ON THE $X$-RANK OF A POINTS OF THE TANGENT DEVELOPABLE OF A CURVE IN A PROJECTIVE SPACE

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Abstract: Let $X \subset \mathbb{P}^n$ be a smooth curve. For any $P \in \mathbb{P}^n$ the $X$-rank of $P$ is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \rangle$ denote the linear span. Let $\tau(X) \subset \mathbb{P}^n$ be the tangent developable of $X$. We compute upper bounds for the $X$-rank of all $P \in \tau(X)$ or of the general $P \in \tau(X)$, mainly if $X$ is a canonically embedded curve. To do that we define some invariants for the pair $(X, \mathcal{O}_X(1))$ and compute them if $X$ is canonically embedded and either $X$ is a smooth plane curve or it has general moduli.

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1. Introduction

Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ the $X$-rank $r_X(P)$ of $P$ is the minimal cardinality of a subset $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \rangle$ denote the linear span ([16], [8], [15]). Now assume that $X$ is a smooth curve and that $n \geq 3$. Let $\tau(X) \subset \mathbb{P}^n$ be the tangent developable of $X$, i.e. the unions of all tangent lines $T_O X$. Quite often the maximal among the
of all system of Pic gon denote the set of all line bundles $h \in \mathcal{L}(C)$ have $\subset C$ the minimal degree of a line bundle $X \subset$ a smooth curve, usually when $X$ is a smooth curve, usually when $X$ is the canonical model of a non-hyperelliptic smooth curve. There are also some easy and well-known lower bound and sometimes we get in this way a good picture). As a byproduct of these definitions we prove the following result.

Corollary 1. Let $C \subset \mathbb{P}^2$, $d \geq 4$, be a smooth plane curve of degree $d$. Let $X \subset \mathbb{P}^{g-1}$, $g = (d - 1)(d - 2)/2$, be the canonical model of $C$ and $\phi : C \to X$ the canonical map Then $r_X(P) \leq 2d - 5$ for all $P \in \tau(X)$ and $r_X(P) = d - 2$ for a general $P \in \tau(X)$ and $r_X(P) = d - 3$ for some $P \in \tau(X)$. If $d \geq 5$ and $O \in C$ is such that the tangent line to $C$ at $O$ is neither a flex nor a multitangent, then $r_X(P) = d - 3$ for at least one and at most $d - 2$ points of $T_\phi(O)(X) \setminus \{\phi(O)\}$ and $r_X(P) = d - 2$ for all other points of $T_\phi(O)(X) \setminus \{\phi(O)\}$. We have $r_X(P) \geq 2d - 6$ for at least one $P \in \tau(X)$.

In section 2 we study the lowest integers $r_X(P)$, $P \in \tau(X) \setminus X$. In section 3 we study $r_X(P) \in \tau(X)$ (see Proposition 6 for curves with general moduli, Propositions 2, 3 and 5 for an arbitrary curve and Theorem 1 for a result giving, with a little effort, Corollary 1).

We work over an algebraically closed field $\mathbb{K}$ such that $\text{char}(\mathbb{K}) = 0$.

2. The Lowest Ranks of Points of $\tau(X) \setminus X$

Notation 1. Let $C \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. Let $\beta(C)$ be the maximal integer such that every zero-dimensional subscheme of $C$ with degree at most $\beta(C)$ is linearly independent.

Let $X$ be a smooth curve of genus $g \geq 3$. The gonality $\text{gon}(X)$ of $X$ is the minimal degree of a line bundle $L$ on $X$ with $h^0(X, L) \geq 2$. Let $\text{Gon}(X)$ denote the set of all line bundles $L$ on $X$ with degree $\text{gon}(X)$ and $h^0(X, L) \geq 2$. By the definition of gonality we have $\text{Gon}(X) \neq \emptyset$. For each $L \in \text{Gon}(X)$ we have $h^0(X, L) = 2$ and $X$ is base point free. The set $\text{Gon}(X)$ is a closed subset of $\text{Pic}^{\text{gon}(X)}(X)$ and $\dim(\text{Gon}(X)) \leq 1$ ([11]). If $X$ is not hyperelliptic and $C \subset \mathbb{P}^{g-1}$ is the canonical model of $X$, then $\beta(C) = \text{gon}(X) - 1$.

