FACTORIZATIONS OF ANALYTIC FUNCTIONS VIA GENERALIZED INVERSE POWER PRODUCT EXPANSIONS

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Abstract: Given an arbitrary sequence of complex numbers \(\{a_n\}_{n=1}^{\infty}\) and an arbitrary nonzero sequence of complex numbers \(\{r_n\}_{n=1}^{\infty}\), we study the expansion of the Taylor series \(1 + \sum_{n=1}^{\infty} a_n x^n\) into infinite products of the form \(\prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}\). Algebraic properties, convergence criteria, and combinatorial interpretations of the infinite products are investigated. We also provide an asymptotic formula for the majorizing product expansion associated with \(1 - \sum_{n=1}^{\infty} s^n x^n\), \(s := \sup_{n \geq 1} |a_n|\).

AMS Subject Classification: 41A10, 30E10, 11P81, 05A17
Key Words: power series, expansions, analytic functions, power products, generalized power products, generalized inverse power products, convergence, asymptotics, multi-sets, partitions, compositions

1. Introduction

Let \(f(x)\) be a complex function with power series representation \(f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n\). In [9] the authors discovered algebraic, analytic, and combinatorial
properties of the generalized power product expansion \( f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n)^{r_n} \), where \( \{r_n\} \) is an arbitrary set of nonzero complex numbers. Such work complements and extends the results of Borofsky, Feld, Hertzog, Ketchum, Kolberg, A. Knopfmacher, Indelkofer, Ritt, and Warlimont [1, 2, 3, 4, 12, 14, 15, 16, 17, 10, 5, 8, 7, 13, 19, 18], who analyzed the particular case of \( r_n = 1 \). These authors, besides exploring \( f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n) \), also investigated the inverse power product expansion

\[
\prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}.
\]

The purpose of this current work is to develop algebraic, analytic, and combinatorial properties of the generalized inverse power product expansion \( f(x) = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \). Many of the algebraic properties obeyed by \( \{h_n\}_{n=1}^{\infty} \) are similar in nature to the algebraic properties of \( \{g_n\}_{n=1}^{\infty} \) discussed in Section 2 of [9]. But there is an important difference. Whereas the negativity of all \( a_n \) translates to the negativity of all \( g_n \) whenever \( r_n \geq 1 \) (see Theorem 2.2. of [9]), in order for the negativity to transfer to all \( h_n \), it is necessary that all \( r_n \) be a positive integers.

This paper is presented in a self-contained manner, with no assumption that the reader has previously read [9]. Section 2 derives the algebraic properties of \( h_n \) in term of \( \{a_n\}_{n=1}^{\infty} \) and \( \{r_n\}_{n=1}^{\infty} \). The useful property, to be called the Structure Property, writes \( h_n \) as a polynomial in the variables \( \{a_i\}_{i=1}^{\infty} \), whose coefficients are rational expressions of the form \( \frac{p(r_1, r_2, ..., r_n)}{q(r_1, r_2, ..., r_n)} \). If each \( r_n \) is a positive integer, and each \( a_n \leq 0 \), the Structure Property ensures that \( h_n \leq 0 \). We exploit the Structure Property in Section 3 when determining a lower bound for the radius of convergence of \( \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \). See Theorem 3.2. Section 3 also contains an asymptotic approximation for the generalized inverse power product expansion associated with \( 1 - \sum_{n=1}^{\infty} s^n x^n \) where \( s = \sup_{n \geq 1} |a_n|^{1/n} \), namely the majorizing product expansion used in the proof of Theorem 3.2. The paper ends with a section dedicated to combinatorial interpretations of \( \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \) and \( \prod_{n=1}^{\infty} (1 + g_n x^n)^{r_n} \), where \( \{r_n\}_{n=1}^{\infty} \) is assumed to be a fixed set of positive integers.
Given a formal power series \( 1 + \sum_{n=1}^{\infty} a_n x^n \) or an analytic function \( f(x) \) with \( f(0) = 1 \) and a Taylor power series representation
\[
f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n,
\]
we define the \textit{Generalized Power Product Expansion}, GPPE, of \( f(x) \) as
\[
f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n)^{r_n} = (1 + g_1 x^1)^{r_1} (1 + g_2 x^2)^{r_2} (1 + g_3 x^3)^{r_3} \ldots,
\]
and the \textit{Generalized Inverse Power Product Expansion}, GIPPE, of \( f(x) \) as
\[
f(x) = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} = (1 - h_1 x^1)^{-r_1} (1 - h_2 x^2)^{-r_2} (1 - h_3 x^3)^{-r_3} \ldots,
\]
where \( \{g_n\}_{n=1}^{\infty}, \{h_n\}_{n=1}^{\infty}, \) and \( \{r_n\}_{n=1}^{\infty} \) are arbitrary nonzero complex numbers.

In Section 2 of [9] various algebraic formulas relating \( \{a_n\}_{n=1}^{\infty} \) to \( \{g_n\}_{n=1}^{\infty} \) and \( \{r_n\}_{n=1}^{\infty} \) of the GPPE were developed. We now derive the analogous formulas for the GIPPE. We start by expanding the right side of Equation (2.2) via Newton’s Binomial Theorem.
\[
1 + \sum_{k=1}^{\infty} a_k x^k = \sum_{k_1=0}^{\infty} \binom{-r_1}{k_1} (-h_1 x^1)^{k_1} \sum_{k_2=0}^{\infty} \binom{-r_2}{k_2} (-h_2 x^2)^{k_2} \sum_{k_3=0}^{\infty} \binom{-r_3}{k_3} (-h_3 x^3)^{k_3} \ldots
\]
Comparing the coefficient of \( x^n \) on both sides of Equation (2.3) gives
\[
a_n = \binom{-r_n}{1} (-h_n) + \sum_{\substack{l,v=n \\ l_j \leq n \\ j \leq v}} (-1)^{v_1 + \ldots + v_\theta} \binom{-r_{l_1}}{v_1} \ldots \binom{-r_{l_\theta}}{v_\theta} \prod_{j=1}^{\theta} h_{l_j}^{v_j}, \tag{2.4}
\]
where \( l = [l_1, l_2, \ldots l_\theta] \) and \( v = [v_1, v_2, \ldots v_\theta] \). Equation (2.4) is equivalent to
\[
h_n = \frac{1}{r_n} \left[ a_n - \sum_{\substack{l,v=n \\ l_j \leq n \\ j \leq v}} (-1)^{v_1 + \ldots + v_\theta} \binom{-r_{l_1}}{v_1} \ldots \binom{-r_{l_\theta}}{v_\theta} \prod_{j=1}^{\theta} h_{l_j}^{v_j} \right]. \tag{2.5}
\]
We formalize the above discussion in the following proposition. It is a statement about a bijection between the sequence of the coefficients in a given power series and the sequence of coefficients in its GIPPE expansion.

**Proposition 2.1.** Let \( \{r_k\}_{k=1}^{\infty} \) denote a sequence of nonzero complex numbers. Let \( h_k \in \mathbb{C} \), \( k = 1, 2, \ldots \), be an infinite sequence. Let the symbol \( \prod_{k=1}^{\infty} (1 - h_k x^k)^{-r_k} \) stand for the infinite product

\[
\prod_{k=1}^{\infty} (1 - h_k x^k)^{-r_k} := (1 - h_1 x)^{-r_1} (1 - h_2 x^2)^{-r_2} \cdots (1 - h_k x^k)^{-r_k} \cdots .
\]

(2.6)

Then there exists a unique sequence \( a_n \in \mathbb{C}, n = 1, 2, \ldots \), such that in the sense of power series the following holds

\[
1 + \sum_{n=1}^{\infty} a_n x^n := \prod_{k=1}^{\infty} (1 - h_k x^k)^{-r_k}.
\]

(2.7)

Conversely, let \( a_n \in \mathbb{C}, n = 1, 2, \ldots \), be an infinite sequence. Then there exists a unique sequence of elements \( h_k \in \mathbb{C}, k = 1, 2, \ldots \), such that the identity (2.7) holds. Moreover, the elements \( h_k \) have the representation provided by Equation (2.5).

To develop a number theoretic formula for \( h_n \), we define \( \log(1 - h_n x^n) := -\sum_{k=1}^{\infty} \left( \frac{h_n x^n}{k} \right)^k \), where we say \( \log(1 - h_n x^n) \) exists and is well-defined if the series itself converges. Next define

\[
\log \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} := \sum_{n=1}^{\infty} r_n \log (1 - h_n x^n) := \sum_{n=1}^{\infty} r_n \sum_{k=1}^{\infty} \frac{(h_n x^n)^k}{k},
\]

(2.8)

by the convention that \( \sum_{n=1}^{\infty} r_n \log (1 - h_n x^n) \) exists and is well-defined if the double sum on the right side of Equation (2.8) converges. If the double sum on the right of Equation (2.8) is absolutely convergent both \( \sum_{n=1}^{\infty} r_n \log (1 - h_n x^n) \) and \( \log (1 - h_n x^n) \) are also absolutely convergent.

Equation (2.8), when combined with Equation (2.2), allows us to define \( \log f(x) \) as

\[
\log f(x) := -\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n).
\]

(2.9)
Assume that $\sum_{n=1}^{\infty} r_n \log(1 - h_n x^n)$ is absolutely convergent. Differentiate both sides of Equation (2.9) to find that
\[
\frac{f'(x)}{f(x)} = \sum_{n=1}^{\infty} \frac{nr_n h_n x^{n-1}}{1 - h_n x^n} = \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} (h_n x^n)^s r_n h_n^{s+1} x^{n s + n - 1}.
\]

Since $\frac{f'(x)}{f(x)} = d_0 + \sum_{k=1}^{\infty} d_k x^k = r_1 h_1 + \sum_{k=1}^{\infty} d_k x^k$, we may compare the coefficient of $x^k$ in this series with the coefficient of $x^k$ in $\sum_{s=0}^{\infty} \sum_{n=1}^{\infty} nr_n h_n^{s+1} x^{n s + n - 1}$ to obtain
\[
d_k = \sum_{n: n|(k+1)} nr_n h_n^{k+1}.
\]

(2.10)

If $k + 1$ is prime the sum of Equation (2.10) consists of only two terms, namely those associated with $n = 1$ and $n = k + 1$. Equation (2.10) becomes $d_k = r_1 h_1^{k+1} + (k + 1)r_{k+1} h_{k+1}$, which is equivalent to saying
\[
h_{k+1} = \frac{d_k - r_1 h_1^{k+1}}{(k + 1)r_{k+1}}, \quad k + 1 \text{ prime.}
\]

(2.11)

If $k + 1$ is not prime we use Equation (2.10) to solve for $h_{k+1}$ as
\[
h_{k+1} = \frac{d_k - \sum_{n: n|(k+1)} nr_n h_n^{k+1}}{(k + 1)r_{k+1}}.
\]

(2.12)

Equation (2.12) allows us to prove a theorem which demonstrates the connection between a GPPE of $f(x)$ and its associated GIPPE.

