve of genus g. We recall the following definitions ([10], [5], [6]). For each integer $r \geq 1$ the *r-gonality* $\mathrm{gon}(C, r)$ of C is the minimal integer d such that there is a degree d line bundle L on C with $h^0(C, L) \geq r + 1$ (note that $h^0(C, L) = r + 1$ and that L has no base points for each such L). The integer $\mathrm{gon}(C, 1)$ is the gonality $\mathrm{gon}(C)$ of C. For every integer $r \geq 2$ the *r-birational gonality* $\mathrm{birgon}(C, r)$ is the minimal integer d such that there is a degree d line bundle L on C with $h^0(C, L) \geq r+1$ and the rational map induced by $|L|$ is birational onto its image (notice that $h^0(C, L) = r + 1$ and L has no base points for each such L). For every integer $r \geq 3$ the *r-embedding gonality* $\mathrm{embgon}(C, r)$ is the minimal integer d such that there is a degree d line bundle L on C with $h^0(C, L) \geq r + 1$ and the rational map induced by $|L|$ is an embedding.

If C has general moduli and $1 \leq r \leq g - 2$, then $\mathrm{gon}(C, r) = \lceil rg/(r - 1) \rceil + r - 1$ and the set $W^r_d(C)$ of line bundles on C evincing $d := \mathrm{gon}(C, r)$ has pure dimension $\rho(g, r, d) := (r+1)d - rg - r(r+1)$; this set $W^r_d(C)$ is also irreducible if $\rho(g, r, d) > 0$ ([3]). A dimensional count shows that if C is general and $r \geq 2$, then $\mathrm{gon}(C, r) = \mathrm{birgon}(C, r)$ and that if $\rho(g, r, d) > 0$, then a general element of $W^r_d(C)$ induces a morphism $C \to \mathbb{P}^r$ which is birational onto its image. It is also well-known that if $r \geq 3$, then $\mathrm{gon}(C, r) = \mathrm{embgon}(C, r)$.

Let C be a smooth and connected projective curve of genus g. For each $R \in \mathrm{Pic}(C)$ and each $O \in C$ let $d(C, R, O)$ be the minimal integer d such that there is $S \subset C$, S is reduced, $\sharp(S) = d$, $h^0(C, R(-2O - S)) = h^0(R(-2O)) - \sharp(S) + 1$ and $h^0(C, R(-O - S)) = h^0(C, R(-O)) - \sharp(S)$ (if there is at least one such set S; if no set S exists, then we say that $d(C, R, O)$ is not defined). Assume that $d(C, R, O)$ is defined. By Riemann-Roch the last condition is equivalent to $h^1(R(-2O - S)) = h^1(R(-2O)) + 1$. If R is very ample, then $h^0(C, R(-2O)) = h^0(R) - 2$ and hence $h^1(C, R) = h^1(C, R(-2O))$. Therefore if R is very ample and $D(C, R, O)$ is defined, then $d(C, R, O)$ is the minimal cardinality of a finite set S with $O \notin S$, $h^1(C, R(-2O - S)) > h^1(C, R)$ and $h^1(C, R(-O - S)) = h^1(C, R)$. If $d(C, R, O)$ is defined for all $O \in C$ let $d(C, R, 2)$ be the maximum of all integers $d(C, R, O)$, $O \in C$. Let $d'(C, R, 2)$ be the minimal integer d such that there is a finite set $A \subset C$ with $d(C, R, O)$ defined and $d \geq d(C, R, O)$ for all $O \in C \setminus A$. Now assume that C has genus $g \geq 3$ and that C is not hyperelliptic., i.e. assume that ω_X is very ample. Set $d(C, O) := d(C, \omega_C, O)$, $d(C, 2) := d(C, \omega_C, 2)$ and $d'(C, 2) := d'(C, \omega_C, 2)$ (if these integers are defined).