Fix $L \in \text{Gon}(X)$. Since $L$ is spanned and $h^0(X, L) = 2$, the complete linear system $|L|$ induces a degree $\text{gon}(X)$ morphism $h_L : X \to \mathbb{P}^1$. Let $R(L)$ be the set of all $O \in X$ such that $h^0(X, L(-2O)) > 0$. Since $L$ has no base points, we have
$h^0(X, L(-2O)) = 1$ (i.e. $h^1(X, L(-2O)) = h^1(X, L) + 1$) for each $O \in R(L)$ and $h^0(X, L(-2O)) = 0$ for each $O \in X \setminus R(L)$. For each $O \in R(L)$ let $D_{L,O}$ be the unique effective divisor such that $D_{L,O} + 2O \in |L|$. Since $g > 0$ we have $R(L) \neq \emptyset$ for each $L \in \text{gon}(X)$. Let $R(L)'$ be the set of all $O \in R(L)$ such that $D_{L,O}$ is reduced (we allow the case in which $O$ appears in $D_{L,O}$, i.e. we allow the case $h^0(X, L(-3O)) = 1$, but of course $h^0(X, L(-4O)) = 0$ if $O \in R(L)'$. Let $R(L)''$ be the set of all $O \in R(L)'$ such that $D_{L,O} - O$ is effective; $R(L)'''$ is the set of all $O \in X$ such that $h^0(X(L - 3O)) = 1$, the unique divisor $D'_{L,O}$ of $|L(-3O)|$ is reduced and $O \notin D'_{L,O}$. Set $R(L)_1 := R(L)' \setminus R(L)'', i.e. let $R(L)_1$ be the set of all $O \in X$ with $2O + S \in |L|$ with $S$ reduced and $O \notin S$. Let $\Delta(X)_1$ be the union of all $O \in R(L)_1, L \in \text{Gon}(X)$. For each $O \in \Delta_1(X)$ let $\mathbb{M}(X, O)$ denote the set of all $L \in \text{Gon}(X)$ such that $O \in R(L)_1$. Set $\mu(X, O) := \sharp(\mathbb{M}(X, O))$ with the convention $\mu(X, O) = +\infty$ if $\mathbb{M}(X, O)$ is infinite.

**Remark 1.** Easy examples (standard cyclic coverings with degree $k$ and large $g$) show that we may have $R'(L) = \emptyset$ for all $L \in \text{Gon}(X)$. In particular we may have $\Delta(X)_1 = \emptyset$.

**Remark 2.** Let $C \subset \mathbb{P}^n$ be a smooth curve. Take zero-dimensional schemes $A, B \subset C$. Therefore $A$ and $B$ are effective divisors of $C$. For each $P \in C$ let $a_P$ (resp. $b_P$) be the degree of the connected component of $A$ (resp. $B$). The effective divisor $A + B$ has degree $\deg(A) + \deg(B)$ (each $P$ appears with multiplicity $a_P + b_P$ in $A + B$). The scheme $A \cup B$ is the minimal subscheme of $C$ containing both $A$ and $B$. Each $P \in C$ appears with multiplicity $\max\{a_P, b_P\}$ in $A \cup B$. Therefore $A + B = A \cup B$ if and only if $A \cap B = \emptyset$.

**Proposition 1.** Assume $\text{gon}(X) \geq 6$. We have $r_X(P) \geq \text{gon}(X) - 2$ for all $P \in \tau(X) \setminus X$. Fix $O \in X$.

(i) If $O \notin \Delta(X)_1$, then $r_X(P) \geq \text{gon}(X) - 1$ for all $P \in T_OX \setminus \{O\}$.