**Theorem 2.1.** Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$. Let $\{r_n\}_{n=1}^{\infty}$ be a given set of nonzero complex numbers. Suppose $f(x)$ has a GPPE of the form $f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n)^{r_n}$ and an GIPPE of the form $f(x) = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}$. Then $h_{2l+1} = g_{2l+1}$ for all nonnegative integers $l$.

**Proof.** We induct on $l$. A derivation similar to the one used for Equation (2.12) implies that
\[
g_{k+1} = \frac{d_k + \sum_{n: n|(k+1)} nr_n (-g_n)^{k+1}}{(k + 1)r_{k+1}},
\]

(2.13)
where \( f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \), \( \frac{f'(x)}{f(x)} = \sum_{n=0}^{\infty} d_0 x^n \), and \( f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n)^{r_n} \) for a given set of nonzero complex numbers \( \{r_n\}_{n=0}^{\infty} \). Let \( l = 0 \). Equation (2.13) with \( f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n)^{r_n} \) implies \( g_1 = \frac{d_0}{r_1} \), while Equation (2.12) implies \( h_1 = \frac{d_0}{r_1} \). Hence \( g_1 = h_1 \). Now assume that \( g_{2l+1} = h_{2l+1} \) for all \( 0 \leq l \leq L \) where \( L \) is a fixed nonnegative integer. In Equation (2.13) take \( k = 2L + 2 \) to obtain

\[
g_{2L+3} = \frac{d_{2L+2} + \sum_{n \neq 2L+3} n r_n (-g_n)^{2L+3}}{(2L + 3)r_{2L+3}}
\]

where the last equality follows from Equation (2.12).

**Remark 2.1.** Proposition 3.1 of [8] is now a special case of Theorem 2.1.

Our next task is to develop a recursive formula for \( h_n \). Let \( f(x) = 1 + \sum_{n=1}^{\infty} B_{1,n} x^n = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \). By definition \( a_n = B_{1,n} \). We define a recursive system of equations as follows.

\[
1 + \sum_{n=1}^{\infty} B_{1,n} x^n = (1 - h_1 x)^{-r_1} \prod_{n=2}^{\infty} (1 - h_n x^n)^{-r_n}
= (1 - h_1 x)^{-r_1} \left[ 1 + \sum_{n=2}^{\infty} B_{2,n} x^n \right]
\]
\[
1 + \sum_{n=2}^{\infty} B_{2,n} x^n = (1 - h_2 x^2)^{-r_2} \prod_{n=3}^{\infty} (1 - h_n x^n)^{-r_n}
= (1 - h_2 x^2)^{-r_2} \left[ 1 + \sum_{n=3}^{\infty} B_{3,n} x^n \right]
\]
\[
\vdots
\]
\[
1 + \sum_{n=j}^{\infty} B_{j,n} x^n = (1 - h_j x^j)^{-r_j} \prod_{n=j+1}^{\infty} (1 - h_n x^n)^{-r_n}
\]
\[(1 - h_j x^j)^{-r_j} \left[ 1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n \right] \]

\[\vdots\]

Newton’s Binomial Theorem implies that

\[1 + \sum_{n=j}^{\infty} B_{j,n} x^n = \left[ 1 + \sum_{k=1}^{\infty} \binom{-r_j}{k} (-h_j)^k x^{jk} \right] \left[ 1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n \right]. \tag{2.14}\]

We expand the right side of Equation (2.14) and compare the coefficient of \(x^s\) with that of \(x^s\) in \(1 + \sum_{n=j}^{\infty} B_{j,n} x^n\) to obtain

\[B_{j,s} = \sum_{N+jk=s} \binom{-r_j}{k} (-h_j)^k B_{j+1,N} \tag{2.15}\]

\[= B_{j+1,s} + \sum_{k=1}^{\left\lfloor \frac{s}{j} \right\rfloor} \binom{r_j + k - 1}{k} h_j^k B_{j+1,s-jk}, \tag{2.16}\]

since \(\binom{z}{k} = (-1)^k \binom{-z+k-1}{k}\).

Take Equation (2.15) and set \(s = j\) where \(j \geq 1\). The sum has only two terms, namely when \(N = 0\) and \(k = 1\), or \(N = j\) and \(k = 0\). By definition \(B_{j+1,j} = 0\) and \(B_{j+1,0} = 1\), and Equation (2.15) becomes

\[B_{j,j} = \binom{-r_j}{1} (-h_j)^k B_{j+1,0} = r_j h_j. \tag{2.17}\]

Equation (2.17) demonstrates the connection between \(B_{j,j}\) and \(h_j\). We use this connection to derive a structure property of \(\{h_j\}_{j=1}^{\infty}\). To discover this intriguing structure rewrite \(1 + \sum_{n=j}^{\infty} B_{j,n} x^n = (1 - h_j x^j)^{-r_j} \left[ 1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n \right]\) as

\[1 + \sum_{n=j+1}^{\infty} B_{j+1,n} x^n = (1 - h_j x^j)^{r_j} \left[ 1 + \sum_{n=j}^{\infty} B_{j,n} x^n \right] = \left[ 1 + \sum_{k=1}^{\infty} \binom{r_j}{k} (-h_j)^k x^{jk} \right] \left[ 1 + \sum_{n=j}^{\infty} B_{j,n} x^n \right] \]
$$= \left[1 + \sum_{k=1}^{\infty} (-1)^k \left( \frac{r_j}{k} \right) \left( \frac{B_{j,j}}{r_j} \right)^k x^k \right] \left[1 + \sum_{n=j}^{\infty} B_{j,n}x^n \right].$$

If we compare the coefficient of $x^s$ on both sides of the previous equation we find that

$$B_{j+1,s} = \sum_{jk+n=s} (-1)^k \left( \frac{r_j}{k} \right) B_{j,n}B_{j,j}^k, \quad 0 \leq k \leq \frac{s}{j}. \quad (2.18)$$

Equation (2.18) is key to proving the following theorem.

**Theorem 2.2.** Let $j$ be any positive integer. Define $B_{j,0} = 1$ and $B_{j,N} = 0$ for $1 \leq N \leq j-1$. Assume that $r_j$ is a positive integer for all $j$ and that $B_{j,N} \leq 0$ for all $j \leq N$. Then $B_{j+1,N} \leq 0$ whenever $j + 1 \leq N$.

**Proof.** Equation (2.18) is equivalent to

$$B_{j+1,s} = \sum_{n+jk=s} \left( -1 \right)^k \left( \frac{r_j}{k} \right) B_{j,j}^k B_{j,n} + \frac{\left( r_j \right)}{\hat{s}} \left( -B \right)^{\hat{s}_{j,j}} + \left( -1 \right)^{\hat{s}_{j,j}-1} \left( \frac{r_j}{\hat{s}_{j,j}-1} \right) B_{j,j}^{\hat{s}_{j,j}}. \quad (2.19)$$

Rewrite Equation (2.19) as $B_{j+1,s} = A + B$, where

$$A := \sum_{n+jk=s} \left( -1 \right)^k \left( \frac{r_j}{k} \right) B_{j,j}^k B_{j,n}, \quad B := \frac{\left( r_j \right)}{\hat{s}} \left( -B \right)^{\hat{s}_{j,j}} + \left( -1 \right)^{\hat{s}_{j,j}-1} \left( \frac{r_j}{\hat{s}_{j,j}-1} \right) B_{j,j}^{\hat{s}_{j,j}}. \quad (2.20)$$

Since $r_j$ is a positive integer $\left( \frac{r_j}{k} \right) \geq 0$. By hypothesis $B_{j,n}B_{j,j}^k$ is the product of $k + 1$ nonpositive numbers and is either zero or has a sign of $(-1)^{k+1}$. Thus $(-1)^kB_{j,n}B_{j,j}^k$ is either zero or negative, and each summand in $A$ is nonpositive.

It remains to show that $B$ is also nonpositive. Notice that $B$ only exists if $\frac{s}{j}$ is a positive integer, say $\hat{s}_{j} = \hat{k}$. Then $B$ becomes

$$B = (-1)^{\hat{k}} \left( \frac{r_j}{\hat{k}} \right) B_{j,j}^{\hat{k}} + (-1)^{\hat{k}-1} \left( \frac{r_j}{\hat{k} - 1} \right) B_{j,j}^{\hat{k}-1}$$

$$= (-1)^{\hat{k}} \frac{r_j}{\hat{k}} \left( \frac{r_j - 1}{\hat{k} - 1} \right) B_{j,j}^{\hat{k}} + (-1)^{\hat{k}-1} \left( \frac{r_j}{\hat{k} - 1} \right) B_{j,j}^{\hat{k}-1}.$$
Since \( r_j \) and \( \hat{k} \) are positive integers \( \binom{r_j - 1}{\hat{k} - 1} \geq 0 \). By hypothesis \( B^k_{j,j} \) is either zero or has a sign of \((-1)^k\). Thus \( \frac{(-1)^{k-1}}{r_j - \hat{k} - 1} \binom{r_j - 1}{\hat{k} - 1} B^k_{j,j} \) is nonpositive. It remains to analyze the sign of the rational expression inside the square bracket at (2.21). The sign of this expression depends only on the sign of \( r_j - \hat{k} - 1 \) since the other three factors are always nonnegative. If we assume \( r_j - \hat{k} + 1 > 0 \), or that \( r_j + 1 > \hat{k} \), then the rational expression is nonnegative, and the quantity at (2.21) becomes nonpositive. If \( r_j + 1 - \hat{k} < 0 \), then \( 1 \leq r_j \leq \hat{k} - 1 \), which in turn implies that \( \binom{r_j - 1}{\hat{k} - 1} = 0 \). So once again the quantity at Line (1) is nonpositive. Only one case remains, that of \( r_j + 1 = \hat{k} \). Notice that \( 1 \leq r_j = \hat{k} - 1 \). The definition of \( B \) provided by Equation (2.20) implies that \( B = (-1)^{k-1} \frac{B^k_{j,j}}{r_j - 1} \), a quantity which is either zero or has a sign of \((-1)^{k-1}(-1)^{\hat{k}} = -1 \). In all three cases we have shown that \( B \) is nonpositive.

