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# FACTORIZATIONS OF ANALYTIC FUNCTIONS VIA GENERALIZED INVERSE POWER PRODUCT EXPANSIONS

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**Abstract:** Given an arbitrary sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$ and an arbitrary nonzero sequence of complex numbers $\{r_n\}_{n=1}^{\infty}$, we study the expansion of the Taylor series $1+\sum_{n=1}^{\infty} a_n x^n$ into infinite products of the form $\prod_{n=1}^{\infty}(1-h_n x^n)^{-r_n}$. Algebraic properties, convergence criteria, and combinatorial interpretations of the infinite products are investigated. We also provide an asymptotic formula for the majorizing product expansion associated with $1-\sum_{n=1}^{\infty} s^n x^n$, $s := \sup_{n\geq 1} |a_n|^{\frac{1}{n}}$.

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## 1. Introduction

Let $f(x)$ be a complex function with power series representation $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$. In [9] the authors discovered algebraic, analytic, and combinatorial

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properties of the generalized power product expansion $f(x) = \prod_{n=1}^{\infty}(1+g_n x^n)^{r_n}$, where $\{r_n\}$ is an arbitrary set of nonzero complex numbers. Such work complements and extends the results of Borofsky, Feld, Hertzog, Ketchum, Kolberg, A. Knopfmacher, Indelkofer, Ritt, and Warlimont [1, 2, 3, 4, 12, 14, 15, 16, 17, 10, 5, 8, 7, 13, 19, 18], who analyzed the particular case of $r_n = 1$. These authors, besides exploring $f(x) = \prod_{n=1}^{\infty}(1+g_n x^n)$, also investigated the inverse power product expansion

$$f(x) = \prod_{n=1}^{\infty}(1-h_n x^n)^{-1}.$$

The purpose of this current work is to develop algebraic, analytic, and combinatorial properties of the generalized inverse power product expansion $f(x) = \prod_{n=1}^{\infty}(1-h_n x^n)^{-r_n}$. Many of the algebraic properties obeyed by $\{h_n\}_{n=1}^{\infty}$ are similar in nature to the algebraic properties of $\{g_n\}_{n=1}^{\infty}$ discussed in Section 2 of [9]. But there is an important difference. Whereas the negativity of all $a_n$ translates to the negativity of all $g_n$ whenever $r_n \geq 1$, (see Theorem 2.2. of [9]), in order for the negativity to transfer to all $h_n$, it is necessary that all $r_n$ be a positive integers.

This paper is presented in a self-contained manner, with no assumption that the reader has previously read [9]. Section 2 derives the algebraic properties of $h_n$ in term of $\{a_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$. The useful property, to be called the Structure Property, writes $h_n$ as a polynomial in the variables $\{a_i\}_{i=1}^{n}$, whose coefficients are rational expressions of the form $\frac{p(r_1,r_2,..,r_n)}{q(r_1,r_2,..,r_n)}$. If each $r_n$ is a positive integer, and each $a_n \leq 0$, the Structure Property ensures that $h_n \leq 0$. We exploit the Structure Property in Section 3 when determining a lower bound for the radius of convergence of $\prod_{n=1}^{\infty}(1-h_n x^n)^{-r_n}$. See Theorem 3.2. Section 3 also contains an asymptotic approximation for the generalized inverse power product expansion associated with $1 - \sum_{n=1}^{\infty} s^n x^n$ where $s = \sup_{n\geq 1}|a_n|^{\frac{1}{n}}$, namely the majorizing product expansion used in the proof of Theorem 3.2. The paper ends with a section dedicated to combinatorial interpretations of $\prod_{n=1}^{\infty}(1-h_n x^n)^{-r_n}$ and $\prod_{n=1}^{\infty}(1+g_n x^n)^{r_n}$, where $\{r_n\}_{n=1}^{\infty}$ is assumed to be a fixed set of positive integers.

## 2. Algebraic Formulas for the Coefficients of a Generalized Inverse Power Product Expansion

Given a formal power series $1+\sum_{n=1}^{\infty} a_n x^n$ or an analytic function $f(x)$ with $f(0)=1$ and a Taylor power series representation

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n,$$

we define the *Generalized Power Product Expansion*, GPPE, of $f(x)$ as

$$f(x) = \prod_{n=1}^{\infty} (1+g_n x^n)^{r_n} = (1+g_1 x^1)^{r_1}(1+g_2 x^2)^{r_2}(1+g_3 x^3)^{r_3}\ldots, \tag{2.1}$$

and the *Generalized Inverse Power Product Expansion*, GIPPE, of $f(x)$ as

$$f(x) = \prod_{n=1}^{\infty} (1-h_n x^n)^{-r_n} = (1-h_1 x^1)^{-r_1}(1-h_2 x^2)^{-r_2}(1-h_3 x^3)^{-r_3}\ldots, \tag{2.2}$$

where $\{g_n\}_{n=1}^{\infty}$, $\{h_n\}_{n=1}^{\infty}$, and $\{r_n\}_{n=1}^{\infty}$ are arbitrary nonzero complex numbers.

In Section 2 of [9] various algebraic formulas relating $\{a_n\}_{n=1}^{\infty}$ to $\{g_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ of the GPPE were developed. We now derive the analogous formulas for the GIPPE. We start by expanding the right side of Equation (2.2) via Newton's Binomial Theorem.

$$\begin{aligned} &1+\sum_{k=1}^{\infty} a_k x^k \\ &= \sum_{k_1=0}^{\infty}\binom{-r_1}{k_1}(-h_1x_1)^{k_1}\sum_{k_2=0}^{\infty}\binom{-r_2}{k_2}(-h_2x_2)^{k_2}\sum_{k_3=0}^{\infty}\binom{-r_3}{k_1}(-h_3x_3)^{k_3}\ldots \end{aligned} \tag{2.3}$$

Comparing the coefficient of $x^n$ on both sides of Equation (2.3) gives

$$a_n = \binom{-r_n}{1}(-h_n) + \sum_{\substack{l\cdot v=n\\ l_j<n}} (-1)^{v_1+\cdots+v_\theta}\binom{-r_{l_1}}{v_1}\ldots\binom{-r_{l_\theta}}{v_\theta} h_{l_1}^{v_1}\ldots h_{l_\theta}^{v_\theta}, \tag{2.4}$$

where $l=[l_1,l_2,\ldots l_\theta]$ and $v=[v_1,v_2,\ldots v_\theta]$. Equation (2.4) is equivalent to

$$h_n = \frac{1}{r_n}\left[a_n - \sum_{\substack{l\cdot v=n\\ l_j<n}} (-1)^{v_1+\cdots+v_\theta}\binom{-r_{l_1}}{v_1}\ldots\binom{-r_{l_\theta}}{v_\theta} h_{l_1}^{v_1}\ldots h_{l_\theta}^{v_\theta}\right]. \tag{2.5}$$

We formalize the above discussion in the following proposition. It is a statement about a bijection between the sequence of the coefficients in a given power series and the sequence of coefficients in its GIPPE expansion.

**Proposition 2.1.** *Let* $\{r_k\}_{k=1}^{\infty}$ *denote a sequence of nonzero complex numbers. Let* $h_k \in \mathbb{C}$, $k = 1, 2, \ldots$, *be an infinite sequence. Let the symbol* $\prod_{k=1}^{\infty}(1 - h_k x^k)^{-r_k}$ *stand for the infinite product*

$$\prod_{k=1}^{\infty}(1 - h_k x^k)^{-r_k} := (1 - h_1 x)^{-r_1}(1 - h_2 x^2)^{-r_2} \cdots (1 - h_k x^k)^{-r_k} \cdots . \tag{2.6}$$

*Then there exists a unique sequence* $a_n \in \mathbb{C}, n = 1, 2, \ldots$, *such that in the sense of power series the following holds*

$$1 + \sum_{n=1}^{\infty} a_n x^n := \prod_{k=1}^{\infty}(1 - h_k x^k)^{-r_k}. \tag{2.7}$$

*Conversely, let* $a_n \in \mathbb{C}, n = 1, 2, \ldots$, *be an infinite sequence. Then there exists a unique sequence of elements* $h_k \in \mathbb{C}$,
$k = 1, 2, \ldots$, *such that the identity (2.7) holds. Moreover, the elements* $h_k$ *have the representation provided by Equation (2.5).*

To develop a number theoretic formula for $h_n$, we define $\log(1 - h_n x^n) := -\sum_{k=1}^{\infty} \frac{(h_n x^n)^k}{k}$, where we say $\log(1 - h_n x^n)$ exists and is well-defined if the series itself converges. Next define

$$\log \prod_{n=1}^{\infty}(1 - h_n x^n)^{-r_n} := \sum_{n=1}^{\infty} r_n \log(1 - h_n x^n) := \sum_{n=1}^{\infty} r_n \sum_{k=1}^{\infty} \frac{(h_n x^n)^k}{k}, \tag{2.8}$$

by the convention that $\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n)$ exists and is well-defined if the double sum on the right side of Equation (2.8) converges. If the double sum on the right of Equation (2.8) is absolutely convergent both $\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n)$ and $\log(1 - h_n x^n)$ are also absolutely convergent.

Equation (2.8), when combined with Equation (2.2), allows us to define $\log f(x)$ as

$$\log f(x) := -\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n). \tag{2.9}$$

Assume that $\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n)$ is absolutely convergent. Differentiate both sides of Equation (2.9) to find that

$$\frac{f'(x)}{f(x)} = \sum_{n=1}^{\infty} \frac{n r_n h_n x^{n-1}}{1 - h_n x^n} = \sum_{n=1}^{\infty} n r_n h_n x^{n-1} \sum_{s=0}^{\infty} (h_n x^n)^s$$
$$= \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} n r_n h_n^{s+1} x^{ns+n-1}.$$

Since $\frac{f'(x)}{f(x)} = d_0 + \sum_{k=1}^{\infty} d_x x^k = r_1 h_1 + \sum_{k=1}^{\infty} d_k x^k$, we may compare the coefficient of $x^k$ in this series with the coefficient of $x^k$ in $\sum_{s=0}^{\infty} \sum_{n=1}^{\infty} n r_n h_n^{s+1} x^{ns+n-1}$ to obtain

$$d_k = \sum_{n: n|(k+1)} n r_n h_n^{\frac{k+1}{n}}. \tag{2.10}$$

If $k+1$ is prime the sum of Equation (2.10) consists of only two terms, namely those associated with $n = 1$ and $n = k+1$. Equation (2.10) becomes $d_k = r_1 h_1^{k+1} + (k+1) r_{k+1} h_{k+1}$, which is equivalent to saying

$$h_{k+1} = \frac{d_k - r_1 h_1^{k+1}}{(k+1) r_{k+1}}, \qquad k+1 \text{ prime}. \tag{2.11}$$

If $k+1$ is not prime we use Equation (2.10) to solve for $h_{k+1}$ as

$$h_{k+1} = \frac{d_k - \sum_{\substack{n: n|(k+1) \\ n \neq k+1}} n r_n h_n^{\frac{k+1}{n}}}{(k+1) r_{k+1}}. \tag{2.12}$$

Equation (2.12) allows us to prove a theorem which demonstrates the connection between a GPPE of $f(x)$ and its associated GIPPE.