(ii) Assume $O \in \Delta(X)_1$. Then $\text{gon}(X) - 2 \leq r_X(P) \leq \text{gon}(X) - 1$ for all $P \in T_OX \setminus \{O\}$. Let $\mathcal{W}$ be the set of all $P \in T_OX \setminus \{O\}$ such that $r_X(P) = \text{gon}(X) - 2$. We have $\mathcal{W} \neq \emptyset$ and $\sharp(\mathcal{W}) \leq \mu(X, O)$.

**Proof.** We have $\beta(X) = \text{gon}(X) - 1 \geq 5$. Since $\beta(X) \geq 4$, for each $P \in \tau(X)$ there is a unique zero-dimensional scheme $O \in X$ with $P \in T_OX$. Fix $P \in \tau(X) \setminus X$ and take the unique $O \in X$ such that $P \in T_OX$, i.e. such that $P \in (2O)$. Fix $S \subset X$ evincing $r_X(P)$, i.e. fix $S \subset X$ with $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. By [7, Lemma 1] we have $h^1(\mathbb{P}^g - 1, \mathcal{I}_{2O \cup S}(1)) > 0$, i.e. $h^1(X, \omega_X(-2O \cup S)) \geq 2$ and hence either $\sharp(S) \geq \text{gon}(X) - 1$ or $\sharp(S) = \text{gon}(X) - 2$ and $O \notin S$ (Remark 2). We get part (i).

Now assume $O \in \Delta(X)_1$. Fix $L \in \mathbb{M}(X, O)$ and let $S \subset X$ be the set such
that $2O + S \in |L|$. Since $O \notin S$, we have $2O + S = 2O \cup S$, i.e. $2O + S$ is the minimal zero-dimensional scheme containing both $2O$ and $S$. Since $2O + S$ is a divisor of a line bundle evincing the gonality of $X$, Grassmann’s formula gives that $(2O) \cap (S)$ is a single point, $P_L$. We get $W \neq \emptyset$ and the existence of a surjection $\mathbb{M}(X, O) \rightarrow W$. Hence $\#(W) \leq \mu(X, O)$. Fix any $P' \in W$, say associated to $L \in \Delta(X)_1$, and take the only $S \subset X$ such that $2O + S \in |L|$. We have $O \notin S$ and we saw that $\{P'\} = (2O) \cap (S)$. Fix any $P \in T_O X$. Since $P' \neq O$, we have $P \in \langle \{O\} \cup S \rangle$ and hence $r_X(P) \leq \text{gon}(X) - 1$. \qed

3. The Rank of a General Point of the Tangent Developable

Let $C$ be a smooth and connected projective curve of genus $g$. We recall the following definitions ([10], [5], [6]). For each integer $r \geq 1$ the $r$-gonality $\text{gon}(C, r)$ of $C$ is the minimal integer $d$ such that there is a degree $d$ line bundle $L$ on $C$ with $h^0(C, L) \geq r + 1$ (note that $h^0(C, L) = r + 1$ and that $L$ has no base points for each such $L$). The integer $\text{gon}(C, 1)$ is the gonality $\text{gon}(C)$ of $C$. For every integer $r \geq 2$ the $r$-birational gonality $\text{birgon}(C, r)$ is the minimal integer $d$ such that there is a degree $d$ line bundle $L$ on $C$ with $h^0(C, L) \geq r + 1$ and the rational map induced by $|L|$ is birational onto its image (notice that $h^0(C, L) = r + 1$ and $L$ has no base points for each such $L$). For every integer $r \geq 3$ the $r$-embedding gonality $\text{embgon}(C, r)$ is the minimal integer $d$ such that there is a degree $d$ line bundle $L$ on $C$ with $h^0(C, L) \geq r + 1$ and the rational map induced by $|L|$ is an embedding.

If $C$ has general moduli and $1 \leq r \leq g - 2$, then $\text{gon}(C, r) = \lceil rg/(r - 1) \rceil + r - 1$ and the set $W^r_d(C)$ of line bundles on $C$ evincing $d := \text{gon}(C, r)$ has pure dimension $\rho(g, r, d) := (r + 1)d - rg - r(r + 1)$; this set $W^r_d(C)$ is also irreducible if $\rho(g, r, d) > 0$ ([3]). A dimensional count shows that if $C$ is general and $r \geq 2$, then $\text{gon}(C, r) = \text{birgon}(C, r)$ and that if $\rho(g, r, d) > 0$, then a general element of $W^r_d(C)$ induces a morphism $C \rightarrow \mathbb{P}^r$ which is birational onto its image. It is also well-known that if $r \geq 3$, then $\text{gon}(C, r) = \text{embgon}(C, r)$.