If we use the notation of [6], we may transform Theorem (2.2) into a theorem about the structure of the \( B_{j+1,s} \). Define \( \alpha = (j_1, j_2, \ldots, j_n) \) to be a vector with \( n \) components where each component is a positive integer. Let \( \lambda = \lambda(\alpha) \) be the length of \( \alpha \), i.e. \( \lambda = n \). Let \( |\alpha| \) denote the sum of the components, namely \( |\alpha| = \sum_{s=1}^{n} j_s \). The symbol \( B_{j,\alpha} \) represents the expression \( B_{j,j_1}B_{j,j_2} \ldots B_{j,j_n} \). For example if \( \alpha = (2, 3, 4, 3) \), then \( \lambda = 4 \), \( |\alpha| = 12 \), and \( B_{j,(2,3,4,3)} = B_{j,2}B_{j,3}B_{j,4}B_{j,3} = B_{j,2}B_{j,3}^2B_{j,4} \).

**Theorem 2.3.** (Structure Property) Let \( j \) be a positive integer. Then

\[
B_{j+1,s} = \sum_{l} (-1)^{\lambda(\alpha(l)) - 1} |c(\alpha(l), j, s)| B_{j,\alpha(l)}
= \sum_{l} (-1)^{\lambda(\alpha(l)) + 1} |c(\alpha(l), j, s)| B_{j,\alpha(l)}, \tag{2.22}
\]
where the sum is over all \( \alpha(l) = (j_1, j_2, \ldots j_\lambda) \) such that \( |\alpha(l)| = s \) and at most one \( j_i \neq j \). The expression \( |c(\alpha(l), j, s)| \) denotes a rational expression in terms of \( j, s \) and \( r_j \) which is nonnegative whenever \( r_j \) is a positive integer. Furthermore, define \( B_{j, \alpha(l)} = B_{j, j_1} B_{j, j_2} \ldots B_{j, j_\lambda} \). If \( B_{j, s} \leq 0 \) for all nonnegative integers \( j \) and all \( s \geq j \), Equation (2.22) is equivalent to

\[
B_{j+1, s} = -\sum |c(\alpha(l), j, s)||B_{j, j_1}||B_{j, j_2}||B_{j, j_3}|, \tag{2.23}
\]

where the sum is over all \( \alpha(l) = (j_1, j_2, \ldots j_\lambda) \) such that \( |\alpha(l)| = s \) and at most one \( j_i \neq j \).

**Proof.** Take the first term on the right side of Equation (2.19), represent \( B_{k, j}^k \) as \( B_{j, \alpha(l)} \) and \( \frac{\lambda}{r_j} \) as \( |c(\alpha(l), j, s)| \). Notice that \( (-1)^k = (-1)^{\lambda(\alpha(l))-1} \). The remaining terms on the right side of Equation (2.19) are \( B \), and we combine them via (2.21) by setting \( \hat{B}_{j, j}^k = B_{j, \alpha(l)} \), and \( |c(\alpha(l), j, s)| = \frac{\lambda}{r_j} \frac{(\hat{r}_j+1)(\hat{k}-1)}{k(\hat{k}-\hat{r}_j+1)} \) as long as \( r_j \neq \hat{k} + 1 \). If \( r_j = \hat{k} + 1 \), then

\[
B = (-1)^{\hat{k}+1} \frac{B_{j, j}^k}{r_j^{\hat{k}+1}} \text{ and } \hat{B}_{j, j}^k = B_{j, \alpha(l)} \text{ while } |c(\alpha(l), j, s)| = \frac{1}{r_j^{\hat{k}+1}}.
\]

If we take Equation (2.22) and iterate \( j \) times we discover that

\[
B_{j+1, s} = \sum_{\alpha(l)}(-1)^{\lambda(\alpha(l))+1}|c(\alpha(l), j, s)||a_{\alpha(l)}
\]

\[= -\sum_{\alpha(l)}|c(\alpha(l), j, s)||a_{j_1}||a_{j_2}||a_{j_3}|, \quad \tag{2.24}
\]

where the sum is over all \( \alpha(l) = (j_1, j_2, \ldots j_\lambda) \) such that \( |\alpha(l)| = s \) and \( |c(\alpha(l), j, s)| \) is a rational expression in \( j, s \) and \( \{r_i\}_{i=1}^{j+1} \) which is nonnegative whenever \( r_i \) is a positive integer.

If \( s = j + 1 \) Equation (2.24) becomes

\[
B_{j+1, j+1} = r_{j+1} h_{j+1} = \sum_{\alpha(l)}(-1)^{\lambda(\alpha(l))+1}|c(\alpha(l), j)|a_{\alpha(l)}
\]

\[= -\sum_{\alpha(l)}|c(\alpha(l), j)||a_{j_1}||a_{j_2}||a_{j_3}|, \quad \tag{2.25}
\]

where the sum is over all \( \alpha(l) = (j_1, j_2, \ldots j_\lambda) \) such that \( |\alpha(l)| = j + 1 \). If \( \{r_i\}_{i=1}^{j+1} \) is a set of positive integers, each coefficient is nonnegative. Below we explicitly
list $h_i$ for $1 \leq i \leq 6$.

\[ h_1 = (-1)^0 \frac{1}{r_1} a_1 \quad h_2 = (-1)^1 \frac{r_1 + 1}{2r_1 r_2} a_1^2 + (-1)^0 \frac{1}{r_2} a_2 \]

\[ h_3 = (-1)^1 \frac{r_1^2 - 1}{3r_1^2 r_3} a_4^3 + (-1)^1 \frac{1}{r_3} a_1 a_2 + (-1)^0 \frac{1}{r_3} a_3 \]

\[ h_4 = (-1)^1 \frac{r_2 + 1}{2r_2 r_4} a_2^2 + (-1)^1 \frac{2r_3^2 r_2 + r_1 + 1}{2r_1 r_2 r_4} a_1 a_2 \]

\[ + (-1)^3 \frac{2r_2 + 2r_3^3 r_2 + r_3^2 + 2r_1^3 + r_1}{8r_1^3 r_2 r_4} a_1^4 + (-1)^1 \frac{1}{r_4} a_1 a_3 + (-1)^0 \frac{1}{r_4} a_4 \]

\[ h_5 = (-1)^2 \frac{1}{r_5} a_1^2 a_3 + (-1)^1 \frac{1}{r_5} a_2 a_3 + (-1)^2 \frac{1}{r_5} a_1 a_2 \]

\[ + (-1)^3 \frac{1}{r_5} a_1 a_2 + (-1)^1 \frac{1}{r_5} a_1 a_4 + (-1)^4 \frac{r_1^4 - 1}{5r_1^4 r_5} a_4^5 + (-1)^0 \frac{1}{r_5} a_5 \]

\[ h_6 = (-1)^2 \frac{1}{r_6} a_1^2 a_4 + (-1)^1 \frac{1}{r_6} a_2 a_4 + (-1)^1 \frac{r_3 + 1}{2r_3 r_6} a_3 \]

\[ + (-1)^3 \frac{r_1^2 + 3r_1^3 r_3 - 1}{3r_1^2 r_3 r_6} a_3 a_3 + (-1)^2 \frac{2r_3 + 1}{r_3 r_6} a_1 a_2 a_3 + (-1)^2 \frac{r_2^2 - 1}{3r_2^2 r_6} a_3 \]

\[ + (-1)^3 \frac{3r_1^3 r_2 r_3 + r_1^3 r_2 - r_3 - r_1 r_3}{2r_1 r_2^2 r_3 r_6} a_1^2 a_2 \]

\[ + (-1)^4 \frac{12r_1^2 r_2^2 r_3 - 3r_3 - 6r_3 r_1 + 4r_1^2 r_2^2 - 4r_2^2 - 3r_1^2 r_3}{12r_1^2 r_2^2 r_3 r_6} a_1^4 a_2 \]

\[ + (-1)^5 \frac{(12r_1^2 r_2^3 r_3 + 12r_2^3 r_3 - 9r_1^4 r_3 - 3r_1^2 r_3 - 8r_1^2 r_3^3 + 4r_1 r_2^2 - 3r_1^2 r_3 + 4r_1^3 r_2^2 - 9r_1^3 r_3)}{72r_1^2 r_2^2 r_3 r_6} a_1^6 \]

Table 1: The expressions for $h_i$, $1 \leq i \leq 6$.

**Remark 2.2.** It is reasonable to expect that the number of terms occurring in each $B_{j,n}$ of a GPPE or GIPPE, where at least one $r_j \neq 1$, to be larger than the number of terms in a GPPE or GIPPE with all $r_j = 1$. This observation translates into the number theoretic property that the sum of the integral coefficients in the last two terms of $h_6$ are zero when $r_1 = r_2 = r_3 = r_6 = 1$.

**Remark 2.3.** It is instructive to compare the properties of the GPPE with those of the GIPEE. For a GPPE we have

\[
r_{j+1}g_{j+1} = \sum_l (-1)^{l(\alpha(l)) + 1} |c(\alpha(l), j)| a_{\alpha(l)}
\]

\[= - \sum_l |c(\alpha(l), j)| |a_{j_1}||a_{j_2}| \ldots |a_{j_N}|, \quad (2.26)\]

where the sum is over all $\alpha(l) = (j_1, j_2, \ldots j_N)$ such that $|\alpha(l)| = j + 1$ and $\{r_i\}_{i=1}^{j+1}$ is a set of real numbers such that $r_i \geq 1$. See [12]. Since Theorem
2.1 shows that \( g_{2l+1} = h_{2l+1} \) for all nonnegative integers \( l \), we conclude that Equation (2.25) is valid for \( \{r_i\}_{i=1}^{j+1} \), a set of real numbers with \( r_i \geq 1 \), whenever \( j \) is an even nonnegative integer. This observation does not follow from the techniques used to prove Theorem 2.2. In that proof it was crucial that \( r_j \) was a positive integer since this condition ensured that \( \binom{r_j}{k} \geq 0 \). As the following counterexample demonstrates, if \( s \neq j+1 \), Equation (2.22) is not true for \( r_j \geq 1 \) unless \( r_j \) is a positive integer. Take Equation (2.18) and let \( j = 1 \) and \( s = 5 \) to obtain

\[
B_{2,5} = \left( \frac{\binom{r_1}{4}}{r_1^4} - \frac{\binom{r_5}{5}}{r_5^5} \right) B_{1,1}^5 - \frac{\binom{r_1}{3}}{r_1^3} B_{1,1}^3 B_{1,2} + \frac{\binom{r_2}{2}}{r_2^2} B_{1,1}^2 B_{1,3}
- B_{1,1} B_{1,4} + B_{1,5}.
\]

Then set \( r_1 = \frac{3}{2} \). After simplification we find that

\[
B_{2,5} = \frac{1}{162} B_{1,1}^5 + \frac{1}{54} B_{1,1}^3 B_{1,2} + \frac{1}{6} B_{1,1}^2 B_{1,3} - B_{1,1} B_{1,4} + B_{1,5}.
\quad (2.27)
\]

The term \( \frac{1}{54} B_{1,1}^3 B_{1,2} \) does not have proper sign of \( (-1)^{4-1} = -1 \).