Lemma 1. *Let C be a smooth curve and let $R \in Pic(C)$ be a line bundle with $\deg(R) \geq 4$. Fix $O \in C$ and assume $h^0(R(-2O)) = h^0(R) - 2$, that $R(-2O)$ is spanned and that the morphism ϕ associated to $R(-2O)$ is birational onto its image. Then $d(C, R, O)$ is defined and $d(C, R, O) \leq h^0(R) - 1$.*

Proof. Since $R(-2O)$ is spanned, R is spanned at each point of $C \setminus \{O\}$. Since $h^0(R(-2O)) = h^0(R) - 2$, R is spanned at O. Let $\psi : C \to \mathbb{P}^r$, $r = h^0(R) - 1$, be the morphism associated to $|R|$. Since ϕ is birational onto its image, ψ is birational onto its image. Set $Y := \psi(C)$ and $B := \mathrm{Sing}(Y)$. Set $B' := \psi^{-1}(B)$. Notice that ψ induces an isomorphism between $C \setminus B'$ and $Y \setminus B$. Since $h^0(R(-2O)) = h^0(R) - 2$, the zero-dimensional scheme $Z := \psi(2O)$ has degree two. Hence $\langle Z \rangle$ is a line. Since $\deg(R(-2O)) \geq 2$, we have $Y \neq \langle Z \rangle$ and hence there is $P \in \langle Z \rangle \setminus \langle Z \rangle \cap Y$. Let $H \subset \mathbb{P}^r$ be a general hyperplane containing P. Since $P \notin Y$, Bertini's theorem implies that the scheme $Y \cap H$ is formed by $\deg(R)$ distinct points, none of them being in $B \cup \{\phi(O)\}$. Since $P' \in \langle Z \rangle \setminus \{\psi(O)\}$, Therefore there is a set $A \subset C \setminus (\{O\} \cup B')$ with $\psi(A) = Y \cap H$. Since $P' \in \langle Z \rangle \setminus \{\psi(O)\}$, we have $\langle Y \cap H \cap \langle Z \rangle = \{P'\}$ and $\langle (Y \cap H) \cup \{\psi(O)\} \rangle$. Hence $h^0(R(-A)) = 1$, $h^0(R(-O - A)) = 0$ and $h^0(R(-2O - A)) = 0$. Therefore $h^1(R(-A)) = h^1(R(-A - O)) < h^1(R(-A - 2O))$. Hence $d(C, R, O)$ is defined and $d(C, R, O) \leq h^0(R) - 1$. □

Corollary 2. *Let C be a smooth curve of genus $g \geq 5$ which is neither hyperelliptic, nor trigonal nor bielliptic. Then $d(C, \omega_C, 2)$ is defined.*

Proof. Fix $O \in C$. Since C is not hyperelliptic, the line bundle ω_C is very ample and hence $h^0(C, \omega_C(-2O)) = g - 2$. Since C is not trigonal, $\omega_C(-2O)$ is spanned. Let $\phi : C \to \mathbb{P}^{g-3}$ be the morphism induced by $|\omega_C(-2O)|$. By Lemma 1 it is sufficient to prove that ϕ is birational onto its image. Assume $\deg(\phi) \geq 2$. Since $\phi(C)$ spans $\mathbb{P}^{g-3}$, we have $\deg(\phi(C)) \geq g - 3$. Assume for the moment $g \geq 6$. Since $\deg(\phi) \cdot \deg(\phi(C)) = 2g - 4$ and $g \geq 5$, we get $\deg(\phi) = 2$ and $\deg(\phi(C)) = g - 2$. Since C is not hyperelliptic and $\deg(\phi) = 2$, the normalization of $\phi(C)$ is not rational. Since $p_a(\phi(C)) \leq 1$ by the upper bound for the arithmetic genera of non-degenerate curve in $\mathbb{P}^r$ with degree $\le r+1$, we get that $\phi(C)$ is an elliptic curve. Hence C is a bielliptic curve, a contradiction. Now assume $g = 5$. We need to exclude the case $\deg(\phi) = 3$. This is excluded, because in this case $\phi(C)$ would be a smooth conic and hence C would be a trigonal curve. □

Proposition 2. *Let $X \subset \mathbb{P}^{g-1}$, $g \ge 4$, be a canonically embedded curve such that $d(X, 2)$ is defined. Then $r_X(P) \le d(X, 2) - 1$ for all $P \in \tau(X)$.*