Let $C$ be a smooth and connected projective curve of genus $g$. For each $R \in \text{Pic}(C)$ and each $O \in C$ let $d(C, R, O)$ be the minimal integer $d$ such that there is $S \subset C$, $S$ is reduced, $\sharp(S) = d$, $h^0(C, R(-2O - S)) = h^0(R(-2O)) - \sharp(S) + 1$ and $h^0(C, R(-O - S)) = h^0(C, R(-O)) - \sharp(S)$ (if there is at least one such set $S$; if no set $S$ exists, then we say that $d(C, R, O)$ is not defined). Assume that $d(C, R, O)$ is defined. By Riemann-Roch the last condition is equivalent to $h^1(R(-2O - S)) = h^1(R(-2O)) + 1$. If $R$ is very ample, then $h^0(C, R(-2O)) = h^0(R) - 2$ and hence $h^1(C, R) = h^1(C, R(-2O))$. Therefore if $R$ is very ample
and $D(C,R,O)$ is defined, then $d(C,R,O)$ is the minimal cardinality of a finite set $S$ with $O \notin S$, $h^1(C,R(-2O-S)) > h^1(C,R)$ and $h^1(C,R(-O-S)) = h^1(C,R)$. If $d(C,R,O)$ is defined for all $O \in C$ let $d(C,R,2)$ be the maximum of all integers $d(C,R,O)$, $O \in C$. Let $d'(C,R,2)$ be the minimal integer $d$ such that there is a finite set $A \subset C$ with $d(C,R,O)$ defined and $d \geq d(C,R,O)$ for all $O \in C \setminus A$. Now assume that $C$ has genus $g \geq 3$ and that $C$ is not hyperelliptic, i.e. assume that $\omega_X$ is very ample. Set $d(C,O) := d(C,\omega_C,O)$, $d(C,2) := d(C,\omega_C,2)$ and $d'(C,2) := d'(C,\omega_C,2)$ (if these integers are defined).

**Lemma 1.** Let $C$ be a smooth curve and let $R \in \text{Pic}(C)$ be a line bundle with $\deg(R) \geq 4$. Fix $O \in C$ and assume $h^0(R(-2O)) = h^0(R) - 2$, that $R(-2O)$ is spanned and that the morphism $\phi$ associated to $R(-2O)$ is birational onto its image. Then $d(C,R,O)$ is defined and $d(C,R,O) \leq h^0(R) - 1$.

**Proof.** Since $R(-2O)$ is spanned, $R$ is spanned at each point of $C \setminus \{O\}$. Since $h^0(R(-2O)) = h^0(R) - 2$, $R$ is spanned at $O$. Let $\psi : C \to \mathbb{P}^r$, $r = h^0(R) - 1$, be the morphism associated to $|R|$. Since $\phi$ is birational onto its image, $\psi$ is birational onto its image. Set $Y := \psi(C)$ and $B := \text{Sing}(Y)$. Set $B' := \psi^{-1}(B)$. Notice that $\psi$ induces an isomorphism between $C \setminus B'$ and $Y \setminus B$. Since $h^0(R(-2O)) = h^0(R) - 2$, the zero-dimensional scheme $Z := \psi(2O)$ has degree two. Hence $\langle Z \rangle$ is a line. Since $\deg(R(-2O)) \geq 2$, we have $Y \neq \langle Z \rangle$ and hence there is $P \in \langle Z \rangle \setminus \langle Z \rangle \cap Y$. Let $H \subset \mathbb{P}^r$ be a general hyperplane containing $P$. Since $P \notin Y$, Bertini’s theorem implies that the scheme $Y \cap H$ is formed by $\deg(R)$ distinct points, none of them being in $B \cup \{\phi(O)\}$. Since $P' \in \langle Z \rangle \setminus \{\psi(O)\}$, Therefore there is a set $A \subset C \setminus \{(O) \cup B'\}$ with $\psi(A) = Y \cap H$. Since $P' \in \langle Z \rangle \setminus \{\psi(O)\}$, we have $\langle Y \cap H \cap \langle Z \rangle \rangle = \{P'\}$ and $\langle (Y \cap H) \cup \{\psi(O)\}\rangle$. Hence $h^0(R(-A)) = 1$, $h^0(R(-O-A)) = 0$ and $h^0(R(-2O-A)) = 0$. Therefore $h^1(R(-A)) = h^1(R(-A - O)) < h^1(R(-A - 2O))$. Hence $d(C,R,O)$ is defined and $d(C,R,O) \leq h^0(R) - 1$. 