Equation (2.27) also demonstrates that \( B_{2,5} \) may be positive even if \( \{B_{1,k}\}_{k=1}^5 = \{a_k\}_{k=1}^5 \) are assumed to be nonpositive. For example, take \( B_{1,1} = B_{1,3} = B_{1,4} = B_{1,5} = -1 \), and let \( B_{1,2} = -200 \). Equation (2.27) becomes \( \frac{124}{81} \). If we fix \( B_{1,n} \) for \( n \in \{1, 3, 4, 5\} \) we may choose a value of \( B_{1,2} \leq 0 \) such that \( B_{2,5} \) is arbitrarily large in the positive sense.

There is one other observation which follows from Table 1. For \( h_i \) with \( 1 \leq i \leq 6 \) we observe that \( h_i \) may be written as a polynomial in \( \{r_i^{-1}, \ldots, r_i^{-1}\} \). We deduce that \( h_j \) is a polynomial in \( \{r_1^{-1}, \ldots, r_j^{-1}\} \) for all nonnegative integers \( j \). This and more actually follow Equation (2.22). If we take Equation (2.22) and induct on \( j \) we prove the following proposition.

**Proposition 2.2.** Let \( n \) and \( j \) be positive integers. Define \( B_{j,n} \) via Equation (2.14). Then \( B_{j,n} \) is a polynomial in \( \{r_1^{-1}, r_1^{-2}, \ldots, r_j^{-1}\} \). In particular \( h_j = \frac{B_{j,j}}{r_j} \) is also a polynomial in \( \{r_1^{-1}, r_1^{-2}, \ldots, r_j^{-1}\} \). Moreover, \( h_{2l+1} \) is a polynomial in \( \{r_{2l+1}^{-1}\}_{l=0}^{l} \).
3. Convergence Criterion for Equation (2.2)

The structure of $h_j$ provided by Equation (2.25) allows us to prove the following theorem.

**Theorem 3.1.** Let $\{r_n\}_{n=1}^{\infty}$ be a set of positive integers. Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$. Then $f(x)$ has the GIPPE

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}. \quad (3.1)$$

Consider the auxiliary functions

$$C(x) = 1 - \sum_{n=1}^{\infty} |a_n| x^n = \prod_{n=1}^{\infty} (1 + H_n x^n)^{-r_n}. \quad (3.2)$$

$$M(x) = 1 - \sum_{n=1}^{\infty} M_n x^n = \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}. \quad (3.3)$$

Assume that $|a_n| \leq M_n$ for all $n$. Then $|h_n| \leq H_n \leq F_n$ for all $n$.

**Proof.** By Equation (2.25) we have

$$h_n = \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))} + 1 |c(\alpha(l), n)| a_{\alpha(l)}$$

$$= \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))} + 1 |c(\alpha(l), n)| a_{j_1} a_{j_2} \cdots a_{j_{\lambda}}. \quad (3.4)$$

Equation (3.4) implies that

$$|h_n| = \left| \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))} + 1 |c(\alpha(l), n)| a_{j_1} a_{j_2} \cdots a_{j_{\lambda}} \right|$$

$$\leq \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)||a_{j_1}| |a_{j_2}| \cdots |a_{j_{\lambda}}| \quad (3.5)$$

Equation (2.25) when applied to Equation (3.2) implies that

$$0 \leq H_n = \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))}|c(\alpha(l), n)| (-|a_{j_1}|)(-|a_{j_2}|) \cdots (-|a_{j_{\lambda}}|)$$
\[
\begin{align*}
= & \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(2\alpha(l))} |c(\alpha(l), n)|([a_{j1}])([a_{j2}]) \ldots ([a_{j\lambda}]) \\
= & \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)|[a_{j1}] [a_{j2}] \ldots [a_{j\lambda}] \\
\end{align*}
\]

Combining Equations (3.5) and (3.6) shows that \( |h_n| \leq H_n \). Since \(|a_n| \leq M_n\) we also have

\[
0 \leq H_n = \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)||a_{j1}| |a_{j2}| \ldots |a_{j\lambda}| \\
\leq \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)|M_{j1} M_{j2} \ldots M_{j\lambda} = F_n. \quad \Box
\]

**Remark 3.1.** Equation (2.24) allows for the following generalization of Theorem (3.1). Let \( f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \) and \( \{r_n\}_{n=1}^{\infty} \) be a set of positive integers. Consider the partial product expansion

\[
f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{p} (1 - h_n x^n)^{-r_n} \left[ 1 + \sum_{j=p+1}^{\infty} B_{p+1,j} x^j \right].
\]

Also consider the partial product expansions of the auxiliary functions

\[
C(x) = 1 - \sum_{n=1}^{\infty} |a_n| x^n = \prod_{n=1}^{p} (1 + H_n x^n)^{r_n} \left[ 1 - \sum_{j=p+1}^{\infty} \hat{B}_{p+1,j} x^j \right],
\]

\[
M(x) = 1 - \sum_{n=1}^{\infty} M_n x^n = \prod_{n=1}^{p} (1 + F_n x^n)^{r_n} \left[ 1 - \sum_{j=p+1}^{\infty} F_{p+1,j} x^j \right].
\]

If \(|a_n| \leq M_n\) for all \( n \), we conclude that \(|h_k| \leq H_k \leq F_k\) for \( 1 \leq k \leq p \) and that \(|B_{p+1,j}| \leq \hat{B}_{p+1,j} \leq F_{p+1,j}\) for all \( j \geq p + 1 \).

Let

\[
M(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}, \quad s := \sup_{n \geq 1} |a_n|^n. \quad (3.7)
\]

Our first task is to determine the radius of convergence of the GIPPE in Equation (3.7). Define

\[
\log(1 + F_n x^n)^{-r_n} := -r_n \log(1 + F_n x^n) := r_n \sum_{l=1}^{\infty} \frac{(-1)^l(F_n x^n)^l}{l}
\]
by the convention that $-r_n \log(1 + F_n x^n)$ is well-defined whenever the series converges. Next define

$$-\sum_{n=1}^{\infty} r_n \log (1 + F_n x^n) = \sum_{n=1}^{\infty} [-r_n \log (1 + F_n x^n)]$$

$$:= \sum_{n=1}^{\infty} r_n \sum_{l=1}^{\infty} \frac{(-1)^l (F_n x^n)^l}{l}, \quad (3.8)$$

by the convergence of the double series. Equation (3.8) implies that if the double series is absolutely convergent, then both $\sum_{n=1}^{\infty} [-r_n \log (1 + F_n x^n)]$ and $-r_n \log(1 + F_n x^n)$ are absolutely convergent. The absolute convergence of the double series implies the absolute convergence of $\prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}$ since

$$e^{\sum_{n=1}^{\infty} [-r_n \log(1+F_n x^n)]} = e^{\sum_{n=1}^{\infty} \log(1+F_n x^n)^{-r_n}} = \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}.$$  

Define $\log \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n} := \sum_{n=1}^{\infty} [-r_n \log (1 + F_n x^n)]$ via Equation (3.8). If we take the logarithm of Equation (3.7) we find that

$$\sum_{n=1}^{\infty} [-r_n \log (1 + F_n x^n)] = \log \left( 1 - \sum_{n=1}^{\infty} s^n x^n \right) \quad (3.9)$$

Now

$$1 - \sum_{n=1}^{\infty} s^n x^n = 1 - sx \sum_{n=0}^{\infty} (sx)^n = 1 - \frac{sx}{1-sx} = \frac{1-2sx}{1-sx}.$$

Therefore

$$\log \left( \frac{1-2sx}{1-sx} \right) = \log(1-2sx) - \log(1-sx)$$

$$= -\sum_{n=1}^{\infty} \frac{(2sx)^n}{n} + \sum_{n=1}^{\infty} \frac{(sx)^n}{n} = \sum_{n=1}^{\infty} \frac{1-2^n}{n} (sx)^n.$$

By the Ratio Test we know that $\sum_{n=1}^{\infty} \frac{1-2^n}{n} (sx)^n$ converges whenever

$$\lim_{n \to \infty} \left| \frac{n(1-2^{n+1})}{(n+1)(1-2^n)} \right| |sx| < 1.$$

This is ensured by requiring $|x| < \frac{1}{2s}$. 

To determine the radius of convergence of the GIPPE of Equation (2.2) it suffices to determine the radius of convergence of

$$\log \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} := \sum_{n=1}^{\infty} [-r_n \log(1 - h_n x^n)] = - \sum_{n=1}^{\infty} r_n \log(1 - h_n x^n),$$

where the series is defined via the convergence of the double series in Equation (2.8). By definition we have

$$\left| \log \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \right| = \left| \sum_{n=1}^{\infty} [-r_n \log(1 - h_n x^n)] \right| \leq \sum_{n=1}^{\infty} r_n \left| \log(1 - h_n x^n) \right| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(h_n x^n)^k}{k} \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(|h_n|\|x\|^n)^k}{k},$$

where the last inequality follows from (3.1). These calculations imply that if \( \sum_{n=1}^{\infty} [-r_n \log(1 + F_n x^n)] \) is absolutely convergent then so is

$$\sum_{n=1}^{\infty} [-r_n \log(1 - h_n x^n)]$$

and

$$\prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}$$

absolutely convergent. Since \( \sum_{n=1}^{\infty} [-r_n \log(1 + F_n x^n)] \) is absolutely convergent whenever \( |x| < \frac{1}{2^s} \), we have proven the following theorem.