Proof. It is sufficient to prove the inequality for all $P \in \tau(X) \setminus X$. Fix $O \in X$ such that $P \in \langle 2O \rangle \setminus \{O\}$. Take a line bundle L on X evincing $d(X, O)$. Therefore we have $2O + S \in |L|$ with S a reduced divisor and O not in the support of S. Hence $\langle 2O \rangle \cap \langle S \rangle$ is a point, P', in $\langle 2O \rangle \setminus \{O\}$. Since $P \in \{P', O\}$, we get $r_X(P) \le \sharp(S) - 1 = d(X, O) - 1$. □

In the same way we prove the following result.

Proposition 3. *Let $X \subset \mathbb{P}^{g-1}$, $g \ge 4$, be a canonically embedded curve such that $d'(X, 2)$ is defined. We have $r_X(P) \le d'(X, 2) - 1$ for a general $P \in \tau(X)$.*

The proof of Proposition 2 and 3 gives the following result.

Proposition 4. *Fix a linearly normal smooth curve C. If $d(C, \mathcal{O}_C(1), O)$ is defined for all $O \in C$, then $r_C(P) \le d(C, \mathcal{O}_C(1), 2) + 1$ for all $P \in \sigma_2(C)$. If $d'(C, \mathcal{O}_C(1), 2)$ is defined, then $r_C(P) \le d'(C, \mathcal{O}_C(1), 2)$ for a general $P \in \tau(C)$.*

Proposition 5. *We have $d'(C, 2) \le$ birgon$(C, 2) - 3$ for every non-hyperelliptic smooth curve C of genus $g \ge 3$.*

Proof. Fix $L \in \mathrm{Pic}(C)$ evincing $\mathrm{birgon}(C, 2)$ and let $f : C \to \mathbb{P}^2$ be the associated morphism. Fix a general $O \in C$. Since O is general, O is an ordinary point with respect to f in the sense of [14], i.e. $h^0(C, L(-2O)) = 1$ and $h^0(C, L(-3O)) = 0$. Therefore the only divisor $D \in |L(-2O)$ has not O in its support. Since we are in characteristic zero, $f(C)$ is not a strange curve, i.e. there is no $o \in \mathbb{P}^2$ contained in every tangent line to a smooth point of C. Since O is general, we get $T_{f(O)} f(C) \cap \mathrm{Sing}(C) = \emptyset$. Since a general tangent line of $f(C)$ is tangent at a unique smooth point of $f(C)$ and f is birational onto its image, D contains no point of C with multiplicity ≥ 2. Since C has positive genus, we have $\mathrm{birgon}(C, 2) \ge 3$. Hence the tangent line of $f(C)$ at $f(O)$ contains another smooth point, say $f(Q)$, of $f(C)$. Take $L' := L(-Q)$ and $S := D - Q$. We have $h^0(\mathcal{O}_C(2O)) = 1$, since C is not hyperelliptic. By construction we have $h^0(C, \mathcal{O}_C(2O + S)) = h^0(L') = 2$. □

Theorem 1. *Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d \geq 4$. We have $d'(C,2) = d-3$. If the tangent line to C at O is neither a flex nor a multitangent, then $d(C,O)$ is defined and $d(C,O) = d-3$. If the tangent line to C at O is either a flex or a multitangent, then $d(C,O)$ is defined and $d(C,O) = 2d-6$. We have $d(C,2) = 2d-6$.*