**Corollary 2.** Let $C$ be a smooth curve of genus $g \geq 5$ which is neither hyperelliptic, nor trigonal nor bielliptic. Then $d(C,\omega_C,2)$ is defined.

**Proof.** Fix $O \in C$. Since $C$ is not hyperelliptic, the line bundle $\omega_C$ is very ample and hence $h^0(C,\omega_C(-2O)) = g - 2$. Since $C$ is not trigonal, $\omega_C(-2O)$ is spanned. Let $\phi : C \to \mathbb{P}^{g-3}$ be the morphism induced by $|\omega_C(-2O)|$. By Lemma 1 it is sufficient to prove that $\phi$ is birational onto its image. Assume $\deg(\phi) \geq 2$. Since $\phi(C)$ spans $\mathbb{P}^{g-3}$, we have $\deg(\phi(C)) \geq g - 3$. Assume for the moment $g \geq 6$. Since $\deg(\phi) \cdot \deg(\phi(C)) = 2g - 4$ and $g \geq 5$, we get $\deg(\phi) = 2$ and $\deg(\phi(C)) = g - 2$. Since $C$ is not hyperelliptic and $\deg(\phi) = 2$, the normalization of $\phi(C)$ is not rational. Since $p_a(\phi(C)) \leq 1$ by the upper bound.
for the arithmetic genera of non-degenerate curve in $\mathbb{P}^r$ with degree $\leq r+1$, we get that $\phi(C)$ is an elliptic curve. Hence $C$ is a bielliptic curve, a contradiction. Now assume $g = 5$. We need to exclude the case $\deg(\phi) = 3$. This is excluded, because in this case $\phi(C)$ would be a smooth conic and hence $C$ would be a trigonal curve.

\textbf{Proposition 2.} Let $X \subset \mathbb{P}^{g-1}$, $g \geq 4$, be a canonically embedded curve such that $d(X, 2)$ is defined. Then $r_X(P) \leq d(X, 2) - 1$ for all $P \in \tau(X)$.

\textbf{Proof.} It is sufficient to prove the inequality for all $P \in \tau(X) \setminus X$. Fix $O \in X$ such that $P \in \langle 2O \rangle \setminus \{O\}$. Take a line bundle $L$ on $X$ evincing $d(X, O)$. Therefore we have $2O + S \in |L|$ with $S$ a reduced divisor and $O$ not in the support of $S$. Hence $\langle 2O \rangle \cap \langle S \rangle$ is a point, $P'$, in $\langle 2O \rangle \setminus \{O\}$. Since $P \in \{P', O\}$, we get $r_X(P) \leq \sharp(S) - 1 = d(X, O) - 1$.

In the same way we prove the following result.

\textbf{Proposition 3.} Let $X \subset \mathbb{P}^{g-1}$, $g \geq 4$, be a canonically embedded curve such that $d'(X, 2)$ is defined. We have $r_X(P) \leq d'(X, 2) - 1$ for a general $P \in \tau(X)$.

The proof of Proposition 2 and 3 gives the following result.