**Theorem 3.2.** Let \( f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \) Let \( s := \sup_{n \geq 1} |a_n|^{\frac{1}{n}} \). Then both \( f(x) \) and its GIPPE,

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 - g_n x^n)^{-r_n},$$

where
and the auxiliary function, along with its GIPPE,

\[ M(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \prod_{n=1}^{\infty} (1 + F_n x^n)^{-r_n}, \quad (3.10) \]

will be absolutely convergent whenever \( |x| \leq \frac{1}{2s} \).

We now provide an asymptotic estimate for the majorizing GIPPE of Equation (3.10).

**Theorem 3.3.** Let \( f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \frac{1-2sx}{1-sx} \) where \( s > 0 \). Let \( \{r_n\}_{n=1}^{\infty} \) be a sequence of positive integers. For this particular \( f(x) \) and its associated GIPPE \( \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \) we have

\[ r_n h_n \sim \frac{(1 - 2^n)s^n}{n}, \quad n \to \infty. \quad (3.11) \]

Before we prove Theorem (3.3) we need the following lemma.

**Lemma 3.1.** Let \( f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \frac{1-2sx}{1-sx} \) where \( s > 0 \). Let \( \{r_n\}_{n=1}^{\infty} \) be a sequence of positive integers. For this particular \( f(x) \) and its associated GIPPE \( \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} \) there exists \( \alpha \) with \( 1 < \alpha < 2 \) such that

\[ mr_m |h_m| \leq \alpha 2^m s^m. \quad (3.12) \]

**Proof.** It is possible to verify through a straight forward calculation that \( \frac{mr_m |h_m|}{(2s)^m} \leq 1.6875 \) whenever \( 1 \leq m \leq 22 \). To prove Equation (3.12) for arbitrary \( m \) assume inductively that \( jr_j |h_j| \leq \alpha 2^j s^j \) is true for \( 1 \leq j < m \). Our analysis shows that we may assume \( m \geq 16 \). To that end we proceed as follows. The first step in our induction proof is to rewrite Equation (3.1). Start with Equation (2.9) and note that

\[ \log f(x) = - \sum_{n=1}^{\infty} r_n \log(1 - h_n x^n) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{r_n h_n^k}{k} x^{nk} = \sum_{m=1}^{\infty} D_m x^m. \]

Comparing the coefficient of \( x^m \) shows that

\[ D_m = \frac{1}{m} \sum_{n: n|m} nr_n h_n^m. \quad (3.13) \]
For the specific case of \( f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \frac{1-2sx}{1-sx} \) we find that

\[
\log f(x) = \log \left( \frac{1-2sx}{1-sx} \right) = \sum_{k=1}^{\infty} \frac{-(2^k-1)s^k}{k} x^k = \sum_{m=1}^{\infty} D_m x^m,
\]

which implies that \( D_m = \frac{-(2s)^m (1-2^{-m})}{m} \).

Take Equation (3.13) and rewrite it as

\[
m[D_m - T_1 - T_2 - T_3 - T_4 - T_5 - T_6 - T_7 - \Delta] = m r_m h_m,
\]

where

\[
T_j = \frac{jr_j}{m} (h_j)^{\frac{m}{j}}, \quad 1 \leq j \leq 7, \quad \Delta = \frac{1}{m} \sum_{\frac{n|m}{2 \geq n \geq 8}} n r_n h_m^{\frac{m}{n}}.
\]

The range of summation of \( \Delta \) implies that \( m \geq 16 \). In order to prove Equation (3.12) it suffices to show that

\[
\frac{m r_m |h_m|}{(2s)^m} = \frac{m}{(2s)^m} |D_m - T_1 - T_2 - T_3 - T_4 - T_5 - T_6 - T_7 - \Delta| \leq \frac{m}{(2s)^m} |D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta| < 2,
\]

whenever \( m \geq 16 \). We must approximate \( \frac{m}{(2s)^m}|D_m|, \frac{m}{(2s)^m}|T_j| \) for \( 1 \leq j \leq 7 \), and \( \frac{m}{(2s)^m}|\Delta| \). Begin with \( \frac{m}{(2s)^m}|D_m| \) and observe that

\[
\frac{m}{(2s)^m}|D_m| = \frac{m}{(2s)^m} \cdot \frac{(2s)^m (1-2^{-m})}{m} < 1.
\]

We now work with \( \frac{m}{(2s)^m}|T_j| \). Take Table 1, let \( a_i = -s^i \), and simplify the results to find that

\[
\begin{align*}
&h_1 = -\frac{s}{r_1}, \quad h_2 = -\frac{s^2 (3r_1 + 1)}{2r_1 r_2}, \quad h_3 = -\frac{s^3 (7r_1^2 - 1)}{3r_1^2 r_3}, \\
&h_7 = -\frac{s^7 (127r_1^6 - 1)}{7r_1^6 r_7}, \quad h_4 = -\frac{s^4 (9r_1^3 + 30r_1^3 r_2 + 6r_1^2 + 2r_2 + r_1)}{8r_1^3 r_2 r_4}, \\
&h_5 = -\frac{s^5 (31r_1^4 - 1)}{5r_1^4 r_5}.
\end{align*}
\]
\[ h_6 = -s^6(4r_1^2 + 12r_2^2r_3 - 3r_1^2r_3 - 56r_2^2r_3 - 81r_3^2 + 756r_1^2r_3 + 196r_1r_2^2 - 81r_1^2r_3 - 27r_3r_3) / 72r_1^2r_3r_6. \]

We use this data to approximate \( m \) for 1 \( \leq j \leq 7 \). All approximations use \( m \geq 16 \) and \( r_j \geq 1. \)

\[
\frac{m}{(2s)^m} |T_1| = \frac{r_1}{(2s)^m} \left( \frac{s}{r_1} \right)^m = \frac{1}{2(2r_1)^{m-1}} \leq \frac{1}{2^{16}} \leq 0.00002 \quad (3.16)
\]

\[
\frac{m}{(2s)^m} |T_2| = \frac{2r_2}{(2s)^m} |h_2|^{\frac{m}{2}} = \frac{2r_2}{4} \left( \frac{3r_1 + 1}{2r_1 r_2} \right)^{\frac{m}{2}} = 2r_2 \left( \frac{3r_1 + 1}{8r_1 r_2} \right) = \left( \frac{3 + r_1^{-1}}{8r_2} \right)^{\frac{m}{2} - 1} \leq \left( \frac{1}{2} \right)^{\frac{16}{7}} \leq 0.08 \quad (3.17)
\]

When approximating \( m \) for \( T_3 \) use the fact that \( T_3 = 0 \) if \( 3 \nmid m \).

\[
\frac{m}{(2s)^m} |T_3| = \frac{3r_3}{(2s)^m} |h_3|^{\frac{m}{3}} = \frac{3r_3}{2} \left( \frac{7r_2^2 - 1}{3r_1^2r_3} \right)^{\frac{m}{3}} \leq \frac{3r_3}{8} \left( \frac{7}{3} \right)^{\frac{m}{3}} = \frac{3r_3}{8} \left( \frac{7}{24} \right)^{\frac{m}{3} - 1} \leq \frac{7}{8} \left( \frac{7}{24} \right)^{\frac{16}{7} - 1} = \frac{7}{8} \left( \frac{7}{24} \right)^{5} \leq 0.002 \quad (3.18)
\]

\[
\frac{m}{(2s)^m} |T_4| = \frac{4r_4}{(2^4)^{\frac{m}{4}}} \left( \frac{9r_1^3 + 30r_1^3r_2 + 6r_1^2 + 2r_2 + r_1}{8r_1^2r_4} \right)^{\frac{m}{4}} = 4r_4 \left( \frac{9r_1^3 + 30r_1^3r_2 + 6r_1^2 + 2r_2 + r_1}{2^7r_1^2r_4} \right)^{\frac{m}{4}} \leq \frac{48}{2^7} \left( \frac{3}{2} \right)^{\frac{m}{4} - 1} \leq \frac{3}{2} \left( \frac{3}{8} \right)^{\frac{16}{7} - 1} \leq \frac{3}{2} \left( \frac{3}{8} \right)^{3} \leq 0.08 \quad (3.19)
\]

When approximating \( m \) for \( T_5 \) use the fact that \( T_5 = 0 \) if \( 5 \nmid m \).

\[
\frac{m}{(2s)^m} |T_5| = \frac{5r_5}{(2^5)^{\frac{m}{5}}} \left( \frac{31r_4^4 - 1}{5r_1^2r_5} \right)^{\frac{m}{5}} \leq 5r_5 \left( \frac{31}{160r_5} \right)^{\frac{m}{5}} = \frac{31}{32} \left( \frac{31}{160r_5} \right)^{\frac{m}{5} - 1}
\]
\[
\leq \frac{31}{32} \left( \frac{31}{160} \right)^{\frac{m}{5} - 1} \leq \frac{31}{32} \left( \frac{31}{160} \right)^{\frac{20}{5} - 1} = \frac{31}{32} \left( \frac{31}{160} \right)^3 \leq 0.008 \quad (3.20)
\]

When approximating \( \frac{m}{(2s)^m} |T_6| \) use the fact that \( T_6 = 0 \) if \( 6 \nmid m \).

\[
\frac{m}{(2s)^m} |T_6| =
\]

\[
= \left. \frac{6r_6}{(2^6)^{\frac{m}{7}}} \left( \frac{4r_1r_2^2 + 12r_2^2r_3 + 3r_2^2r_3 - 56r_2^2r_1 - 81r_3r_4 + 756r_1^3r_2^2r_3 + 196r_1^3r_2^2 - 81r_1^3r_3 - 27r_1^3r_3}{72r_1^3r_2^2r_3r_6} \right)^{\frac{m}{7}}
\]

\[
\leq \left. \frac{6r_6}{(2^6)^{\frac{m}{7}}} \left( \frac{4r_1r_2^2 + 12r_2^2r_3 + 756r_1^3r_2^2r_3 + 196r_1^3r_2^2}{3^22^6r_1^2r_2^2r_3r_6} \right)^{\frac{m}{7}}
\]

\[
\leq \left. \frac{968}{3^22^6} \left( \frac{968}{3^22^6} \right)^{\frac{18}{12} - 1} = \frac{11^2}{3^22^6} \left( \frac{11^2}{3^22^6} \right)^2 \leq 0.056 \quad (3.21)
\]

When approximating \( \frac{m}{(2s)^m} |T_7| \) use the fact that \( T_7 = 0 \) if \( 7 \nmid m \).