**Theorem 2.1.** *Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$. Let $\{r_n\}_{n=1}^{\infty}$ be a given set of nonzero complex numbers. Suppose $f(x)$ has a GPPE of the form $f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n)^{r_n}$ and an GIPPE of the form $f(x) = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}$. Then $h_{2l+1} = g_{2l+1}$ for all nonnegative integers $l$.*

*Proof.* We induct on $l$. A derivation similar to the one used for Equation (2.12) implies that

$$g_{k+1} = \frac{d_k + \sum_{\substack{n|(k+1) \\ n \neq k+1}} n r_n (-g_n)^{\frac{k+1}{n}}}{(k+1) r_{k+1}}, \tag{2.13}$$

where $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$, $\frac{f'(x)}{f(x)} = \sum_{n=0}^{\infty} d_0 x^n$, and $f(x) = \prod_{n=1}^{\infty}(1 + g_n x^n)^{r_n} = \prod_{n=1}^{\infty}(1 - h_n x^n)^{-r_n}$ for a given set of nonzero complex numbers $\{r_n\}_{n=0}^{\infty}$. Let $l = 0$. Equation (2.13) with $f(x) = \prod_{n=1}^{\infty}(1 + g_n x^n)^{r_n}$ implies $g_1 = \frac{d_0}{r_1}$, while Equation (2.12) implies $h_1 = \frac{d_0}{r_1}$. Hence $g_1 = h_1$. Now assume that $g_{2l+1} = h_{2l+1}$ for all $0 \leq l \leq L$ where $L$ is a fixed nonnegative integer. In Equation (2.13) take $k = 2L + 2$ to obtain

$$\begin{aligned} g_{2L+3} &= \frac{d_{2L+2} + \sum_{\substack{n|(2L+3)\\ n\neq 2L+3}} n r_n (-g_n)^{\frac{2L+3}{n}}}{(2L+3) r_{2L+3}} \\ &= \frac{d_{2L+2} - \sum_{\substack{n|(2L+3)\\ n\neq 2L+3}} n r_n (g_n)^{\frac{2L+3}{n}}}{(2L+3) r_{2L+3}} \\ &= \frac{d_{2L+2} - \sum_{\substack{n|(2L+3)\\ n\neq 2L+3}} n r_n (h_n)^{\frac{2L+3}{n}}}{(2L+3) r_{2L+3}} = h_{2L+3}, \end{aligned}$$

where the last equality follows from Equation (2.12). □

**Remark 2.1.** Proposition 3.1 of [8] is now a special case of Theorem 2.1.

Our next task is to develop a recursive formula for $h_n$. Let $f(x) = 1 + \sum_{n=1}^{\infty} B_{1,n} x^n = \prod_{n=1}^{\infty}(1 - h_n x^n)^{-r_n}$. By definition $a_n = B_{1,n}$. We define a recursive system of equations as follows.

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} B_{1,n} x^n &= (1 - h_1 x)^{-r_1} \prod_{n=2}^{\infty} (1 - h_n x^n)^{-r_n} \\ &= (1 - h_1 x)^{-r_1} \left[1 + \sum_{n=2}^{\infty} B_{2,n} x^n\right] \\ 1 + \sum_{n=2}^{\infty} B_{2,n} x^n &= (1 - h_2 x^2)^{-r_2} \prod_{n=3}^{\infty} (1 - h_n x^n)^{-r_n} \\ &= (1 - h_2 x^2)^{-r_2} \left[1 + \sum_{n=3}^{\infty} B_{3,n} x^n\right] \\ &\vdots \\ 1 + \sum_{n=j}^{\infty} B_{j,n} x^n &= (1 - h_j x^j)^{-r_j} \prod_{n=j+1}^{\infty} (1 - h_n x^n)^{-r_n} \end{aligned}$$

$$= (1 - h_j x^j)^{-r_j} \left[ 1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n \right]$$

$$\vdots$$

Newton's Binomial Theorem implies that

$$1 + \sum_{n=j}^{\infty} B_{j,n} x^n = \left[ 1 + \sum_{k=1}^{\infty} \binom{-r_j}{k} (-h_j)^k x^{jk} \right] \left[ 1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n \right]. \quad (2.14)$$

We expand the right side of Equation (2.14) and compare the coefficient of $x^s$ with that of $x^s$ in $1 + \sum_{n=j}^{\infty} B_{j,n} x^n$ to obtain

$$B_{j,s} = \sum_{N+jk=s} \binom{-r_j}{k} (-h_j)^k B_{j+1,N} \quad (2.15)$$

$$= B_{j+1,s} + \sum_{k=1}^{\lfloor \frac{s}{j} \rfloor} \binom{r_j + k - 1}{k} h_j^k B_{j+1,s-jk}, \quad (2.16)$$

since $\binom{z}{k} = (-1)^k \binom{-z+k-1}{k}$.

Take Equation (2.15) and set $s = j$ where $j \geq 1$. The sum has only two terms, namely when $N = 0$ and $k = 1$, or $N = j$ and $k = 0$. By definition $B_{j+1,j} = 0$ and $B_{j+1,0} = 1$, and Equation (2.15) becomes

$$B_{j,j} = \binom{-r_j}{1} (-h_j)^k B_{j+1,0} = r_j h_j. \quad (2.17)$$

Equation (2.17) demonstrates the connection between $B_{j,j}$ and $h_j$. We use this connection to derive a structure property of $\{h_j\}_{j=1}^{\infty}$. To discover this intriguing structure rewrite $1+\sum_{n=j}^{\infty} B_{j,n} x^n = (1-h_j x^j)^{-r_j} \left[ 1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n \right]$ as

$$1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n = (1 - h_j x^j)^{r_j} \left[ 1 + \sum_{n=j}^{\infty} B_{j,n} x^n \right]$$

$$= \left[ 1 + \sum_{k=1}^{\infty} \binom{r_j}{k} (-h_j)^k x^{jk} \right] \left[ 1 + \sum_{n=j}^{\infty} B_{j,n} x^n \right]$$

$$= \left[1 + \sum_{k=1}^{\infty} (-1)^k \binom{r_j}{k} \left(\frac{B_{j,j}}{r_j}\right)^k x^{jk}\right] \left[1 + \sum_{n=j}^{\infty} B_{j,n} x^n\right].$$

If we compare the coefficient of $x^s$ on both sides of the previous equation we find that

$$B_{j+1,s} = \sum_{jk+n=s} (-1)^k \frac{\binom{r_j}{k}}{r_j^k} B_{j,n} B_{j,j}^k, \qquad 0 \le k \le \frac{s}{j}. \tag{2.18}$$

Equation (2.18) is key to proving the following theorem.

**Theorem 2.2.** *Let $j$ be any positive integer. Define $B_{j,0} = 1$ and $B_{j,N} = 0$ for $1 \le N \le j-1$. Assume that $r_j$ is a positive integer for all $j$ and that $B_{j,N} \le 0$ for all $j \le N$. Then $B_{j+1,N} \le 0$ whenever $j+1 \le N$.*

*Proof.* Equation (2.18) is equivalent to

$$B_{j+1,s} = \sum_{\substack{n+jk=s \\ n \ne 0,j}} (-1)^k \frac{\binom{r_j}{k}}{r_j^k} B_{j,j}^k B_{j,n} + \frac{\binom{r_j}{\frac{s}{j}}}{r_j^{\frac{s}{j}}} (-B)_{j,j}^{\frac{s}{j}} + (-1)^{\frac{s}{j}-1} \frac{\binom{r_j}{\frac{s}{j}-1}}{r^{\frac{s}{j}-1}} B_{j,j}^{\frac{s}{j}}. \tag{2.19}$$

Rewrite Equation (2.19) as $B_{j+1,s} = A + B$, where

$$A := \sum_{\substack{n+jk=s \\ n \ne 0,j}} (-1)^k \frac{\binom{r_j}{k}}{r_j^k} B_{j,j}^k B_{j,n}, \;\; B := \frac{\binom{r_j}{\frac{s}{j}}}{r_j^{\frac{s}{j}}} (-B)_{j,j}^{\frac{s}{j}} + (-1)^{\frac{s}{j}-1} \frac{\binom{r_j}{\frac{s}{j}-1}}{r^{\frac{s}{j}-1}} B_{j,j}^{\frac{s}{j}}. \tag{2.20}$$

Since $r_j$ is a positive integer $\frac{\binom{r_j}{k}}{r_j^k} \ge 0$. By hypothesis $B_{j,n} B_{j,j}^k$ is the product of $k+1$ nonpositive numbers and is either zero or has a sign of $(-1)^{k+1}$. Thus $(-1)^k B_{j,n} B_{j,j}^k$ is either zero or negative, and each summand in $A$ is nonpositive.

It remains to show that $B$ is also nonpositive. Notice that $B$ only exists if $\frac{s}{j}$ is a positive integer, say $\frac{s}{j} = \hat{k}$. Then $B$ becomes

$$\begin{aligned} B &= (-1)^{\hat{k}} \binom{r_j}{\hat{k}} \frac{B_{j,j}^{\hat{k}}}{r_j^{\hat{k}}} + (-1)^{\hat{k}-1} \binom{r_j}{\hat{k}-1} \frac{B_{j,j}^{\hat{k}}}{r_j^{\hat{k}-1}} \\ &= (-1)^{\hat{k}} \frac{r_j}{\hat{k}} \binom{r_j - 1}{\hat{k}-1} \frac{B_{j,j}^{\hat{k}}}{r_j^{\hat{k}}} + (-1)^{\hat{k}-1} \binom{r_j}{\hat{k}-1} \frac{B_{j,j}^{\hat{k}}}{r_j^{\hat{k}-1}} \end{aligned}$$

$$= \frac{(-1)^{\hat{k}-1}}{r_j^{\hat{k}-1}} B_{j,j}^{\hat{k}} \left[ -\frac{1}{\hat{k}} \binom{r_j - 1}{\hat{k} - 1} + \binom{r_j}{\hat{k} - 1} \right]$$

$$= \frac{(-1)^{\hat{k}-1}}{r_j^{\hat{k}-1}} \binom{r_j - 1}{\hat{k} - 1} B_{j,j}^{\hat{k}} \left[ -\frac{1}{\hat{k}} + \frac{r_j}{r_j - \hat{k} + 1} \right]$$

$$= \frac{(-1)^{\hat{k}-1}}{r_j^{\hat{k}-1}} \binom{r_j - 1}{\hat{k} - 1} B_{j,j}^{\hat{k}} \left[ \frac{(r_j + 1)(\hat{k} - 1)}{\hat{k}(r_j - \hat{k} + 1)} \right]. \tag{2.21}$$