\textbf{Proposition 4.} Fix a linearly normal smooth curve $C$. If $d(C, O_C(1), O)$ is defined for all $O \in C$, then $r_C(P) \leq d(C, O_C(1), 2) + 1$ for all $P \in \sigma_2(C)$. If $d'(C, O_C(1), 2)$ is defined, then $r_C(P) \leq d'(C, O_C(1), 2)$ for a general $P \in \tau(C)$.

\textbf{Proposition 5.} We have $d'(C, 2) \leq \text{birgon}(C, 2) - 3$ for every non-hyperelliptic smooth curve $C$ of genus $g \geq 3$.

\textbf{Proof.} Fix $L \in \text{Pic}(C)$ evincing $\text{birgon}(C, 2)$ and let $f : C \to \mathbb{P}^2$ be the associated morphism. Fix a general $O \in C$. Since $O$ is general, $O$ is an ordinary point with respect to $f$ in the sense of [14], i.e. $h^0(C, L(-2O)) = 1$ and $h^0(C, L(-3O)) = 0$. Therefore the only divisor $D \in |L(-2O)|$ has not $O$ in its support. Since we are in characteristic zero, $f(C)$ is not a strange curve, i.e. there is no $o \in \mathbb{P}^2$ contained in every tangent line to a smooth point of $C$. Since $O$ is general, we get $T_{f(O)} f(C) \cap \text{Sing}(C) = \emptyset$. Since a general tangent line of $f(C)$ is tangent at a unique smooth point of $f(C)$ and $f$ is birational onto its image, $D$ contains no point of $C$ with multiplicity $\geq 2$. Since $C$ has positive genus, we have $\text{birgon}(C, 2) \geq 3$. Hence the tangent line of $f(C)$ at $f(O)$ contains another smooth point, say $f(Q)$, of $f(C)$. Take $L' := L(-Q)$ and $S := D - Q$. We have $h^0(O_C(2O)) = 1$, since $C$ is not hyperelliptic. By construction we have $h^0(C, O_C(2O + S)) = h^0(L') = 2$. \qed
Theorem 1. Let \( C \subset \mathbb{P}^2 \) be a smooth curve of degree \( d \geq 4 \). We have \( d'(C, 2) = d - 3 \). If the tangent line to \( C \) at \( O \) is neither a flex nor a multitangent, then \( d(C, O) \) is defined and \( d(C, O) = d - 3 \). If the tangent line to \( C \) at \( O \) is either a flex or a multitangent, then \( d(C, O) \) is defined and \( d(C, O) = 2d - 6 \). We have \( d(C, 2) = 2d - 6 \).