\[
\frac{m}{(2s)^m} |T_7| = \left. \frac{7r_7}{(2^7)^{\frac{m}{7}}} \left| \frac{127r_7^6 - 1}{7r_7^6} \right| \right| \leq \frac{7r_7}{(2^7)^{\frac{m}{7}}} \left( \frac{127}{7r_7} \right)^{\frac{m}{7}} = \frac{7r_7}{(2^7)^{\frac{m}{7}}} \left( \frac{127}{7r_7} \right)^{\frac{m}{7}}
\]

\[
= \frac{127}{2^7} \left( \frac{127}{7127r_7} \right)^{\frac{m}{7} - 1} \leq \frac{127}{2^7} \left( \frac{127}{7127} \right)^{\frac{21}{7} - 1}
\]

\[
= \frac{127}{2^7} \left( \frac{127}{7127} \right)^2 \leq 0.02 \quad (3.22)
\]

It remains to approximate \( \frac{m}{(2s)^m} \Delta \). Here is where we make use of the induction hypothesis. We also use the fact the \( \frac{m}{2} \geq n \) implies \( \frac{m}{n} \geq 2 \). By definition we have

\[
\frac{m}{(2s)^m} \Delta \leq \left( \frac{1}{(2s)^m} \sum_{n|m \geq 2} \frac{nr_m|n|}{n} \right)^m \leq \left( \frac{1}{(2s)^m} \sum_{n|m \geq 2} \frac{nr_m}{n} \right)^m \leq \frac{1}{(2s)^m} \sum_{n|m \geq 2} \frac{nr_n}{n} \left( \frac{\alpha n^s m}{n} \right)^m \leq \alpha \sum_{n|m \geq 2} \frac{nr_n}{\alpha} \left( \frac{1}{nr_n} \right)^m \leq \alpha \sum_{n|m \geq 2} \left( \frac{\alpha}{nr_n} \right)^{m - 1}
\]

\[
= \alpha \sum_{n|m \geq 2} \frac{nr_n}{\alpha} \left( \frac{1}{nr_n} \right)^m = \alpha \sum_{n|m \geq 2} \left( \frac{\alpha}{nr_n} \right)^{m - 1}
\]
\[ \leq \alpha \sum_{n|m, m \geq 2} \left( \frac{\alpha}{8} \right)^{n-1} \leq \alpha \left[ \frac{\alpha}{8} \right] \]

\[ = \alpha \left[ \frac{\alpha}{8 - \alpha} \right] \leq \alpha \left[ \frac{2}{8 - 2} \right] = \frac{\alpha}{3} \leq \frac{2}{3} \quad (3.23) \]

We now take Equations (3.16) through Equation (3.23) and place them in

\[ \frac{m r_m h_m}{(2s)^m} \leq \frac{m}{(2s)^m} \left[ |D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta| \right] \]

to find that

\[ \frac{m r_m h_m}{(2s)^m} \leq \frac{m}{(2s)^m} \left[ |D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta| \right] \]

\[ \leq 1 + 0.00002 + 0.08 + 0.002 + 0.08 + 0.008 + 0.056 + 0.02 + \frac{2}{3} \]

\[ = 1.91268667 < 2. \]

In other words Equation (3.14) is valid and our proof is complete. \( \square \)

**Proof of Theorem 3.3.** Equation (2.12) implies that

\[ (k + 1) r_{k+1} h_{k+1} = d_k - r_1 h_{k+1} - \sum_{n|(k+1), k+1 \geq n \geq 2} n r_n h_n^{k+1}. \quad (3.24) \]

Since \( f(x) = 1 - \sum_{n=1}^{\infty} (sx)^n = \frac{1-2sx}{1-sx} \) we discover that

\[ \frac{f'(x)}{f(x)} = \frac{-2s}{1-2sx} + \frac{s}{1-sx} = s \left[ -2 \sum_{k=0}^{\infty} 2^k s^k x^k + \sum_{k=0}^{\infty} s^k x^k \right] \]

\[ = s \sum_{k=0}^{\infty} (-2^{k+1} + 1)s^k x^k. \]

By definition \( \frac{f'(x)}{f(x)} = \sum_{k=0}^{\infty} d_k x^k \). Hence \( d_k = (-2^{k+1} + 1)s^k x^k \), and Equation (3.24) becomes

\[ (k + 1) r_{k+1} h_{k+1} = (-2^{k+1} + 1)s^{k+1} - r_1 \left( \frac{s}{r_1} \right)^{k+1} - \sum_{n|(k+1), k+1 \geq n \geq 2} n r_n h_n^{k+1}. \quad (3.25) \]
Define
\[ T_1 := (-2^{k+1} + 1)s^{k+1}, \quad T_2 := r_1 \left( -\frac{s}{r_1} \right)^{k+1}, \]
and
\[ \Delta := \sum_{n|\{k+1\}}^{|\{k+1\}|} n r_n h_n^{k+1}. \]

Equation (3.25) is equivalent to \((k + 1) r_{k+1} h_{k+1} = T_1 - T_2 - \Delta.\) Lemma (3.1) implies there exist \(\alpha\) with \(1 < \alpha < 2\) such that
\[ n |h_n| \leq nr_n |h_n| \leq \alpha 2^n s^n. \quad (3.26) \]

By definition
\[ |\Delta| = \left| \sum_{n|\{k+1\}}^{|\{k+1\}|} n r_n h_n^{k+1} \right| \leq \sum_{n|\{k+1\}}^{|\{k+1\}|} n r_n |h_n|^{k+1} \leq \sum_{n|\{k+1\}}^{|\{k+1\}|} n r_n \left[ \frac{\alpha 2^n s^n}{nr_n} \right]^{k+1} \]
\[ = \alpha (2s)^{k+1} \sum_{n|\{k+1\}}^{|\{k+1\}|} \frac{1}{n |n/\alpha|^{k+1}} = \alpha (2s)^{k+1} \sum_{n|\{k+1\}}^{|\{k+1\}|} \frac{1}{n |n/\alpha|^{k+1}} \]
\[ \leq \alpha (2s)^{k+1} \sum_{n|\{k+1\}}^{|\{k+1\}|} \frac{1}{n |n/\alpha|^{k+1}} \leq \alpha (2s)^{k+1} \sum_{n|\{k+1\}}^{|\{k+1\}|} \frac{1}{n |n/\alpha|^{k+1}} \]
\[ = \alpha (2s)^{k+1} \left[ \frac{1}{2\alpha^{k+1} - 1} + \frac{2\alpha}{k+1} + \sum_{k+1 \geq n \geq 3} \frac{1}{n |n/\alpha|^{k+1}} \right] \]
\[ \leq \alpha (2s)^{k+1} \left[ \frac{1}{2\alpha^{k+1} - 1} + \frac{2\alpha}{k+1} + \sum_{k+1 \geq n \geq 3} \frac{1}{n |n/\alpha|^{k+1}} \right], \quad (3.27) \]

where the last equality reflects the fact that \(\frac{1}{2} < \frac{1}{\alpha} < 1.\)

Define \(M := \sum_{k+1 \geq n \geq 3} \frac{1}{n |n/\alpha|^{k+1}} = \frac{1}{2} \left( \frac{1}{\alpha^{k+1} - 1} + \frac{1}{(\frac{k+1}{2})^{k+1} - 1} + \frac{1}{(\frac{k+1}{3})^{k+1} - 1} + \frac{1}{(\frac{k+1}{5})^{k+1} - 1} + \frac{1}{(\frac{k+1}{7})^{k+1} - 1} \right) + \sum_{k+1 \geq n \geq 3} \frac{1}{n |n/\alpha|^{k+1}} \)
\(b(n, k) := -\ln \left( \frac{k+1}{n} \right)^{k+1} = -\left( \frac{k+1}{n} - 1 \right) \ln \frac{n}{2}.\) Then
\[ \frac{\partial b(n, k)}{\partial n} = \frac{k+1}{n^2} \ln \frac{n}{2} - \left( \frac{k+1}{n} - 1 \right) \frac{1}{n} \]
\[ \frac{k + 1}{n} \left[ \frac{1}{n} \left( \ln \frac{n}{2} - 1 \right) + \frac{1}{k + 1} \right] > 0, \quad n \geq 6. \quad (3.28) \]

Line (3.28) shows that \( b(n, k) \) is an increasing function with respect to \( n \) whenever \( n \geq 6 \). Hence

\[
b(n, k) < b \left( \frac{k + 1}{3}, k \right) = -(3 - 1) \ln \frac{k + 1}{6} = -2 \ln \frac{k + 1}{6},
\]

which implies each term of \( M \) satisfies \( e^{b(n, k)} \leq e^{-2 \ln \frac{k + 1}{6}} = \frac{36}{(k + 1)^2} \) whenever \( n = 6, 7, 8, \ldots \). Therefore

\[
\sum_{k + 1 \geq n \geq 3} \frac{1}{(\frac{n}{2})^{k + 1 - 1}} \leq \frac{1}{(\frac{3}{2})^{k + 1 - 1}} + \frac{1}{(\frac{4}{2})^{k + 1 - 1}} + \frac{1}{(\frac{5}{2})^{k + 1 - 1}} + \sum_{k + 1 \geq n \geq 6} \frac{36}{(k + 1)^2}
\]

\[
\leq \frac{1}{(\frac{3}{2})^{k + 1 - 1}} + \frac{1}{(\frac{4}{2})^{k + 1 - 1}} + \frac{1}{(\frac{5}{2})^{k + 1 - 1}} + (k + 1) \frac{36}{(k + 1)^2}
\]

\[
\leq \frac{3}{(\frac{3}{2})^{k + 1 - 1}} + \frac{1}{(\frac{4}{2})^{k + 1 - 1}} + \frac{1}{(\frac{5}{2})^{k + 1 - 1}} + \frac{36}{k + 1}. \quad (3.29)
\]

Equation (3.29) shows that \( \lim_{k \to \infty} M = 0 \). By combining this result with Equation (3.27) we conclude that

\[
\lim_{k \to \infty} \left\{ \frac{\Delta}{(-2^{k+1} + 1)s^{k+1}} \right\} = \lim_{k \to \infty} \left\{ \frac{|\Delta|}{(1 + 2^{k-1})|(2s)^{k+1}|} \right\}
\]

\[
= \lim_{k \to \infty} \frac{\alpha(2s)^{k+1}}{|(1 + 2^{k-1})|(2s)^{k+1}} \left[ \frac{1}{(\frac{2}{\alpha})^{k+1}} + \frac{2\alpha}{(k + 1)} + M \right] = 0.
\]