Since $r_j$ and $\hat{k}$ are positive integers $\binom{r_j-1}{\hat{k}-1} \geq 0$. By hypothesis $B_{j,j}^{\hat{k}}$ is either zero or has a sign of $(-1)^k$. Thus $\frac{(-1)^{\hat{k}-1}}{r_j^{\hat{k}-1}} \binom{r_j-1}{\hat{k}-1} B_{j,j}^{\hat{k}}$ is nonpositive. It remains to analyze the sign of the rational expression inside the square bracket at (2.21). The sign of this expression depends only on the sign of $r_j - \hat{k} - 1$ since the other three factors are always nonnegative. If we assume $r_j - \hat{k} + 1 > 0$, or that $r_j + 1 > \hat{k}$, then the rational expression is nonnegative, and the quantity at (2.21) becomes nonpositive. If $r_j + 1 - \hat{k} < 0$, then $1 \leq r_j < \hat{k} - 1$, which in turn implies that $\binom{r_j-1}{\hat{k}-1} = 0$. So once again the quantity at Line (1) is nonpositive. Only one case remains, that of $r_j + 1 = \hat{k}$. Notice that $1 \leq r_j = \hat{k} - 1$. The definition of $B$ provided by Equation (2.20) implies that $B = (-1)^{\hat{k}-1} \frac{B_{j,j}^{\hat{k}}}{r_j^{\hat{k}-1}}$, a quantity which is either zero or has a sign of $(-1)^{\hat{k}-1}(-1)^{\hat{k}} = -1$. In all three cases we have shown that $B$ is nonpositive. □

If we use the notation of [6], we may transform Theorem (2.2) into a theorem about the structure of the $B_{j+1,s}$. Define $\alpha = (j_1, j_2, \ldots, j_n)$ to be a vector with $n$ components where each component is a positive integer. Let $\lambda = \lambda(\alpha)$ be the length of $\alpha$, i.e. $\lambda = n$. Let $|\alpha|$ denote the sum of the components, namely $|\alpha| = \sum_{s=1}^{n} j_s$. The symbol $B_{j,\alpha}$ represents the expression $B_{j,j_1} B_{j,j_2} \ldots B_{j,j_n}$. For example if $\alpha = (2, 3, 4, 3)$, then $\lambda = 4$, $|\alpha| = 12$, and $B_{j,(2,3,4,3)} = B_{j,2} B_{j,3} B_{j,4} B_{j,3} = B_{j,2} B_{j,3}^2 B_{j,4}$.

**Theorem 2.3.** *(Structure Property) Let $j$ be a positive integer. Then*

$$B_{j+1,s} = \sum_l (-1)^{\lambda(\alpha(l))-1} |c(\alpha(l), j, s)| B_{j,\alpha(l)}$$

$$= \sum_l (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), j, s)| B_{j,\alpha(l)}, \tag{2.22}$$

*where the sum is over all $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = s$ and at most one $j_i \neq j$ . The expression $|c(\alpha(l), j, s)|$ denotes a rational expression in terms of $j$, $s$ and $r_j$ which is nonnegative whenever $r_j$ is a positive integer. Furthermore, define $B_{j,\alpha(l)} = B_{j,j_1} B_{j,j_2} \ldots B_{j,j_\lambda}$. If $B_{j,s} \leq 0$ for all nonnegative integers $j$ and all $s \geq j$, Equation (2.22) is equivalent to*

$$B_{j+1,s} = -\sum |c(\alpha(l), j, s)|\,|B_{j,j_1}|\,|B_{j,j_2}| \ldots |B_{j,j_\lambda}|, \tag{2.23}$$

*where the sum is over all $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = s$ and at most one $j_i \neq j$.*

*Proof.* Take the first term on the right side of Equation (2.19), represent $B^k_{j,j} B_{j,n}$ as $B_{j,\alpha(l)}$ and $\frac{\binom{r_j}{k}}{r_j^k}$ as $|c(\alpha(l), j, s)|$. Notice that $(-1)^k = (-1)^{\lambda(\alpha(l))-1}$. The remaining terms on the right side of Equation (2.19) are $B$, and we combine them via (2.21) by setting $B^{\hat{k}}_{j,j} = B_{j,\alpha(l)}$, and $|c(\alpha(l), j, s)| = \frac{\binom{r_j-1}{\hat{k}-1}}{r_j^{\hat{k}-1}} \frac{(r_j+1)(\hat{k}-1)}{\hat{k}(r_j-\hat{k}+1)} = \frac{(\hat{k}-1)(r_j+1)(r_j-1)(r_j-2)\ldots(r_j-\hat{k}+2)}{r_j^{\hat{k}-1}\hat{k}!}$ as long as $r_j \neq \hat{k}+1$. If $r_j = \hat{k}+1$, then $B = (-1)^{\hat{k}-1} \frac{B^{\hat{k}}_{j,j}}{r_j^{\hat{k}-1}}$ and $B^{\hat{k}}_{j,j} = B_{j,\alpha(l)}$ while $|c(\alpha(l), j, s)| = \frac{1}{r_j^{\hat{k}-1}}$. □

If we take Equation (2.22) and iterate $j$ times we discover that

$$\begin{aligned} B_{j+1,s} &= \sum_l (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), j, s)| a_{\alpha(l)} \\ &= -\sum_l |c(\alpha(l), j, s)||a_{j_1}||a_{j_2}| \ldots |a_{j_\lambda}|, \end{aligned} \tag{2.24}$$

where the sum is over all $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = s$ and $|c(\alpha(l), j, s)|$ is a rational expression in $j$, $s$, and $\{r_i\}_{i=1}^{j+1}$ which is *nonnegative* whenever $r_i$ is a positive integer.

If $s = j+1$ Equation (2.24) becomes

$$\begin{aligned} B_{j+1,j+1} = r_{j+1} h_{j+1} &= \sum_l (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), j)| a_{\alpha(l)} \\ &= -\sum_l |c(\alpha(l), j)||a_{j_1}||a_{j_2}| \ldots |a_{j_\lambda}|, \end{aligned} \tag{2.25}$$

where the sum is over all $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = j+1$. If $\{r_i\}_{i=1}^{j+1}$ is a set of positive integers, each coefficient is nonnegative. Below we explicitly list $h_i$ for $1 \le i \le 6$.

$$h_1 = (-1)^0 \frac{1}{r_1} a_1 \qquad h_2 = (-1)^1 \frac{r_1+1}{2r_1r_2} a_1^2 + (-1)^0 \frac{1}{r_2} a_2$$

$$h_3 = (-1)^2 \frac{r_1^2-1}{3r_1^2r_3} a_1^3 + (-1)^1 \frac{1}{r_3} a_1a_2 + (-1)^0 \frac{1}{r_3} a_3$$

$$\begin{aligned} h_4 &= (-1)^1 \frac{r_2+1}{2r_2r_4} a_2^2 + (-1)^2 \frac{2r_1r_2+r_1+1}{2r_1r_2r_4} a_1^2a_2 \\ &+ (-1)^3 \frac{2r_2+2r_1^3r_2+r_1^3+2r_1^2+r_1}{8r_1^3r_2r_4} a_1^4 + (-1)^1 \frac{1}{r_4} a_1a_3 + (-1)^0 \frac{1}{r_4} a_4 \end{aligned}$$

$$\begin{aligned} h_5 &= (-1)^2 \frac{1}{r_5} a_1^2a_3 + (-1)^1 \frac{1}{r_5} a_2a_3 + (-1)^2 \frac{1}{r_5} a_1a_2^2 \\ &+ (-1)^3 \frac{1}{r_5} a_1^3a_2 + (-1)^1 \frac{1}{r_5} a_1a_4 + (-1)^4 \frac{r_1^4-1}{5r_1^4r_5} a_1^5 + (-1)^0 \frac{1}{r_5} a_5 \end{aligned}$$

$$\begin{aligned} h_6 &= (-1)^2 \frac{1}{r_6} a_1^2a_4 + (-1)^1 \frac{1}{r_6} a_2a_4 + (-1)^1 \frac{r_3+1}{2r_3r_6} a_3^2 \\ &+ (-1)^3 \frac{r_1^2+3r_1^2r_3-1}{3r_1^2r_3r_6} a_1^3a_3 + (-1)^2 \frac{2r_3+1}{r_3r_6} a_1a_2a_3 + (-1)^2 \frac{r_2^2-1}{3r_2^2r_6} a_2^3 \\ &+ (-1)^3 \frac{3r_1r_2^2r_3+r_1r_2^2-r_3-r_1r_3}{2r_1r_2^2r_3r_6} a_1^2a_2^2 + (-1)^0 \frac{1}{r_6} a_6 + (-1)^1 \frac{1}{r_6} a_1a_5 \\ &+ (-1)^4 \frac{12r_1^2r_2^2r_3-3r_3-6r_3r_1+4r_1^2r_2^2-4r_2^2-3r_1^2r_3}{12r_1^2r_2^2r_3r_6} a_1^4a_2 \\ &+ \frac{(-1)^5(12r_1^5r_2^2r_3+12r_2^2r_3-9r_1^4r_3-3r_1^2r_3-8r_2^2r_1^3+4r_1r_2^2-3r_1^5r_3+4r_1^5r_2^2-9r_1^3r_3)}{72r_1^5r_2^2r_3r_6} a_1^6 \end{aligned}$$

Table 1: The expressions for $h_i$, $1 \le i \le 6$.

**Remark 2.2.** It is reasonable to expect that the number of terms occurring in each $B_{j,n}$ of a GPPE or GIPPE, where at least one $r_j \neq 1$, to be larger than the number of terms in a GPPE or GIPPE with all $r_j = 1$. This observation translates into the number theoretic property that the sum of the integral coefficients in the last two terms of $h_6$ are zero when $r_1 = r_2 = r_3 = r_6 = 1$.

**Remark 2.3.** It is instructive to compare the properties of the GPPE with those of the GIPEE. For a GPPE we have

$$\begin{aligned} r_{j+1}g_{j+1} &= \sum_l (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), j)| a_{\alpha(l)} \\ &= -\sum_l |c(\alpha(l), j)||a_{j_1}||a_{j_2}| \ldots |a_{j_\lambda}|, \end{aligned} \tag{2.26}$$

where the sum is over all $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = j+1$ and $\{r_i\}_{i=1}^{j+1}$ is a set of real numbers such that $r_i \ge 1$. See [12]. Since Theorem

2.1 shows that $g_{2l+1} = h_{2l+1}$ for all nonnegative integers $l$, we conclude that Equation (2.25) is valid for $\{r_i\}_{i=1}^{j+1}$, a set of real numbers with $r_i \geq 1$, whenever $j$ is an *even* nonnegative integer. This observation *does not* follow from the techniques used to prove Theorem 2.2. In that proof it was crucial that $r_j$ was a positive integer since this condition ensured that $\binom{r_j}{k} \geq 0$. As the following counterexample demonstrates, if $s \neq j+1$, Equation (2.22) is not true for $r_j \geq 1$ unless $r_j$ is a positive integer. Take Equation (2.18) and let $j = 1$ and $s = 5$ to obtain

$$B_{2,5} = \left(\frac{\binom{r_1}{4}}{r_1^4} - \frac{\binom{r_1}{5}}{r_1^5}\right) B_{1,1}^5 - \frac{\binom{r_1}{3}}{r_1^3} B_{1,1}^3 B_{1,2} + \frac{\binom{r_1}{2}}{r_1^2} B_{1,1}^2 B_{1,3} - B_{1,1}B_{1,4} + B_{1,5}.$$

Then set $r_1 = \frac{3}{2}$. After simplification we find that

$$B_{2,5} = \frac{1}{162} B_{1,1}^5 + \frac{1}{54} B_{1,1}^3 B_{1,2} + \frac{1}{6} B_{1,1}^2 B_{1,3} - B_{1,1}B_{1,4} + B_{1,5}. \tag{2.27}$$

The term $\frac{1}{54} B_{1,1}^3 B_{1,2}$ does not have proper sign of $(-1)^{4-1} = -1$.