Proof. The inequality \( d'(C, 2) \leq d - 3 \) follows from Proposition 5. We have \( d'(C, 2) \geq d - 3 \), because \( W^1_x(C) = \emptyset \) for all \( x \leq d - 2 \). The proof of Proposition 5 shows that \( d(C, O) = d - 3 \) if the tangent line to \( C \) at \( O \) is neither a flex nor a multitangent. Since \( W^1_{d-1}(C) \) is the set of all \( \mathcal{O}_C(1)(-O) \), \( O \in C \), we get that \( d(C, O) \geq d \) if \( T_O C \) is either a multitangent or an inflexional tangent of \( C \). We have \( d(C, d) \geq d \), because \( C \) has at least one flex. Fix \( O \in C \) and assume that the tangent line to \( C \) at \( O \) is either a flex or a multitangent. Since \( \mathcal{O}_C(1)(-O) \) has no base points, \( \mathcal{O}_C(2)(-2O) \) has no base points. Hence a general \( D \in |\mathcal{O}_C(2)(-2O)| \) is reduced and it does not contain \( O \) in its support. Since \( h^i(\mathbb{P}^2(\mathcal{O}_{\mathbb{P}^2}(2 - d))) = 0 \), \( i = 0, 1 \), there is a unique conic \( T \) with \( T \cap C = 2O + D \) (as schemes). Call \( Z \) the degree two subscheme of \( \mathbb{P}^2 \) with \( 2O \) as its support. A general element of \( |\mathcal{I}_Z(2)| \) is a smooth conic. Since \( D \) is general, \( T \) is a general element of \( |\mathcal{I}_Z(2)| \). Therefore \( T \) is a smooth conic. We have \( h^0(C, \mathcal{O}_C(2)(-2O - S)) = h^0(C, \mathcal{O}_C(2)(-E)) = 2 \), because the smoothness of the conic \( T \) implies that \( E \) is not formed by \( 4 \) collinear points. Therefore \( d(C, \omega_C, O) \) is defined and \( d(C, O) \leq 2d - 6 \). Assume \( d(C, O) \leq 2d - 7 \), i.e. \( d(C, O) + 2 \leq 2d - 5 \). Take \( S \) evincing \( d(C, O) \). We have \( O \notin S \), \( h^0(C, \mathcal{O}_C(2O + S)) = 2 \) and \( \mathcal{O}_C(2O + S) \) has no base points. Hence \( d(C, O) + 2 \) is in the Lüroth semigroup of \( C \). Since \( d(C, O) + 2 \leq 2d - 5 \), S. Greco and G. Raciti proved that \( d(C, O) + 2 \in \{d - 1, d\} \) ([12], [9]). We excluded the case \( d(C, O) + 2 = 3 - 1 \), because the tangent line of \( C \) at \( O \) is either a flex or a multitangent and hence \( O \notin \Delta(C)_1 \). Now assume \( d(C, O) + 2 = d \) and set \( R := \mathcal{O}_C(2O + S) \). By assumption \( R \) is a base point free line bundle of degree \( d \). Since \( T_O C \) is either a flex or a multitangent, \( 2O + S \notin |\mathcal{O}_C(1)| \). Therefore \( R \neq \mathcal{O}_C(1) \). Fix a general \( A \in |R| \). Since \( h^0(R) = 2 \), \( |\omega_C| = |\mathcal{O}_{\mathbb{P}^2}(d - 3)| \), we have \( h^1(\mathbb{P}^2, \mathcal{I}_A(d - 3)) > 0 \). Since \( \deg(A) \leq 2(d - 3) + 1 \), there is a line \( J \subset \mathbb{P}^2 \) such that \( \deg(A \cap J) \geq d - 1 \) ([8, Lemma 34]). Since \( R \) has no base points, we also have \( h^1(\mathbb{P}^2, \mathcal{I}_{A'}(d - 3)) = 0 \) for every \( A' \not\subset A \) (Riemann-Roch and the adjunction formula). Therefore \( A \subset J \cap C \). Since \( \deg(A) = d \), we get \( A \in |\mathcal{O}_C(1)| \) and hence \( R \cong \mathcal{O}_C(1) \), a contradiction.

Proof of Corollary 1. Fix \( O \in C \) such that the tangent line \( T_O C \) of \( C \) at \( O \) is neither a tangent line to \( C \) ad \( O \) is neither a flex of \( C \) nor a multitangent of \( C \). We have \( T_O C \cap C = 2O + S \) with \( S \) a finite set of \( C \) with cardinality \( d - 2 \) and
$O \notin S$. We have $\mathbb{M}(C,O) = \{O_C(1)(-Q)\}_{Q \in S}$. Since $C$ has positive genus, the line bundles $O_C(1)(-Q)$ and $O_C(1)(-Q')$ are not isomorphic if $Q \neq Q'$. Therefore $\mu(C,O) = d - 2$. Use part (ii) of Proposition to get that $r_X(P) = d - 3$ for at least one and at most $d - 2$ points of $T_{\phi(O)}(X) \setminus \{\phi(O)\}$. Fix $Q \in S$ and set $S' := S \setminus Q$. Since $h^0(O_C(2O + S')) = h^0(C,O_C(1)(-Q)) = 0$, the set $T_{\phi(O)}(X) \cap \langle S' \rangle$ is a single point, $P'$, and $P' \neq O$. Fix any $P \in \langle T_{\phi(O)}X \rangle$. Since $P' \neq O$, we have $P \in \langle \{P',\phi(O)\} \rangle$. Therefore $P \in \langle \{O\} \cup S \rangle$. Hence $r_X(P) \leq d - 2$. Now take $O$ such that $T_{\phi(O)}C$ is either a flex or a multitangent of $C$. Theorem 5 gives the existence of $P' \in T_{\phi(O)}X \setminus \phi(O)$ with $r_X(P') = 2d - 6$. Since $P' \neq \phi(O)$ as above we get $r_X(P) \leq 2d - 5$ for all $P' \in T_{\phi(O)}X \setminus \phi(O)$. \hfill \Box