We return to Equation (3.25) and observe that

\[
r_{k+1}h_{k+1} = \frac{T_1}{k + 1} - \frac{T_2}{k + 1} - \frac{\Delta}{k + 1}
\]

\[
= \frac{(-2^{k+1} + 1)s^{k+1}}{k + 1} - \frac{r_1(-1)^{k+1} \left( \frac{s}{r_1} \right)^{k+1}}{k + 1} - \frac{\Delta}{k + 1}
\]

\[
= \frac{(-2^{k+1} + 1)s^{k+1}}{k + 1} \left[ 1 - \frac{(-1)^{k+1}}{r_1^{k}(-2^{k+1} + 1)} - \frac{\Delta}{(-2^{k+1} + 1)s^{k+1}} \right]
\]

\[
= \frac{(-2^{k+1} + 1)s^{k+1}}{k + 1} \left[ 1 + o(1) \right] = \frac{d_k}{k + 1} \left[ 1 + o(1) \right]. \quad \square
\]
Remark 3.2. Define a composition of $a_n$ to be monomial of the form $a_{j_1}a_{j_2}\ldots a_{j_m}$ such that $j_1 + j_2 + \cdots + j_m = n$. By combining Equation (2.5) with Theorem 3.3 we deduce that the absolute value of Equation (3.11) provides an upper bound on the number and weight of compositions of $a_n$ whenever $\{|a_n|^{n^{-1}}\}_{n=1}^{\infty}$ is monotone increasing sequence and $s = \lim_{n\to\infty} |a_n|^{n^{-1}}$.

4. Combinatorial Interpretations for IGPPE

In this section we develop combinatorial interpretations for Equations (2.1) and (2.2). When developing our combinatorial interpretations, we require that $\{r_k\}_{k=1}^{\infty}$ be a set of positive integers. For a fixed set of positive integers $\{r_k\}_{k=1}^{\infty}$ denote the associated multi-set as $1^{r_1}2^{r_2}\ldots k^{r_k}\ldots$. If $r_k$ is the number of ways to color the digit $k$, we require that each copy of $k$ in the multi-set be uniquely colored. For example, $1^{2}2^{3}3^{4}4^{5}$ denotes the multi-set $\{1_R,1_B,2_R,2,B,2,O,2,Y,3_R,3_B,3,O,4_R,4_B,4,O,4,Y,4,G\}$, where $R$ means red, $B$ means blue, $O$ means orange, $Y$ means yellow, and $G$ means green. We form the generating function

$$\prod_{n=1}^{\infty} (1 + x^n)^{r_n} = (1 + x)^{r_1}(1 + x^2)^{r_2}\ldots (1 + x^k)^{r_k}\ldots = \sum_{n=0}^{\infty} \hat{p}_d(n)x^n, \quad (4.1)$$

where $\hat{p}_d(n)$ counts the partitions of $n$ with distinct parts composed from $1^{r_1}2^{r_2}\ldots k^{r_k}\ldots$. Recall that a partition of $n$ is a collection of positive integers whose sums equals $n$, i.e. $n = i_1 + i_2 + \cdots + i_k$ where $1 \leq k \leq n$ and each $i_k$ is a positive integer less than or equal to $n$. The summands in the partition are called parts and the order of summation is immaterial. In the context of colored multi-sets, two parts $i_{j_1}$ and $i_{j_2}$ are distinct if they either have different numerical value, or if they have the same numerical value, they are of different color. For the multi-set $1^{2}2^{4}3^{3}4^{5}$, both $1_R + 3_O$ and $2_R + 2_B$ are partitions of 4 with distinct parts. We generalize Equation (4.1) by introducing collection of weights associated with each part, namely $\{g_n\}_{n=1}^{\infty}$, where $g_n$ is an arbitrary complex number. Equation (4.1) becomes

$$\prod_{n=1}^{\infty} (1 + g_nx^n)^{r_n} = (1 + g_1x)^{r_1}(1 + g_2x)^{r_2}\ldots (1 + g_kx^k)^{r_k}\ldots = \sum_{n=0}^{\infty} \hat{p}_d(\bar{g},n)x^n, \quad (4.2)$$
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where \( \bar{g} \) is a finite polynomial in \( \{g_n\}_{n=1}^{\infty} \), such that each monomial has the form 
\[ g_{1}^{\alpha_1}g_{2}^{\alpha_2} \cdots g_{m}^{\alpha_m} \], where \( \sum_{i=1}^{m} i\alpha_i = n \) and \( \alpha_m \) denotes the number of distinct colored copies of the part \( m \) which appear in the partition. For example, the partition \( 1_R + 3_O \) is represented as \( g_1 g_3 \), while \( 2_R + 2_B \) is represented as \( g_2^2 \).

Equations (4.1) and (4.2) interpret the product side of Equation (2.1). Colored multi-sets also provide means of combinatorially interpreting the product side of Equation (2.2). If \( h_n = 1 \), Equation (2.2) becomes
\[
\prod_{n=1}^{\infty} (1 - x^n)^{-r_n} = (1 - x)^{-r_1}(1 - x^2)^{-r_2}(1 - x^3)^{-r_3} \cdots = \sum_{n=0}^{\infty} \hat{p}(n)x^n, \tag{4.3}
\]
where \( \hat{p}(n) \) is the number of partitions of \( n \) associated with the colored multi-set which contains an unlimited repetition of each integer \( k \) in \( r_k \) colors. The factor \( (1-x^k)^{-r_k} = (1+x^k+x^{2k}+x^{3k}+\ldots)^{r_k} \) corresponds to \( \{k, k+k, k+k+k, \ldots\} \) replicated in \( r_k \) colors.

Equation (4.3) generalizes by assigning a set of weights to each part. In particular, we have
\[
\prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n} = (1 - h_1 x)^{-r_1}(1 - h_2 x^2)^{-r_2}(1 - h_3 x^3)^{-r_3} \cdots = \sum_{n=0}^{\infty} \hat{p}(\bar{h}, n)x^n, \tag{4.4}
\]
where \( \bar{h} \) is a polynomial in \( \{h_n\}_{n=0}^{\infty} \) such that the exponent of \( h_i \) is the number of colored parts \( i \) that appear in a partition of \( n \). In other words, \( \bar{h} \) has the form \( h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_m^{\alpha_m} \), where \( \sum_{i=1}^{m} i\alpha_i = n \) and \( \alpha_m \) denotes the number of colored parts \( m \) which appear in the partition [9].

The combinatorial interpretations of Equations (4.1) through (4.4) originated from the product side of Equations (2.1) and (2.2). Is there way to develop a combinatorial interpretation if we start with the sum side instead? To answer this question define \( f(x) = 1 - \sum_{n=1}^{\infty} a_n x^n \) where \( \{a_n\}_{n=1}^{\infty} \) is a set of positive integers. Equation (2.25), when combined with Equation (2.1), implies that
\[
1 - \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 - g_n x^n)^{r_n}, \quad g_n \text{ a positive integer.} \tag{4.5}
\]

Take Equation (4.5) and form the reciprocal.
\[
\frac{1}{1 - \sum_{n=1}^{\infty} a_n x^n} = \frac{1}{\prod_{n=1}^{\infty} (1 - g_n x^n)^{r_n}} = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}, \quad h_n := g_n \tag{4.6}
\]
Equation (4.6) shows that the reciprocal of \(1 - \sum_{n=1}^{\infty} a_n x^n\) is an IGPPE. Since \(r_n\) is a positive integer we may expand 
\[
\left[1 + \sum_{\mu=1}^{\infty} (g_n x^n)^\mu\right]^{r_n} = \left[1 + \sum_{\mu=1}^{\infty} (g_n x^n)^\mu\right]^{r_n} = 1 + \sum_{n=1}^{\infty} \frac{1}{1 - g_n x^n} = 1 + \sum_{n=1}^{\infty} C_n x^n,
\]
where \(C_n\) is a positive integer determined by a finite number of arithmetic calculations involving a finite number of elements in \(\{g_n\}_{n=1}^{\infty}\). We claim each \(C_n\) has a combinatorial interpretation. Expand the left side of Equation (4.7) as

\[
1 - \sum_{n=1}^{\infty} a_n x^n = 1 - \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} \left[1 + \sum_{\mu=1}^{\infty} (g_n x^n)^\mu\right]^{r_n} = 1 + \sum_{n=1}^{\infty} C_n x^n,
\]

\((4.7)\)

The methodology used to describe \(C_n\) is best exhibited by the particular case of \(a_n = 1\). Equation (4.8) becomes

\[
1 - \sum_{n=1}^{\infty} x^n = \frac{1}{1 - \frac{x}{1-x}} = 1 + \frac{x}{1-x} + \left(\frac{x}{1-x}\right)^2 + \left(\frac{x}{1-x}\right)^3 + \cdots + \left(\frac{x}{1-x}\right)^k + \cdots
\]

\((4.9)\)

= 1 + \sum_{n=1}^{\infty} x^n + \left[\sum_{n=1}^{\infty} x^n\right]^2 + \left[\sum_{n=1}^{\infty} x^n\right]^3 + \cdots + \left[\sum_{n=1}^{\infty} x^n\right]^k + \cdots

\((4.10)\)

We must determine a series expansion for \(\left[\sum_{n=1}^{\infty} x^n\right]^k = \sum_{l=1}^{\infty} \hat{C}(l,k)x^l\) whenever \(k \geq 1\). Clearly \(\hat{C}(l,1) = 1\). A simple exercise in coefficient comparison shows that

\[
\left[\sum_{n=1}^{\infty} x^n\right]^2 = (x + x^2 + x^3 + x^4 + x^5 + \ldots)(x + x^2 + x^3 + x^4 + x^5 + \ldots)
\]

\[
= \sum_{l=2}^{\infty} (l - 1)x^l = \sum_{l=2}^{\infty} \hat{C}(l,2)x^l.
\]
We use this result to determine \( \hat{C}(l, 3) \) as follows

\[
\left[ \sum_{n=1}^{\infty} x^n \right]^3 = \left[ \sum_{n=1}^{\infty} x^n \right]^2 \left[ \sum_{n=1}^{\infty} x^n \right] = (x^2 + 2x^3 + \cdots + (n-1)x^n + \cdots)(x + x^2 + x^3 + \cdots + x^k + \cdots)
\]

\[
= (1)x^3 + (1 + 2)x^4 + (1 + 2 + 3)x^5 + \cdots (1 + 2 + 3 + \cdots + (k-2)) x^k + \cdots
\]

\[
= \sum_{l=3}^{\infty} \binom{l-1}{2} \hat{C}(l, 3) x^l
\]