Equation (2.27) also demonstrates that $B_{2,5}$ may be positive even if $\{B_{1,k}\}_{k=1}^5 = \{a_k\}_{k=1}^5$ are assumed to be nonpositive. For example, take $B_{1,1} = B_{1,3} = B_{1,4} = B_{1,5} = -1$, and let $B_{1,2} = -200$. Equation (2.27) becomes $\frac{124}{81}$. If we fix $B_{1,n}$ for $n \in \{1,3,4,5\}$ we may choose a value of $B_{1,2} \leq 0$ such that $B_{2,5}$ is arbitrarily large in the positive sense.

There is one other observation which follows from Table 1. For $h_i$ with $1 \leq i \leq 6$ we observe that $h_i$ may be written as a polynomial in $\{r_1^{-1}, ..., r_i^{-1}\}$. We deduce that $h_j$ is a polynomial in $\{r_1^{-1}, ..., r_j^{-1}\}$ for all nonnegative integers $j$. This and more actually follow Equation (2.22). If we take Equation (2.22) and induct on $j$ we prove the following proposition.

**Proposition 2.2.** *Let $n$ and $j$ be positive integers. Define $B_{j,n}$ via Equation (2.14). Then $B_{j,n}$ is a polynomial in $\{r_1^{-1}, r_1^{-2}, \ldots, r_j^{-1}\}$. In particular $h_j = \frac{B_{j,j}}{r_j}$ is also a polynomial in $\{r_1^{-1}, r_1^{-2}, \ldots, r_j^{-1}\}$. Moreover, $h_{2l+1}$ is a polynomial in $\{r_{2i+1}^{-1}\}_{i=0}^{l}$.*

## 3. Convergence Criterion for Equation (2.2)

The structure of $h_j$ provided by Equation (2.25) allows us to prove the following theorem.

**Theorem 3.1.** *Let $\{r_n\}_{n=1}^{\infty}$ be a set of positive integers. Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$. Then $f(x)$ has the GIPPE*

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}. \tag{3.1}$$

*Consider the auxiliary functions*

$$C(x) = 1 - \sum_{n=1}^{\infty} |a_n| x^n = \prod_{n=1}^{\infty} \left(1 + H_n x^n\right)^{-r_n}, \tag{3.2}$$

$$M(x) = 1 - \sum_{n=1}^{\infty} M_n x^n = \prod_{n=1}^{\infty} \left(1 + F_n x^n\right)^{-r_n}. \tag{3.3}$$

*Assume that $|a_n| \leq M_n$ for all $n$. Then $|h_n| \leq H_n \leq F_n$ for all $n$.*

*Proof.* By Equation (2.25) we have

$$\begin{aligned} h_n &= \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), n)| a_{\alpha(l)} \\ &= \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), n)| a_{j_1} a_{j_2} \dots a_{j_\lambda}. \end{aligned} \tag{3.4}$$

Equation (3.4) implies that

$$\begin{aligned} |h_n| &= \left| \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), n)| a_{j_1} a_{j_2} \dots a_{j_\lambda} \right| \\ &\leq \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)| |a_{j_1}| |a_{j_2}| \dots |a_{j_\lambda}| \end{aligned} \tag{3.5}$$

Equation (2.25) when applied to Equation (3.2) implies that

$$0 \leq H_n = \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))} |c(\alpha(l), n)| (-|a_{j_1}|)(-|a_{j_2}|) \dots (-|a_{j_\lambda}|)$$

$$= \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(2\alpha(l))}|c(\alpha(l),n)|(|a_{j_1}|)(|a_{j_2}|)\dots(|a_{j_\lambda}|)$$

$$= \sum_{l:|\alpha(l)|=n} |c(\alpha(l),n)||a_{j_1}||a_{j_2}|\dots|a_{j_\lambda}| \qquad (3.6)$$

Combining Equations (3.5) and (3.6) shows that $|h_n| \leq H_n$. Since $|a_n| \leq M_n$ we also have

$$0 \leq H_n = \sum_{l:|\alpha(l)|=n} |c(\alpha(l),n)||a_{j_1}||a_{j_2}|\dots|a_{j_\lambda}|$$

$$\leq \sum_{l:|\alpha(l)|=n} |c(\alpha(l),n)|M_{j_1}M_{j_2}\dots M_{j_\lambda} = F_n. \qquad \square$$

**Remark 3.1.** Equation (2.24) allows for the following generalization of Theorem (3.1). Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ and $\{r_n\}_{n=1}^{\infty}$ be a set of positive integers. Consider the partial product expansion

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{p}(1-h_n x^n)^{-r_n}\left[1 + \sum_{j=p+1}^{\infty} B_{p+1,j}x^j\right].$$

Also consider the partial product expansions of the auxiliary functions

$$C(x) = 1 - \sum_{n=1}^{\infty} |a_n|x^n = \prod_{n=1}^{p}(1+H_n x^n)^{r_n}\left[1 - \sum_{j=p+1}^{\infty} \hat{B}_{p+1,j}x^j\right],$$

$$M(x) = 1 - \sum_{n=1}^{\infty} M_n x^n = \prod_{n=1}^{p}(1+F_n x^n)^{r_n}\left[1 - \sum_{j=p+1}^{\infty} F_{p+1,j}x^j\right].$$

If $|a_n| \leq M_n$ for all $n$, we conclude that $|h_k| \leq H_k \leq F_k$ for $1 \leq k \leq p$ and that $|B_{p+1,j}| \leq \hat{B}_{p+1,j} \leq F_{p+1,j}$ for all $j \geq p+1$.

Let

$$M(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \prod_{n=1}^{\infty}(1+F_n x^n)^{-r_n}, \; s := \sup_{n\geq 1}|a_n|^{\frac{1}{n}}. \qquad (3.7)$$

Our first task is to determine the radius of convergence of the GIPPE in Equation (3.7). Define

$$\log(1+F_n x^n)^{-r_n} := -r_n \log(1+F_n x^n) := r_n \sum_{l=1}^{\infty} \frac{(-1)^l (F_n x^n)^l}{l}$$

by the convention that $-r_n \log(1 + F_n x^n)$ is well-defined whenever the series converges. Next define

$$-\sum_{n=1}^{\infty} r_n \log(1 + F_n x^n) = \sum_{n=1}^{\infty} [-r_n \log(1 + F_n x^n)]$$
$$:= \sum_{n=1}^{\infty} r_n \sum_{l=1}^{\infty} \frac{(-1)^l (F_n x^n)^l}{l}, \quad (3.8)$$

by the convergence of the double series. Equation (3.8) implies that if the double series is absolutely convergent, then both $\sum_{n=1}^{\infty} [-r_n \log(1 + F_n x^n)]$ and $-r_n \log(1 + F_n x^n)$ are absolutely convergent. The absolute convergence of the double series implies the absolute convergence of $\prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}$ since

$$e^{\sum_{n=1}^{\infty}[-r_n \log(1+F_n x^n)]} = e^{\sum_{n=1}^{\infty} \log(1+F_n x^n)^{-r_n}} = \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}.$$

Define $\log \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n} := \sum_{n=1}^{\infty} [-r_n \log(1 + F_n x^n)]$ via Equation (3.8). If we take the logarithm of Equation (3.7) we find that

$$\sum_{n=1}^{\infty} [-r_n \log(1 + F_n x^n)] = \log\left(1 - \sum_{n=1}^{\infty} s^n x^n\right) \quad (3.9)$$

Now

$$1 - \sum_{n=1}^{\infty} s^n x^n = 1 - sx \sum_{n=0}^{\infty} (sx)^n = 1 - \frac{sx}{1 - sx} = \frac{1 - 2sx}{1 - sx}.$$

Therefore

$$\log\left(\frac{1 - 2sx}{1 - sx}\right) = \log(1 - 2sx) - \log(1 - sx)$$
$$= -\sum_{n=1}^{\infty} \frac{(2sx)^n}{n} + \sum_{n=1}^{\infty} \frac{(sx)^n}{n} = \sum_{n=1}^{\infty} \frac{1 - 2^n}{n} (sx)^n.$$

By the Ratio Test we know that $\sum_{n=1}^{\infty} \frac{1-2^n}{n} (sx)^n$ converges whenever

$$\lim_{n\to\infty} \left| \frac{n(1 - 2^{n+1})}{(n + 1)(1 - 2^n)} \right| |sx| < 1.$$

This is ensured by requiring $|x| < \frac{1}{2s}$.

To determine the radius of convergence of the GIPPE of Equation (2.2) it suffices to determine the radius of convergence of

$$\log \prod_{n=1}^{\infty} (1 - h^n x^n)^{-r_n} := \sum_{n=1}^{\infty} [-r_n \log(1 - h_n x^n)] = -\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n),$$

where the series is defined via the convergence of the double series in Equation (2.8). By definition we have

$$\begin{aligned} \left| \log \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \right| &= \left| \sum_{n=1}^{\infty} [-r_n \log(1 - h_n x^n)] \right| \\ &\leq \sum_{n=1}^{\infty} r_n \left| \log(1 - h_n x^n) \right| \\ &= \sum_{n=1}^{\infty} r_n \left| -\sum_{k=1}^{\infty} \frac{(h_n x^n)^k}{k} \right| \\ &\leq \sum_{n=1}^{\infty} r_n \sum_{k=1}^{\infty} \frac{(|h_n||x|^n)^k}{k} \\ &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} r_n \frac{(F_n |x|^n)^k}{k}, \end{aligned}$$

where the last inequality follows from (3.1). These calculations imply that if $\sum_{n=1}^{\infty} [-r_n \log(1 + F_n x^n)]$ is absolutely convergent then so is

$$\sum_{n=1}^{\infty} [-r_n \log(1 - h_n x^n)]$$

and

$$\prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}$$

absolutely convergent. Since $\sum_{n=1}^{\infty} |-r_n \log(1 + F_n x^n)|$ is absolutely convergent whenever $|x| < \frac{1}{2s}$, we have proven the following theorem.

**Theorem 3.2.** *Let* $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ *Let* $s := \sup_{n \geq 1} |a_n|^{\frac{1}{n}}$. *Then both* $f(x)$ *and its GIPPE,*

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 - g_n x^n)^{-r_n},$$

*and the auxiliary function, along with its GIPPE,*

$$M(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}, \tag{3.10}$$

*will be absolutely convergent whenever* $|x| \leq \frac{1}{2s}$.

We now provide an asymptotic estimate for the majorizing GIPPE of Equation (3.10).