Proof. The inequality $d'(C,2) \leq d-3$ follows from Proposition 5. We have $d'(C,2) \geq d-3$, because $W^1_x(C) = \emptyset$ for all $x \leq d-2$. The proof of Proposition 5 shows that $d(C,O) = d-3$ if the tangent line to C at O is neither a flex nor a multitangent. Since $W^1_{d-1}(C)$ is the set of all $\mathcal{O}_C(1)(-O)$, $O \in C$, we get that $d(C,O) \geq d$ if T_OC is either a multitangent or an inflexional tangent of C. We have $d(C,d) \geq d$, because C has at least one flex. Fix $O \in C$ and assume that the tangent line to C at O is either a flex or a multitangent. Since $\mathcal{O}_C(1)(-O)$ has no base points, $\mathcal{O}_C(2)(-2O)$ has no base points. Hence a general $D \in |\mathcal{O}_C(2)(-2O)|$ is reduced and it does not contain O in its support. Since $h^i(\mathbb{P}^2(\mathcal{O}_{\mathbb{P}^2}(2-d)) = 0$, $i = 0,1$, there is a unique conic T with $T \cap C = 2O + D$ (as schemes). Call Z the degree two subscheme of $\mathbb{P}^2$ with $2O$ as its support. A general element of $|\mathcal{I}_Z(2)|$ is a smooth conic. Since D is general, T is a general element of $|\mathcal{I}_Z(2)|$. Therefore T is a smooth conic. We have $h^0(C, \mathcal{O}_C(2)(-2O-S)) = h^0(C, \mathcal{O}_C(2)(-E)) = 2$, because the smoothness of the conic T implies that E is not formed by 4 collinear points. Therefore $d(C, \omega_C, O)$ is defined and $d(C,O) \leq 2d-6$. Assume $d(C,O) \leq 2d-7$, i.e. $d(C,O)+2 \leq 2d-5$. Take S evincing $d(C,O)$. We have $O \notin S$, $h^0(C, \mathcal{O}_C(2O+S)) = 2$ and $\mathcal{O}_C(2O+S)$ has no base points. Hence $d(C,O)+2$ is in the Lüroth semigroup of C. Since $d(C,O)+2 \leq 2d-5$, S. Greco and G. Raciti proved that $d(C,O)+2 \in \{d-1, d\}$ ([12], [9]). We excluded the case $d(C,O)+2 = d-1$, because the tangent line of C at O is either a flex or a multitangent and hence $O \notin \Delta(C)_1$. Now assume $d(C,O)+2 = d$ and set $R := \mathcal{O}_C(2O+S)$. By assumption R is a base point free line bundle of degree d. Since T_OC is either a flex or a multitangent, $2O+S \notin |\mathcal{O}_C(1)|$. Therefore $R \neq \mathcal{O}_C(1)$. Fix a general $A \in |R|$. Since $h^0(R) = 2$, $|\omega_C| = |\mathcal{O}_{\mathbb{P}^2}(d-3))$, we have $h^1(\mathbb{P}^2, \mathcal{I}_A(d-3)) > 0$. Since $\deg(A) \leq 2(d-3)+1$, there is a line $J \subset \mathbb{P}^2$ such that $\deg(A \cap J) \geq d-1$ ([8, Lemma 34]). Since R has no base points, we also have $h^1(\mathbb{P}^2, \mathcal{I}_{A'}(d-3)) = 0$ for every $A' \subsetneq A$ (Riemann-Roch and the adjunction formula). Therefore $A \subset J \cap C$. Since $\deg(A) = d$, we get $A \in |\mathcal{O}_C(1)|$ and hence $R \cong \mathcal{O}_C(1)$, a contradiction. □

Proof of Corollary 1. Fix $O \in C$ such that the tangent line T_OC of C at O is neither a tangent line to C ad O is neither a flex of C nor a multitangent of C. We have $T_OC \cap C = 2O + S$ with S a finite set of C with cardinality $d-2$ and

$O \notin S$. We have $\mathbb{M}(C, O) = \{\mathcal{O}_C(1)(-Q)\}_{Q \in S}$. Since C has positive genus, the line bundles $\mathcal{O}_C(1)(-Q)$ and $\mathcal{O}_C(1)(-Q')$ are not isomorphic if $Q \neq Q'$. Therefore $\mu(C, O) = d-2$. Use part (ii) of Proposition to get that $r_X(P) = d-3$ for at least one and at most $d-2$ points of $T_{\phi(O)}(X)\backslash\{\phi(O)\}$. Fix $Q \in S$ and set $S' := S\backslash Q$. Since $h^0(\mathcal{O}_C(2O+S')) = h^0(C, \mathcal{O}_C(1)(-Q)) = 2$, the set $T_O X \cap \langle S' \rangle$ is a single point, P', and $P' \neq O$. Fix any $P \in \langle T_{\phi(O)} X$. Since $P' \neq O$, we have $P \in \langle\{P', \phi(O)\}\rangle$. Therefore $P \in \langle\{O\} \cup S\rangle$. Hence $r_X(P) \leq d-2$. Now take O such that $T_O C$ is either a flex or a multitangent of C. Theorem 5 gives the existence of $P' \in T_{\phi(O)} X \setminus \phi(O)$ with $r_X(P') = 2d-6$. Since $P' \neq \phi(O)$ as above we get $r_X(P) \leq 2d-5$ for all $P' \in T_{\phi(O)} X \setminus \phi(O)$. $\square$