Since \( \hat{C}(l, 1) = 1 = \binom{l-1}{0} \), \( \hat{C}(l, 2) = \binom{l-1}{1} \), and \( \hat{C}(l, 3) = \binom{l-1}{2} \), we deduce that \( \hat{C}(l, k) = \binom{l-1}{k-1} \). This claim is proven via induction on \( k \). Assume that \( \hat{C}(l, j) = \binom{l-1}{j-1} \) for all positive integers \( j \) with \( j \leq k \). Then

\[
\left[ \sum_{n=1}^{\infty} x^n \right]^{k+1} = \left[ \sum_{n=1}^{\infty} x^n \right]^k \left[ \sum_{n=1}^{\infty} x^n \right] = \sum_{l=k}^{\infty} \hat{C}(l, k) x^l \left[ x + x^2 + \cdots + x^k + \cdots \right]
\]

\[
= \sum_{s=1}^{\infty} \sum_{j=0}^{s-1} \hat{C}(k+j, k) x^{k+s} = \sum_{s=1}^{\infty} \sum_{j=0}^{s-1} \binom{k+j-1}{k-1} x^{k+s}
\]

\[
= \sum_{s=k+1}^{\infty} \binom{s-1}{k} x^{s} = \sum_{s=k+1}^{\infty} \hat{C}(s, k+1) x^s.
\]

We return to Equation (4.10) and write it as

\[
1 - \sum_{n=1}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n + \left[ \sum_{n=1}^{\infty} x^n \right]^2 + \left[ \sum_{n=1}^{\infty} x^n \right]^3 + \cdots + \left[ \sum_{n=1}^{\infty} x^n \right]^k + \cdots
\]

\[
= 1 + \sum_{n=1}^{\infty} \hat{C}(n, 1) x^n + \sum_{n=2}^{\infty} \hat{C}(n, 2) x^n + \sum_{n=3}^{\infty} \hat{C}(n, 3) x^n + \cdots + \sum_{n=k}^{\infty} \hat{C}(n, k) + \cdots
\]

\[
= 1 + \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{n} \hat{C}(n, k) \right] x^n = 1 + \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{n} \binom{n-1}{k-1} \right] x^n = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n.
\]
From these calculations we conclude that \( \frac{1}{1-\sum_{n=1}^{\infty} x^n} = 1 + \sum_{n=1}^{\infty} \hat{C}_n x^n \) where \( \hat{C}_n = 2^{n-1} \). This result could have been obtained by observing that
\[
\frac{1}{1 - \frac{x}{1-x}} = 1 - x = 1 + \frac{x}{1-2x} = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n.
\]

We now turn to the general case and develop a factorization for the ordinary power series whose coefficients are representations of compositions of \( n \). Notice that
\[
\frac{1}{1 - \sum_{n=1}^{\infty} a_n x^n} = 1 + \sum_{n=1}^{\infty} a_n x^n + \left[ \sum_{n=1}^{\infty} a_n x^n \right]^2 + \left[ \sum_{n=1}^{\infty} a_n x^n \right]^3 + \cdots
\]
\[
= 1 + \sum_{n=1}^{\infty} C(n, 1) x^n + \sum_{n=2}^{\infty} C(n, 2) x^n + \sum_{n=3}^{\infty} C(n, 3) x^n + \cdots
\]
\[
= 1 + \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{n} C(n, k) \right] x^n
\]
\[
= 1 + \sum_{n=1}^{\infty} C_n x^n,
\]
where \( C(n, k) \) is a polynomial representation of the compositions of \( n \) with exactly \( k \) parts such that the part \( i \) is represented by \( a_i \) and the + is replace by \( \ast \). In other words, \( C(n, k) \) is composed of monomials \( ca_{i_1}a_{i_2}\ldots a_{i_k} \) such that \( i_1 + i_2 + \ldots i_k \) is a partition of \( n \). Recall that a composition of a positive integer \( n \) with \( k \) parts is a sum \( i_1 + i_2 + \ldots i_k = n \) where each part \( i_j \) is a positive integer with \( 1 \leq i_j \leq n \). The difference between a partition of \( n \) with \( k \) parts and a composition of \( n \) with \( k \) parts is that a composition distinguishes between the order of the parts in the summation. For example, \( 2 + 1 + 1 \) and \( 1 + 2 + 1 \) are two distinct compositions of \( 4 \) with \( 3 \) parts but only one partition of \( 4 \) with \( 3 \) parts [20]. To construct \( C(n, k) \) we list all the compositions of \( n \) with \( k \) parts, replace each part \( i \) with \( a_i \), multiply the terms together, and add the resulting monomials. For example, \( C(n, 1) = a_n \) and \( C(5, 2) = 2a_1a_4 + 2a_2a_3 \) since \( \{4 + 1, 1 + 4, 2 + 3, 3 + 2\} \) are the four compositions of \( 5 \) with \( 2 \) parts.
We justify our combinatorial interpretation of $C(n, k)$ through induction on $k$. Obviously $C(n, 1) = a_n$ satisfies our definition. Now assume for $j \leq k$ that $C(n, j)$ is the polynomial representation of the compositions of $s$ with $k+1$ parts such that each part $i$ is represented by $a_i$ and all + are replaced with *. Then

$$\left[ \sum_{n=1}^{\infty} a_n x^n \right]^{k+1} = \left[ \sum_{n=1}^{\infty} a_n x^n \right]^k \left[ \sum_{n=1}^{\infty} a_n x^n \right]$$

$$= \sum_{l=k}^{\infty} C(l, k) x^l \sum_{l=1}^{\infty} a_l x^l$$

$$= \sum_{s=1}^{\infty} \left[ \sum_{j=0}^{s-1} C(k+j, k) a_{s-j} \right] x^{k+s}$$

$$= \sum_{s=k+1}^{\infty} \left[ \sum_{j=0}^{s-k-1} C(k+j, k) a_{s-k-j} \right] x^s$$

$$:= \sum_{s=k+1}^{\infty} C(s, k+1) x^s.$$

We claim that the polynomial representation of the compositions of $s$ with $k+1$ parts such that each part $i$ is represented by $a_i$ and all + are replaced with * is precisely $\sum_{j=0}^{s-k-1} C(k+j, k) a_{s-k-j}$. Start with a composition of $s$ which has $k+1$ parts. Call this composition $p$ and write $p = p_1 + p_2 = s$, where $p_1$ is the sum of the first $k$ parts and $p_2$ is the $(k+1)^{st}$ part. In other words, $p_1 = i_1 + i_2 + \ldots + i_k$ and $p_2 = i_{k+1}$. Notice that $1 \leq i_{k+1} \leq s-k$ since $i_j \geq 1$ whenever $1 \leq j \leq k$. If $i_{k+1} \geq s-k+1$ we would have the contradiction $p_1 + p_2 \geq k + s - k + 1 = s + 1$. For each choice of $i_{k+1} \in \{1, 2, \ldots, s-k\}$, $p_1$ is a composition $s - i_{k+1}$ with $k$ parts. If $C(s, k+1)$ is the number of compositions of $s$ with $k+1$ parts, the preceding argument shows that $c(s, k+1) = \sum_{j=1}^{s-k} c(s-j, k) j = \sum_{j=0}^{s-k-1} c(s-j-1, k)(j+1) = \sum_{j=0}^{s-k-1} c(k+j, k)(s-k-j)$. It is just a matter of taking each composition represented by $c(k+j, k)(s-k-j)$, replacing the part $i$ with $a_i$, and + with *. The end result is that $\sum_{j=0}^{s-k-1} C(k+j, k) a_{s-k-j} = C(s, k+1)$ is indeed the desired polynomial representation for the compositions of $s$ with $k$ parts.

Since

$$\frac{1}{1 - \sum_{n=1}^{\infty} a_n x^n} = 1 + \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{n} C(n, k) \right] x^n = 1 + \sum_{n=1}^{\infty} C_n x^n, \quad (4.11)$$
we may interpret $C_n$ to be the sum of all non-trivial polynomial representations of the compositions of $n$ with $k$ parts, i.e. $C_n$ is a polynomial representation of the compositions of $n$. Notice that $C_n$ is constructed by taking the set of compositions of $n$, replacing $i$ with $a_i$, replacing $+$ with $\ast$, and summing the monomials. For example, since the compositions of 4 are \{1 + 1 + 1 + 1, 3 + 1 + 1 + 1, 1 + 2 + 1 + 1, 1 + 1 + 2, 4\}, $C_4 = a_4 + 3a_2a_1^2 + a_2^2 + 2a_3a_1 + a_1^4$. If $a_n = 1$, $C(n, k) = \hat{C}(n, k)$ is the number of compositions of $n$ with $k$ parts while $C_n = \hat{C}_n$ is the total number of compositions of $n$. We have shown that $\hat{C}(n, k) = \binom{n-1}{k-1}$ and $\hat{C}_n = 2^{n-1}$. Therefore, our calculations provide yet another proof that the number of compositions of $n$ is $2^{n-1}$, and the number of compositions of $n$ with $k$ parts is $\binom{n-1}{k-1}$. By combining our observations with Equations (4.6) and (4.7), we see that IGPPE $\prod_{n=1}^{\infty} (1 - h_nx^n)^{-r_n}$ provides a way of “factoring” the series $1 + \sum_{n=1}^{\infty} C_n x^n$, where $C_n$ is the polynomial representation of the compositions of $n$.

We should mention a similar result holds for the GIPPE associated with $1 - \sum_{n=1}^{\infty} a_n x^n$ where $a_n$ is a positive integer. Theorem 3.2 implies that

$$1 - \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 + h_n x^n)^{-r_n}, \quad h_n \text{ a positive integer.} \quad (4.12)$$

We then invert Equation (4.12) to obtain

$$\frac{1}{1 - \sum_{n=1}^{\infty} a_n x^n} = \prod_{n=1}^{\infty} (1 + h_n x^n)^{r_n} = 1 + \sum_{n=1}^{\infty} C_n x^n, \quad (4.13)$$

where $C_n$ is the aforementioned polynomial representation for the compositions of $n$.

**References**