**Theorem 3.3.** *Let* $f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \frac{1-2sx}{1-sx}$ *where* $s > 0$. *Let* $\{r_n\}_{n=1}^{\infty}$ *be a sequence of positive integers. For this particular* $f(x)$ *and its associated GIPPE* $\prod_{n=1}^{\infty}(1 - h_n x^n)^{-r_n}$ *we have*

$$r_n h_n \backsim \frac{(1-2^n)s^n}{n}, \qquad n \to \infty. \tag{3.11}$$

Before we prove Theorem (3.3) we need the following lemma.

**Lemma 3.1.** *Let* $f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \frac{1-2sx}{1-sx}$ *where* $s > 0$. *Let* $\{r_n\}_{n=1}^{\infty}$ *be a sequence of positive integers. For this particular* $f(x)$ *and its associated GIPPE* $\prod_{n=1}^{\infty}(1 - h_n x^n)^{-r_n}$ *there exists* $\alpha$ *with* $1 < \alpha < 2$ *such that*

$$m r_m |h_m| \leq \alpha 2^m s^m. \tag{3.12}$$

*Proof.* It is possible to verify through a straight forward calculation that $\frac{m r_m |h_m|}{(2s)^m} \leq 1.6875$ whenever $1 \leq m \leq 22$. To prove Equation (3.12) for arbitrary $m$ assume inductively that $j r_j |h_j| \leq \alpha 2^j s^j$ is true for $1 \leq j < m$. Our analysis shows that we may assume $m \geq 16$. To that end we proceed as follows. The first step in our induction proof is to rewrite Equation (3.1). Start with Equation (2.9) and note that

$$\log f(x) = -\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{r_n h_n^k}{k} x^{nk} = \sum_{m=1}^{\infty} D_m x^m.$$

Comparing the coefficient of $x^m$ shows that

$$D_m = \frac{1}{m} \sum_{n:n|m} n r_n h_n^{\frac{m}{n}}. \tag{3.13}$$

For the specific case of $f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \frac{1-2sx}{1-sx}$ we find that

$$\log f(x) = \log\left(\frac{1-2sx}{1-sx}\right) = \sum_{k=1}^{\infty} \frac{-(2^k-1)s^k}{k} x^k = \sum_{m=1}^{\infty} D_m x^m,$$

which implies that $D_m = \frac{-(2s)^m(1-2^{-m})}{m}$.
Take Equation (3.13) and rewrite it as

$$m\left[D_m - T_1 - T_2 - T_3 - T_4 - T_5 - T_6 - T_7 - \Delta\right] = m r_m h_m,$$

where

$$T_j = \frac{j r_j}{m}(h_j)^{\frac{m}{j}}, \qquad 1 \leq j \leq 7, \qquad \Delta = \frac{1}{m} \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} n r_n h_n^{\frac{m}{n}}.$$

The range of summation of $\Delta$ implies that $m \geq 16$. In order to prove Equation (3.12) it suffices to show that

$$\begin{aligned} \frac{m r_m |h_m|}{(2s)^m} &= \frac{m}{(2s)^m} \left|D_m - T_1 - T_2 - T_3 - T_4 - T_5 - T_6 - T_7 - \Delta\right| \\ &\leq \frac{m}{(2s)^m} \left[|D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta|\right] < 2, \end{aligned} \tag{3.14}$$

whenever $m \geq 16$. We must approximate $\frac{m}{(2s)^m}|D_m|$, $\frac{m}{(2s)^m}|T_j|$ for $1 \leq j \leq 7$, and $\frac{m}{(2s)^m}|\Delta|$. Begin with $\frac{m}{(2s)^m}|D_m|$ and observe that

$$\frac{m}{(2s)^m}|D_m| = \frac{m}{(2s)^m} \cdot \frac{(2s)^m(1-2^{-m})}{m} < 1. \tag{3.15}$$

We now work with $\frac{m}{(2s)^m}|T_j|$. Take Table 1, let $a_i = -s^i$, and simplify the results to find that

$$h_1 = -\frac{s}{r_1} \qquad h_2 = -\frac{s^2(3r_1+1)}{2r_1 r_2} \qquad h_3 = -\frac{s^3(7r_1^2-1)}{3r_1^2 r_3}$$

$$h_7 = -\frac{s^7(127r_1^6-1)}{7r_1^6 r_7} \qquad h_4 = -\frac{s^4(9r_1^3+30r_1^3 r_2+6r_1^2+2r_2+r_1)}{8r_1^3 r_2 r_4}$$

$$h_5 = -\frac{s^5(31r_1^4-1)}{5r_1^4 r_5}$$

$$h_6 = \frac{-s^6(4r_1r_2^2 + 12r_2^2r_3 - 3r_1^2r_3 - 56r_2^2r_1^3 - 81r_3r_1^4 + 756r_1^5r_2^2r_3 + 196r_1^5r_2^2 - 81r_1^5r_3 - 27r_1^3r_3)}{72r_1^5r_2^2r_3r_6}.$$

We use this data to approximate $\frac{m}{(2s)^m}|T_j|$ for $1 \le j \le 7$. All approximations use $m \ge 16$ and $r_j \ge 1$.

$$\frac{m}{(2s)^m}|T_1| = \frac{r_1}{(2s)^m}\left(\frac{s}{r_1}\right)^m = \frac{1}{2(2r_1)^{m-1}} \le \frac{1}{2^m} \le \frac{1}{2^{16}} \le 0.00002 \tag{3.16}$$

$$\begin{aligned}\frac{m}{(2s)^m}|T_2| &= \frac{2r_2}{(2s)^m}|h_2|^{\frac{m}{2}} = \frac{2r_2}{4^{\frac{m}{2}}}\left(\frac{3r_1+1}{2r_1r_2}\right)^{\frac{m}{2}} \\ &= 2r_2\left(\frac{3r_1+1}{8r_1r_2}\right)^{\frac{m}{2}} = \left(\frac{3r_1+1}{4r_1}\right)\left(\frac{3r_1+1}{8r_1r_2}\right)^{\frac{m}{2}-1} \\ &= \left(\frac{3}{4}+\frac{1}{4r_1}\right)\left(\frac{3+r_1^{-1}}{8r_2}\right)^{\frac{m}{2}-1} \le \left(\frac{1}{2}\right)^{\frac{16}{2}-1} \le \left(\frac{1}{2}\right)^7 \le 0.08\end{aligned} \tag{3.17}$$

When approximating $\frac{m}{(2s)^m}|T_3|$ use the fact that $T_3 = 0$ if $3 \nmid m$.

$$\begin{aligned}\frac{m}{(2s)^m}|T_3| &= \frac{3r_3}{(2s)^m}|h_3|^{\frac{m}{3}} = \frac{3r_3}{2^m}\left|\frac{7r_1^2-1}{3r_1^2r_3}\right|^{\frac{m}{3}} \le \frac{3r_3}{8^{\frac{m}{3}}}\left(\frac{7}{3r_3}\right)^{\frac{m}{3}} \\ &= 3r_3\left(\frac{7}{24r_3}\right)^{\frac{m}{3}} = \frac{7}{8}\left(\frac{7}{24r_3}\right)^{\frac{m}{3}-1} \le \frac{7}{8}\left(\frac{7}{24}\right)^{\frac{m}{3}-1} \\ &\le \frac{7}{8}\left(\frac{7}{24}\right)^{\frac{18}{3}-1} = \frac{7}{8}\left(\frac{7}{24}\right)^5 \le 0.002\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{m}{(2s)^m}|T_4| &= \frac{4r_4}{(2^4)^{\frac{m}{4}}}\left(\frac{9r_1^3+30r_1^3r_2+6r_1^2+2r_2+r_1}{8r_1^3r_2r_4}\right)^{\frac{m}{4}} \\ &= 4r_4\left(\frac{9r_1^3+30r_1^3r_2+6r_1^2+2r_2+r_1}{2^7r_1^3r_2r_4}\right)^{\frac{m}{4}} \\ &= \left(\frac{9r_1^3+30r_1^3r_2+6r_1^2+2r_2+r_1}{2^5r_1^3r_2}\right)\left(\frac{9r_1^3+30r_1^3r_2+6r_1^2+2r_2+r_1}{2^7r_1^3r_2r_4}\right)^{\frac{m}{4}-1} \\ &\le \frac{48}{2^5}\left(\frac{48}{2^7}\right)^{\frac{m}{4}-1} \le \frac{3}{2}\left(\frac{3}{8}\right)^{\frac{16}{4}-1} = \frac{3}{2}\left(\frac{3}{8}\right)^3 \le 0.08\end{aligned} \tag{3.19}$$

When approximating $\frac{m}{(2s)^m}|T_5|$ use the fact that $T_5 = 0$ if $5 \nmid m$.

$$\frac{m}{(2s)^m}|T_5| = \frac{5r_5}{(2^5)^{\frac{m}{5}}}\left|\frac{31r_1^4-1}{5r_1^4r_5}\right|^{\frac{m}{5}} \le 5r_5\left(\frac{31}{160r_5}\right)^{\frac{m}{5}} = \frac{31}{32}\left(\frac{31}{160r_5}\right)^{\frac{m}{5}-1}$$

$$\leq \frac{31}{32}\left(\frac{31}{160}\right)^{\frac{m}{5}-1} \leq \frac{31}{32}\left(\frac{31}{160}\right)^{\frac{20}{5}-1} = \frac{31}{32}\left(\frac{31}{160}\right)^{3} \leq 0.008 \quad (3.20)$$

When approximating $\frac{m}{(2s)^m}|T_6|$ use the fact tha $T_6 = 0$ if $6 \nmid m$.

$$\frac{m}{(2s)^m}|T_6| =$$

$$\frac{6r_6}{(2^6)^{\frac{m}{6}}}\left|\frac{4r_1r_2^2 + 12r_2^2r_3 - 3r_1^2r_3 - 56r_2^2r_1^3 - 81r_3r_1^4 + 756r_1^5r_2^2r_3 + 196r_1^5r_2^2 - 81r_1^5r_3 - 27r_1^3r_3}{72r_1^5r_2^2r_3r_6}\right|^{\frac{m}{6}}$$

$$\leq \frac{6r_6}{(2^6)^{\frac{m}{6}}}\left(\frac{4r_1r_2^2 + 12r_2^2r_3 + 756r_1^5r_2^2r_3 + 196r_1^5r_2^2}{72r_1^5r_2^2r_3r_6}\right)^{\frac{m}{6}}$$

$$= 6r_6\left(\frac{4r_1r_2^2 + 12r_2^2r_3 + 756r_1^5r_2^2r_3 + 196r_1^5r_2^2}{3^22^9r_1^5r_2^2r_3r_6}\right)^{\frac{m}{6}}$$

$$\leq \left(\frac{4r_1r_2^2 + 12r_2^2r_3 + 756r_1^5r_2^2r_3 + 196r_1^5r_2^2}{3^12^8r_1^5r_2^2r_3}\right)\left(\frac{4r_1r_2^2 + 12r_2^2r_3 + 756r_1^5r_2^2r_3 + 196r_1^5r_2^2}{3^22^9r_1^5r_2^2r_3r_6}\right)^{\frac{m}{6}-1}$$

$$\leq \frac{968}{3^12^8}\left(\frac{968}{3^22^9}\right)^{\frac{18}{6}-1} = \frac{11^2}{3^12^5}\left(\frac{11^2}{3^22^6}\right)^{2} \leq 0.056 \quad (3.21)$$