It is often easier to compute the integer $d'(C, 2)$ then the integer $d(C, 2)$. It may also be $d(C, 2) \gg d'(C, 2)$.

Lemma 2. *Let C be a smooth curve of genus $g \geq 2$ for which $d'(C, 2)$ is defined. Then* $\dim(W^1_{d'(C,2)+2}(C)) \geq 1$.

Proof. If $W^2_{d'(C,2)+2}(C) \neq \emptyset$, then $\dim(W^1_{d'(C,2)-+1}(C)) \geq 1$ and hence we have $\dim(W^1_{d'(C,2)+1}(C)) \geq 1$ in this case. Now assume $W^2_{d'(C,2)+2}(C) = \emptyset$. In this case we have $G^1_{d'(C,2)+2}(C) = W^1_{d'(C,2)+2}(C)$. Since any non-constant morphism $f : C \to \mathbb{P}^1$ has only finitely many ramification points, then

$$\dim(G^1_{d'(C,2)+2}(C)) \geq 1. \qquad \square$$

Proposition 6. *We have $d'(X, 2) = \lfloor g/2 \rfloor$ for a general curve X of genus $g \geq 7$.*

Proof. Take an odd integer $g \geq 7$ and set $k := (g+3)/2$. We have $\rho(g, 1, k) = 2k - g - 2 = 1$, $\rho(g, 1, k-1) < 0$ and $\rho(g, 2, k) < 0$. Therefore Gieseker-Petri and Fulton-Lazarsfeld theorems say that $W^1_k(X)$ is a smooth and irreducible projective curve and that every $L \in W^1_k(X)$ is base point free and with $h^0(L) = 2$ ([3]). Fix a general $L \in W^1_k(X)$ and call $f : X \to \mathbb{P}^1$ the associated morphism. For a general $L \in W^1_k(X)$ the map f has only ordinary ramification and no two ramification points have the same f-image. The Riemann-Hurwitz formula gives that f has $2k+2g-2 = 3g+1$ ramification points. It is sufficient to prove that these ramification points are not the same for a general pair $(L, R) \in W^1_k(X) \times W^1_k(X)$. Assume that these ramification points are the same for all general (L, R). Let $\mathcal{M}^0_g$ the open part of the moduli space $\mathcal{M}_g$ of smooth curves formed by the curves without non-trivial automorphisms. Over $\mathcal{M}_{g,0}$ we would get a degree $3g+1$ multisection of the morphism $\pi_1 : \mathcal{M}_{g,1} \to \mathcal{M}_g$, contradicting the fact that any such rational multisection has a degree multiple of $2g-2$ ([1], [2], [13], [17]) (we have $3g+1 < 4g-4$, because $g \geq 7$). Since $\rho(g,1,k-1) < 0$, Lemma 2 gives $d'(X,2)(X) \leq k-2$.

Now assume that g is even. We have $\rho(g,1,g/2+1) = 0$ and hence Lemma 2 gives $d'(C,2) \geq g/2+2$. We have $\dim(W^1_{g/2+2}(X)) = \rho(g,r,g/2+2) = 2$. We use the proof of the odd genus case. In this case we have $2(g/2+2)+2g-2 = 3g+2$ ramification points and we use that $3g+2 < 4g-4$, because we assumed $g \geq 8$. □

Question 1. In the set-up of Proposition 6 it is reasonable to conjecture that $d(X,2) = d'(X,2)$ for a general smooth curve of genus $g \geq 7$.

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