It is often easier to compute the integer $d'(C, 2)$ then the integer $d(C, 2)$. It may also be $d(C, 2) \gg d'(C, 2)$.

**Lemma 2.** Let $C$ be a smooth curve of genus $g \geq 2$ for which $d'(C, 2)$ is defined. Then $\dim(W^1_{d'(C, 2)+2}(C)) \geq 1$.

**Proof.** If $W^2_{d'(C, 2)+2}(C) \neq \emptyset$, then $\dim(W^1_{d'(C, 2)+1}(C)) \geq 1$ and hence we have $\dim(W^1_{d'(C, 2)+1}(C)) \geq 1$ in this case. Now assume $W^2_{d'(C, 2)+2}(C) = \emptyset$. In this case we have $G^1_{d'(C, 2)+2}(C) = W^1_{d'(C, 2)+2}(C)$. Since any non-constant morphism $f : C \to \mathbb{P}^1$ has only finitely many ramification points, then
\[
\dim(G^1_{d'(C, 2)+2}(C)) \geq 1. \hfill \Box
\]

**Proposition 6.** We have $d'(X, 2) = \lfloor g/2 \rfloor$ for a general curve $X$ of genus $g \geq 7$.

**Proof.** Take an odd integer $g \geq 7$ and set $k := (g + 3)/2$. We have $ho(g, 1, k) = 2k - g - 2 = 1, \rho(g, 1, k - 1) < 0$ and $\rho(g, 2, k) < 0$. Therefore Gieseker-Petri and Fulton-Lazarsfeld theorems say that $W^1_k(X)$ is a smooth and irreducible projective curve and that every $L \in W^1_k(X)$ is base point free and with $h^0(L) = 2$ ([3]). Fix a general $L \in W^1_k(X)$ and call $f : X \to \mathbb{P}^1$ the associated morphism. For a general $L \in W^1_k(X)$ the map $f$ has only ordinary ramification and no two ramification points have the same $f$-image. The Riemann-Hurwitz formula gives that $f$ has $2k + 2g - 2 = 3g + 1$ ramification points. It is sufficient to prove that these ramification points are not the same for a general pair $(L, R) \in W^1_k(X) \times W^1_k(X)$. Assume that these ramification points are the same for all general $(L, R)$. Let $\mathcal{M}_g^0$ the open part of the moduli space $\mathcal{M}_g$ of smooth curves formed by the curves without non-trivial automorphisms. Over $\mathcal{M}_g^0$ we would get a degree $3g + 1$ multisection of the morphism $\pi_1 : \mathcal{M}_g \to \mathcal{M}_g$, contradicting the fact that any such rational multisection has a
degree multiple of $2g-2$ ([1], [2], [13], [17]) (we have $3g+1 < 4g-4$, because $g \geq 7$). Since $\rho(g, 1, k-1) < 0$, Lemma 2 gives $d'(X, 2)(X) \leq k-2$.

Now assume that $g$ is even. We have $\rho(g, 1, g/2+1) = 0$ and hence Lemma 2 gives $d'(C, 2) \geq g/2+2$. We have $\dim(W^1_{g/2+2}(X)) = \rho(g, r, g/2+2) = 2$. We use the proof of the odd genus case. In this case we have $2(g/2+2)+2g-2 = 3g+2$ ramification points and we use that $3g+2 < 4g-4$, because we assumed $g \geq 8$.

\textbf{Question 1.} In the set-up of Proposition 6 it is reasonable to conjecture that $d(X, 2) = d'(X, 2)$ for a general smooth curve of genus $g \geq 7$.

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\section*{References}