When approximating $\frac{m}{(2s)^m}|T_7|$ use the fact that $T_7 = 0$ if $7 \nmid m$.

$$\frac{m}{(2s)^m}|T_7| = \frac{7r_7}{(2^7)^{\frac{m}{7}}}\left|\frac{127r_1^6 - 1}{7r_1^6r_7}\right|^{\frac{m}{7}} \leq \frac{7r_7}{(2^7)^{\frac{m}{7}}}\left(\frac{127}{7r_7}\right)^{\frac{m}{7}} = 7r_7\left(\frac{127}{7^12^7r_7}\right)^{\frac{m}{7}}$$

$$= \frac{127}{2^7}\left(\frac{127}{7^12^7r_7}\right)^{\frac{m}{7}-1} \leq \frac{127}{2^7}\left(\frac{127}{7^12^7}\right)^{\frac{21}{7}-1}$$

$$= \frac{127}{2^7}\left(\frac{127}{7^12^7}\right)^{2} \leq 0.02 \quad (3.22)$$

It remains to approximate $\frac{m}{(2s)^m}|\Delta|$. Here is where we make use of the induction hypothesis. We also use the fact the $\frac{m}{2} \geq n$ implies $\frac{m}{n} \geq 2$. By definition we have

$$\frac{m}{(2s)^m}|\Delta| \leq \frac{1}{(2s)^m}\sum_{\substack{n|m \\ \frac{m}{2}\geq n\geq 8}} nr_n|h_n|^{\frac{m}{n}} \leq \frac{1}{(2s)^m}\sum_{\substack{n|m \\ \frac{m}{2}\geq n\geq 8}} nr_n\left(\frac{\alpha 2^n s^n}{nr_n}\right)^{\frac{m}{n}}$$

$$= \alpha\sum_{\substack{n|m \\ \frac{m}{2}\geq n\geq 8}} \frac{nr_n}{\alpha}\left(\frac{1}{\frac{nr_n}{\alpha}}\right)^{\frac{m}{n}} = \alpha\sum_{\substack{n|m \\ \frac{m}{2}\geq n\geq 8}} \left(\frac{\alpha}{nr_n}\right)^{\frac{m}{n}-1}$$

$$\le \alpha \sum_{\substack{n|m \\ \frac{m}{2} \ge n \ge 8}} \left(\frac{\alpha}{n}\right)^{\frac{m}{n}-1} \le \alpha \sum_{\substack{n|m \\ \frac{m}{2} \ge n \ge 8}} \left(\frac{\alpha}{8}\right)^{\frac{m}{n}-1}$$

$$\le \alpha \sum_{\frac{m}{n} \ge 2} \left(\frac{\alpha}{8}\right)^{\frac{m}{n}-1} = \alpha \left[\frac{\frac{\alpha}{8}}{1-\frac{\alpha}{8}}\right]$$

$$= \alpha \left[\frac{\alpha}{8-\alpha}\right] \le \alpha \left[\frac{2}{8-2}\right] = \frac{\alpha}{3} \le \frac{2}{3} \tag{3.23}$$

We now take Equations (3.16) through Equation (3.23) and place them in

$$\frac{mr_m|h_m|}{(2s)^m} \le \frac{m}{(2s)^m}\left[|D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta|\right]$$

to find that

$$\frac{mr_m|h_m|}{(2s)^m} \le \frac{m}{(2s)^m}\left[|D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta|\right]$$

$$\le 1 + 0.00002 + 0.08 + 0.002 + 0.08 + 0.008 + 0.056 + 0.02 + \frac{2}{3}$$

$$= 1.91268667 < 2.$$

In other words Equation (3.14) is valid and our proof is complete. □

*Proof of Theorem 3.3.* Equation (2.12) implies that

$$(k+1)r_{k+1}h_{k+1} = d_k - r_1 h_1^{k+1} - \sum_{\substack{n|(k+1) \\ \frac{k+1}{2} \ge n \ge 2}} n r_n h_n^{\frac{k+1}{n}}. \tag{3.24}$$

Since $f(x) = 1 - \sum_{n=1}^{\infty}(sx)^n = \frac{1-2sx}{1-sx}$ we discover that

$$\frac{f'(x)}{f(x)} = \frac{-2s}{1-2sx} + \frac{s}{1-sx} = s\left[-2\sum_{k=0}^{\infty} 2^k s^k x^k + \sum_{k=0}^{\infty} s^k x^k\right]$$

$$= s\sum_{k=0}^{\infty}(-2^{k+1}+1)s^k x^k.$$

By definition $\frac{f'(x)}{f(x)} = \sum_{k=0}^{\infty} d_k x^k$. Hence $d_k = (-2^{k+1}+1)s^{k+1}$, and Equation (3.24) becomes

$$(k+1)r_{k+1}h_{k+1} = (-2^{k+1}+1)s^{k+1} - r_1\left(-\frac{s}{r_1}\right)^{k+1} - \sum_{\substack{n|(k+1) \\ \frac{k+1}{2} \ge n \ge 2}} n r_n h_n^{\frac{k+1}{n}}. \tag{3.25}$$

Define

$$T_1 := (-2^{k+1}+1)s^{k+1}, \quad T_2 := r_1\left(-\frac{s}{r_1}\right)^{k+1},$$

and

$$\Delta := \sum_{\substack{n|(k+1)\\ \frac{k+1}{2}\geq n\geq 2}} nr_n h_n^{\frac{k+1}{n}}.$$

Equation (3.25) is equivalent to $(k+1)r_{k+1}h_{k+1} = T_1 - T_2 - \Delta$. Lemma (3.1) implies there exisst $\alpha$ with $1<\alpha<2$ such that

$$n\left|h_n\right| \leq nr_n\left|h_n\right| \leq \alpha 2^n s^n. \tag{3.26}$$

By definition

$$\begin{aligned}
|\Delta| &= \left|\sum_{\substack{n|(k+1)\\ k+1>n>1}} nr_n h_n^{\frac{k+1}{n}}\right| \leq \sum_{\substack{n|(k+1)\\ \frac{k+1}{2}\geq n\geq 2}} nr_n|h_n|^{\frac{k+1}{n}} \leq \sum_{\substack{n|(k+1)\\ \frac{k+1}{2}\geq n\geq 2}} nr_n\left[\frac{\alpha 2^n s^n}{nr_n}\right]^{\frac{k+1}{n}}\\
&= \alpha(2s)^{k+1}\sum_{\substack{n|(k+1)\\ \frac{k+1}{2}\geq n\geq 2}}\left(\frac{nr_n}{\alpha}\right)\frac{1}{\left(\frac{nr_n}{\alpha}\right)^{\frac{k+1}{n}}} = \alpha(2s)^{k+1}\sum_{\substack{n|(k+1)\\ \frac{k+1}{2}\geq n\geq 2}}\frac{1}{\left(\frac{nr_n}{\alpha}\right)^{\frac{k+1}{n}-1}}\\
&\leq \alpha(2s)^{k+1}\sum_{\substack{n|(k+1)\\ \frac{k+1}{2}\geq n\geq 2}}\frac{1}{\left(\frac{n}{\alpha}\right)^{\frac{k+1}{n}-1}} \leq \alpha(2s)^{k+1}\sum_{\frac{k+1}{2}\geq n\geq 2}\frac{1}{\left(\frac{n}{\alpha}\right)^{\frac{k+1}{n}-1}}\\
&= \alpha(2s)^{k+1}\left[\frac{1}{\left(\frac{2}{\alpha}\right)^{\frac{k+1}{2}-1}} + \frac{2\alpha}{k+1} + \sum_{\frac{k+1}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{\alpha}\right)^{\frac{k+1}{n}-1}}\right]\\
&\leq \alpha(2s)^{k+1}\left[\frac{1}{\left(\frac{2}{\alpha}\right)^{\frac{k+1}{2}-1}} + \frac{2\alpha}{k+1} + \sum_{\frac{k+1}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k+1}{n}-1}}\right],
\end{aligned} \tag{3.27}$$

where the last equality reflects the fact that $\frac{1}{2}<\frac{1}{\alpha}<1$.

Define $M := \sum_{\frac{k+1}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k+1}{n}-1}} = \frac{1}{\left(\frac{3}{2}\right)^{\frac{k+1}{3}-1}} + \frac{1}{\left(\frac{4}{2}\right)^{\frac{k+1}{4}-1}} + \frac{1}{\left(\frac{5}{2}\right)^{\frac{k+1}{5}-1}} + \sum_{\frac{k+1}{3}\geq n\geq 6}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k+1}{n}-1}}$ and $b(n,k) := -\ln\left(\frac{n}{2}\right)^{\frac{k+1}{n}-1} = -\left(\frac{k+1}{n}-1\right)\ln\frac{n}{2}$. Then

$$\frac{\partial b(n,k)}{\partial n} = \frac{k+1}{n^2}\ln\frac{n}{2} - \left(\frac{k+1}{n}-1\right)\frac{1}{n}$$

$$= \frac{k+1}{n}\left[\frac{1}{n}\left[\ln\frac{n}{2}-1\right]+\frac{1}{k+1}\right] > 0, \qquad n \geq 6. \tag{3.28}$$

Line (3.28) shows that $b(n,k)$ is an increasing function with respect to $n$ whenever $n \geq 6$. Hence

$$b(n,k) < b\left(\frac{k+1}{3},k\right) = -(3-1)\ln\frac{k+1}{6} = -2\ln\frac{k+1}{6},$$

which implies each term of $M$ satisfies $e^{b(n,k)} \leq e^{-2\ln\frac{k+1}{6}} = \frac{36}{(k+1)^2}$ whenever $n = 6, 7, 8, \dots$. Therefore

$$\begin{aligned}
\sum_{\frac{k+1}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k+1}{n}-1}} &\leq \frac{1}{\left(\frac{3}{2}\right)^{\frac{k+1}{3}-1}}+\frac{1}{\left(\frac{4}{2}\right)^{\frac{k+1}{4}-1}}+\frac{1}{\left(\frac{5}{2}\right)^{\frac{k+1}{5}-1}}+\sum_{\frac{k+1}{3}\geq n\geq 6}\frac{36}{(k+1)^2}\\
&\leq \frac{1}{\left(\frac{3}{2}\right)^{\frac{k+1}{3}-1}}+\frac{1}{\left(\frac{4}{2}\right)^{\frac{k+1}{4}-1}}+\frac{1}{\left(\frac{5}{2}\right)^{\frac{k+1}{5}-1}}+(k+1)\frac{36}{(k+1)^2}\\
&\leq \frac{1}{\left(\frac{3}{2}\right)^{\frac{k+1}{3}-1}}+\frac{1}{\left(\frac{4}{2}\right)^{\frac{k+1}{4}-1}}+\frac{1}{\left(\frac{5}{2}\right)^{\frac{k+1}{5}-1}}+\frac{36}{k+1}. \qquad (3.29)
\end{aligned}$$

Equation (3.29) shows that $\lim_{k\to\infty} M = 0$. By combining this result with Equation (3.27) we conclude that

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{\Delta}{(-2^{k+1}+1)s^{k+1}}\right| &= \lim_{k\to\infty}\frac{|\Delta|}{|(-1+2^{-k-1})|(2s)^{k+1}}\\
= \lim_{k\to\infty}\frac{\alpha(2s)^{k+1}}{|(-1+2^{-k-1})|(2s)^{k+1}} &\left[\frac{1}{\left(\frac{2}{\alpha}\right)^{\frac{k+1}{2}-1}}+\frac{2\alpha}{(k+1)}+M\right] = 0.
\end{aligned}$$

We return to Equation (3.25) and observe that

$$\begin{aligned}
r_{k+1}h_{k+1} &= \frac{T_1}{k+1}-\frac{T_2}{k+1}-\frac{\Delta}{k+1}\\
&= \frac{(-2^{k+1}+1)s^{k+1}}{k+1}-\frac{r_1(-1)^{k+1}\left(\frac{s}{r_1}\right)^{k+1}}{k+1}-\frac{\Delta}{k+1}\\
&= \frac{(-2^{k+1}+1)s^{k+1}}{k+1}\left[1-\frac{(-1)^{k+1}}{r_1^k(-2^{k+1}+1)}-\frac{\Delta}{(-2^{k+1}+1)s^{k+1}}\right]\\
&= \frac{(-2^{k+1}+1)s^{k+1}}{k+1}\left[1+o(1)\right] = \frac{d_k}{k+1}\left[1+o(1)\right]. \qquad \square
\end{aligned}$$

**Remark 3.2.** Define a composition of $a_n$ to be monomial of the form $a_{j_1}a_{j_2}...a_{j_m}$ such that $j_1+j_2+\cdots+j_m=n$. By combining Equation (2.5) with Theorem 3.3 we deduce that the absolute value of Equation (3.11) provides an upper bound on the number and weight of compositions of $a_n$ whenever $\{|a_n|^{n^{-1}}\}_{n=1}^{\infty}$ is monotone increasing sequence and $s=\lim_{n\to\infty}|a_n|^{n^{-1}}$.

## 4. Combinatorial Interpretations for IGPPE

In this section we develop combinatorial interpretations for Equations (2.1) and (2.2). When developing our combinatorial interpretations, we require that $\{r_k\}_{k=1}^{\infty}$ be a set of *positive integers*. For a fixed set of positive integers $\{r_k\}_{k=1}^{\infty}$ denote the associated multi-set as $1^{r_1}2^{r_2}\ldots k^{r_k}\ldots$, where $k^{r_k}$ denotes $r_k$ distinct copies of the integer $k$. If $r_k$ is the number of ways to color the digit $k$, we require that each copy of $k$ in the multi-set be uniquely colored. For example, $1^2 2^4 3^3 4^5$ denotes the multi-set $\{1_R, 1_B, 2_R, 2_B, 2_O, 2_Y, 3_R, 3_B, 3_O, 4_R, 4_B, 4_O, 4_Y, 4_G\}$, where $R$ means red, $B$ means blue, $O$ means orange, $Y$ means yellow, and $G$ means green. We form the generating function

$$\prod_{n=1}^{\infty}(1+x^n)^{r_n}=(1+x)^{r_1}(1+x^2)^{r_2}\ldots(1+x^k)^{r_k}\cdots=\sum_{n=0}^{\infty}\hat{p}_d(n)x^n, \tag{4.1}$$

where $\hat{p}_d(n)$ counts the partitions of $n$ with *distinct* parts composed from $1^{r_1}2^{r_2}\ldots k^{r_k}\ldots$. Recall that a *partition* of $n$ is a collection of positive integers whose sums equals $n$, i.e. $n=i_1+i_2+\cdots+i_k$ where $1\le k\le n$ and each $i_h$ is a positive integer less than or equal to $n$. The summands in the partition are called *parts* and the order of summation is immaterial. In the context of colored multi-sets, two parts $i_{j_1}$ and $i_{j_2}$ are distinct if they either have different numerical value, or if they have the same numerical value, they are of different color. For the multi-set $1^2 2^4 3^3 4^5$, both $1_R+3_O$ and $2_R+2_B$ are partitions of 4 with distinct parts. We generalize Equation (4.1) by introducing collection of weights associated with each part, namely $\{g_n\}_{n=1}^{\infty}$, where $g_n$ is an arbitrary complex number. Equation (4.1) becomes

$$\begin{aligned}\prod_{n=1}^{\infty}(1+g_nx^n)^{r_n}&=(1+g_1x)^{r_1}(1+g_2x)^{r_2}\ldots(1+g_kx^k)^{r_k}\ldots\\&=\sum_{n=0}^{\infty}\hat{p}_d(\bar{g},n)x^n,\end{aligned}\tag{4.2}$$

where $\bar{g}$ is a finite polynomial in $\{g_n\}_{n=1}^{\infty}$, such that each monomial has the form $g_1^{\alpha_1} g_2^{\alpha_2} \dots g_m^{\alpha_m}$, where $\sum_{i=1}^{m} i\alpha_1 = n$ and $\alpha_m$ denotes the number of distinct colored copies of the part $m$ which appear in the partition. For example, the partition $1_R + 3_O$ is represented as $g_1 g_3$, while $2_R + 2_B$ is represented as $g_2^2$.

Equations (4.1) and (4.2) interpret the product side of Equation (2.1). Colored multi-sets also provide means of combinatorially interpreting the product side of Equation (2.2). If $h_n = 1$, Equation (2.2) becomes

$$\prod_{n=1}^{\infty} (1 - x^n)^{-r_n} = (1 - x)^{-r_1} (1 - x^2)^{-r_2} (1 - x^3)^{-r_3} \cdots = \sum_{n=0}^{\infty} \hat{p}(n) x^n, \qquad (4.3)$$

where $\hat{p}(n)$ is the number of partitions of $n$ associated with the colored multi-set which contains an unlimited repetition of each integer $k$ in $r_k$ colors. The factor $(1-x^k)^{-r_k} = (1+x^k+x^{2k}+x^{3k}+\dots)^{r_k}$ corresponds to $\{k, k+k, k+k+k, \dots\}$ replicated in $r_k$ colors.

Equation (4.3) generalizes by assigning a set of weights to each part. In particular, we have

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} &= (1 - h_1 x)^{-r_1} (1 - h_2 x^2)^{-r_2} (1 - h_3 x^3)^{-r_3} \dots \\ &= \sum_{n=0}^{\infty} \hat{p}(\bar{h}, n) x^n, \end{aligned} \qquad (4.4)$$

where $\bar{h}$ is a polynomial in $\{h_n\}_{n=0}^{\infty}$ such that the exponent of $h_i$ is the number of colored parts $i$ that appear in a partition of $n$. In other words, $\bar{h}$ has the form $h_1^{\alpha_1} h_2^{\alpha_2} \dots h_m^{\alpha_m}$, where $\sum_{i=1}^{m} i\alpha_1 = n$ and $\alpha_m$ denotes the number of colored parts $m$ which appear in the partition [9].

The combinatorial interpretations of Equations (4.1) through (4.4) originated from the product side of Equations (2.1) and (2.2). Is there way to develop a combinatorial interpretation if we start with the sum side instead? To answer this question define $f(x) = 1 - \sum_{n=1}^{\infty} a_n x^n$ where $\{a_n\}_{n=1}^{\infty}$ is a set of *positive* integers. Equation (2.25), when combined with Equation (2.1), implies that

$$1 - \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 - g_n x^n)^{r_n}, \qquad g_n \text{ a positive integer.} \qquad (4.5)$$

Take Equation (4.5) and form the reciprocal.

$$\frac{1}{1 - \sum_{n=1}^{\infty} a_n x^n} = \frac{1}{\prod_{n=1}^{\infty} (1 - g_n x^n)^{r_n}} = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}, \ h_n := g_n \qquad (4.6)$$

Equation (4.6) shows that the reciprocal of $1-\sum_{n=1}^{\infty} a_n x^n$ is an IGPPE. Since $r_n$ is a positive integer we may expand $\frac{1}{(1-g_n x^n)^{r_n}} = \left[\frac{1}{1-g_n x^n}\right]^{r_n} = \left[1+\sum_{\mu=1}^{\infty}(g_n x^n)^{\mu}\right]^{r_n}$ to obtain

$$\frac{1}{1-\sum_{n=1}^{\infty} a_n x^n} = \prod_{n=1}^{\infty}\left[1+\sum_{\mu=1}^{\infty}(g_n x^n)^{\mu}\right]^{r_n} = 1+\sum_{n=1}^{\infty} C_n x^n, \tag{4.7}$$

where $C_n$ is a positive integer determined by a finite number of arithmetic calculations involving a finite number of elements in $\{g_n\}_{n=1}^{\infty}$. We claim each $C_n$ has a combinatorial interpretation. Expand the left side of Equation (4.7) as

$$\frac{1}{1-\sum_{n=1}^{\infty} a_n x^n} \tag{4.8}$$

$$= 1+\sum_{n=1}^{\infty} a_n x^n + \left[\sum_{n=1}^{\infty} a_n x^n\right]^2 + \left[\sum_{n=1}^{\infty} a_n x^n\right]^3 + \cdots + \left[\sum_{n=1}^{\infty} a_n x^n\right]^k + \ldots \quad . \tag{4.9}$$

The methodology used to describe $C_n$ is best exhibited by the particular case of $a_n = 1$. Equation (4.8) becomes

$$\frac{1}{1-\sum_{n=1}^{\infty} x^n} = \frac{1}{1-\left(\frac{x}{1-x}\right)}$$

$$= 1+\frac{x}{1-x}+\left(\frac{x}{1-x}\right)^2+\left(\frac{x}{1-x}\right)^3+\cdots+\left(\frac{x}{1-x}\right)^k+\ldots$$

$$= 1+\sum_{n=1}^{\infty} x^n + \left[\sum_{n=1}^{\infty} x^n\right]^2 + \left[\sum_{n=1}^{\infty} x^n\right]^3 + \cdots + \left[\sum_{n=1}^{\infty} x^n\right]^k + \ldots \quad . \tag{4.10}$$

We must determine a series expansion for $[\sum_{n=1}^{\infty} x^n]^k = \sum_{l=k}^{\infty} \hat{C}(l,k)x^l$ whenever $k \geq 1$. Clearly $\hat{C}(l,1) = 1$. A simple exercise in coefficient comparison shows that

$$\left[\sum_{n=1}^{\infty} x^n\right]^2 = (x+x^2+x^3+x^4+x^5+\ldots)(x+x^2+x^3+x^4+x^5+\ldots)$$

$$= \sum_{l=2}^{\infty}(l-1)x^l = \sum_{l=2}^{\infty}\hat{C}(l,2)x^l.$$

We use this result to determine $\hat{C}(l,3)$ as follows

$$\left[\sum_{n=1}^{\infty} x^n\right]^3 = \left[\sum_{n=1}^{\infty} x^n\right]^2 \left[\sum_{n=1}^{\infty} x^n\right]^1$$
$$= (x^2 + 2x^3 + \cdots + (n-1)x^n + \dots)(x + x^2 + x^3 + \cdots + x^k \dots)$$
$$= (1)x^3 + (1+2)x^4 + (1+2+3)x^5 + \dots (1+2+3+\cdots+(k-2))x^k + \dots$$
$$= \sum_{l=3}^{\infty} \binom{l-1}{2} x^l = \sum_{l=3}^{\infty} \hat{C}(l,3)x^l$$

Since $\hat{C}(l,1) = 1 = \binom{l-1}{0}$, $\hat{C}(l,2) = \binom{l-1}{1}$, and $\hat{C}(l,3) = \binom{l-1}{2}$, we deduce that $\hat{C}(l,k) = \binom{l-1}{k-1}$. This claim is proven via induction on $k$. Assume that $\hat{C}(l,j) = \binom{l-1}{j-1}$ for all positive integers $j$ with $j \leq k$. Then

$$\begin{aligned}
\left[\sum_{n=1}^{\infty} x^n\right]^{k+1} &= \left[\sum_{n=1}^{\infty} x^n\right]^k \left[\sum_{n=1}^{\infty} x^n\right]^1 \\
&= \left[\sum_{l=k}^{\infty} \hat{C}(l,k)x^l\right] \left[x + x^2 + \cdots + x^k + \dots\right] \\
&= \sum_{s=1}^{\infty} \left[\sum_{j=0}^{s-1} \hat{C}(k+j,k)\right] x^{k+s} = \sum_{s=1}^{\infty} \left[\sum_{j=0}^{s-1} \binom{k+j-1}{k-1}\right] x^{k+s} \\
&= \sum_{s=1}^{\infty} \binom{s+k-1}{k} x^{k+s}, \qquad \text{by Eq. (1.48) of [11]} \\
&= \sum_{s=k+1}^{\infty} \binom{s-1}{k} x^s = \sum_{s=k+1}^{\infty} \hat{C}(s,k+1)x^s.
\end{aligned}$$

We return to Equation (4.10) and write it as

$$\frac{1}{1-\sum_{n=1}^{\infty} x^n} = 1 + \sum_{n=1}^{\infty} x^n + \left[\sum_{n=1}^{\infty} x^n\right]^2 + \left[\sum_{n=1}^{\infty} x^n\right]^3 + \cdots + \left[\sum_{n=1}^{\infty} x^n\right]^k + \dots$$
$$= 1 + \sum_{n=1}^{\infty} \hat{C}(n,1)x^n + \sum_{n=2}^{\infty} \hat{C}(n,2)x^n + \sum_{n=3}^{\infty} \hat{C}(n,3)x^n + \cdots + \sum_{n=k}^{\infty} \hat{C}(n,k) + \dots$$
$$= 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \hat{C}(n,k)\right] x^n = 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \binom{n-1}{k-1}\right] x^n = 1 + \sum_{n=1}^{\infty} 2^{n-1}x^n.$$

From these calculations we conclude that $\frac{1}{1-\sum_{n=1}^{\infty} x^n} = 1+\sum_{n=1}^{\infty} \hat{C}_n x^n$ where $\hat{C}_n = 2^{n-1}$. This result could have been obtained by observing that

$$\frac{1}{1-\sum_{n=1}^{\infty} x^n} = \frac{1}{1-\left(\frac{x}{1-x}\right)} = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x} = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n.$$

We now turn to the general case and develop a factorization for the ordinary power series whose coefficients are representations of compositions of $n$. Notice that

$$\begin{aligned}
\frac{1}{1-\sum_{n=1}^{\infty} a_n x^n} =& 1 + \sum_{n=1}^{\infty} a_n x^n + \left[\sum_{n=1}^{\infty} a_n x^n\right]^2 + \left[\sum_{n=1}^{\infty} a_n x^n\right]^3 \\
& + \cdots + \left[\sum_{n=1}^{\infty} a_n x^n\right]^k + \ldots \\
=& 1 + \sum_{n=1}^{\infty} C(n,1)x^n + \sum_{n=2}^{\infty} C(n,2)x^n + \sum_{n=3}^{\infty} C(n,3)x^n \\
& + \cdots + \sum_{n=k}^{\infty} C(n,k)x^n + \ldots \\
=& 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} C(n,k)\right] x^n \\
=& 1 + \sum_{n=1}^{\infty} C_n x^n,
\end{aligned}$$

where $C(n,k)$ is a polynomial representation of the compositions of $n$ with exactly $k$ parts such that the part $i$ is represented by $a_i$ and the $+$ is replace by $*$. In other words, $C(n,k)$ is composed of monomials $ca_{i_1}a_{i_2}\ldots a_{i_k}$ such that $i_1 + i_2 + \ldots i_k$ is a partition of $n$. Recall that a *composition* of a positive integer $n$ with $k$ parts is a sum $i_1 + i_2 + \ldots i_k = n$ where each part $i_j$ is a positive integer with $1 \leq i_j \leq n$. The difference between a partition of $n$ with $k$ parts and a composition of $n$ with $k$ parts is that a composition distinguishes between the order of the parts in the summation. For example, $2+1+1$ and $1+2+1$ are two distinct compositions of 4 with 3 parts but only one partition of 4 with 3 parts [20]. To construct $C(n,k)$ we list all the compositions of $n$ with $k$ parts, replace each part $i$ with $a_i$, multiply the terms together, and add the resulting monomials. For example, $C(n,1) = a_n$ and $C(5,2) = 2a_1a_4 + 2a_2a_3$ since $\{4+1, 1+4, 2+3, 3+2\}$ are the four compositions of 5 with 2 parts.

We justify our combinatorial interpretation of $C(n,k)$ through induction on $k$. Obviously $C(n,1)=a_n$ satisfies our definition. Now assume for $j\le k$ that $C(n,j)$ is the polynomial representation of the compositions of $n$ with exactly $j$ parts such that each part $i$ is represented by $a_i$ and all $+$ are replaced with $*$. Then

$$\begin{aligned}\left[\sum_{n=1}^{\infty}a_nx^n\right]^{k+1} &= \left[\sum_{n=1}^{\infty}a_nx^n\right]^{k}\left[\sum_{n=1}^{\infty}a_nx^n\right]\\ &= \left[\sum_{l=k}^{\infty}C(l,k)x^l\right]\sum_{l=1}^{\infty}a_lx^l\\ &= \sum_{s=1}^{\infty}\left[\sum_{j=0}^{s-1}C(k+j,k)a_{s-j}\right]x^{k+s}\\ &= \sum_{s=k+1}^{\infty}\left[\sum_{j=0}^{s-k-1}C(k+j,k)a_{s-k-j}\right]x^{s}\\ &:= \sum_{s=k+1}^{\infty}C(s,k+1)x^s.\end{aligned}$$

We claim that the polynomial representation of the compositions of $s$ with $k+1$ parts such that each part $i$ is represented by $a_i$ and all $+$ are replaced with $*$ is precisely $\sum_{j=0}^{s-k-1}C(k+j,k)a_{s-k-j}$. Start with a composition of $s$ which has $k+1$ parts. Call this composition $p$ and write $p=p_1+p_2=s$, where $p_1$ is the sum of the first $k$ parts and $p_2$ is the $(k+1)^{st}$ part. In other words, $p_1=i_1+i_2+\dots i_k$ and $p_2=i_{k+1}$. Notice that $1\le i_{k+1}\le s-k$ since $i_j\ge 1$ whenever $1\le j\le k$. If $i_{k+1}\ge s-k+1$ we would have the contradiction $p_1+p_2\ge k+s-k+1=s+1$. For each choice of $i_{k+1}\in\{1,2,\dots,s-k\}$, $p_1$ is a composition $s-i_{k+1}$ with $k$ parts. If $c(s,k+1)$ is the number of compositions of $s$ with $k+1$ parts, the preceding argument shows that $c(s,k+1)=\sum_{j=1}^{s-k}c(s-j,k)j=\sum_{j=0}^{s-k-1}c(s-j-1,k)(j+1)=\sum_{j=0}^{s-k-1}c(k+j,k)(s-k-j)$. It is just a matter of taking each composition represented by $c(k+j,k)(s-k-j)$, replacing the part $i$ with $a_i$, and $+$ with $*$. The end result is that $\sum_{j=0}^{s-k-1}C(k+j,k)a_{s-k-j}=C(s,k+1)$ is indeed the desired polynomial representation for the compositions of $s$ with $k$ parts.

Since

$$\frac{1}{1-\sum_{n=1}^{\infty}a_nx^n}=1+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}C(n,k)\right]x^n=1+\sum_{n=1}^{\infty}C_nx^n, \tag{4.11}$$

we may interpret $C_n$ to be the sum of all non-trivial polynomial representations of the compositions of $n$ with $k$ parts, i.e. $C_n$ is a polynomial representation of the compositions of $n$. Notice that $C_n$ is constructed by taking the set of compositions of $n$, replacing $i$ with $a_i$, replacing $+$ with $*$, and summing the monomials. For example, since the compositions of 4 are $\{1+1+1+1, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 4\}$, $C_4 = a_4 + 3a_2a_1^2 + a_2^2 + 2a_3a_1 + a_1^4$. If $a_n = 1$, $C(n,k) = \hat{C}(n,k)$ is the number of compositions of $n$ with $k$ parts while $C_n = \hat{C}_n$ is the total number of compositions of $n$. We have shown that $\hat{C}(n,k) = \binom{n-1}{k-1}$ and $\hat{C}_n = 2^{n-1}$. Therefore, our calculations provide yet another proof of the fact that number of compositions of $n$ is $2^{n-1}$, and the number of compositions of $n$ with $k$ parts is $\binom{n-1}{k-1}$. By combining our observations with Equations (4.6) and (4.7), we see that IGPPE $\prod_{n=1}^{\infty}(1-h_nx^n)^{-r_n}$ provides a way of "factoring" the series $1+\sum_{n=1}^{\infty} C_nx^n$, where $C_n$ is the polynomial representation of the compositions of $n$.

We should mention a similar result holds for the GIPPE associated with $1-\sum_{n=1}^{\infty} a_nx^n$ where $a_n$ is a positive integer. Theorem 3.2 implies that

$$1-\sum_{n=1}^{\infty} a_nx^n = \prod_{n=1}^{\infty}(1+h_nx^n)^{-r_n}, \qquad h_n \text{ a positive integer.} \tag{4.12}$$

We then invert Equation (4.12) to obtain

$$\frac{1}{1-\sum_{n=1}^{\infty} a_nx^n} = \prod_{n=1}^{\infty}(1+h_nx^n)^{r_n} = 1+\sum_{n=1}^{\infty} C_nx^n, \tag{4.13}$$

where $C_n$ is the aforementioned polynomial representation for the compositions of $n$.

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