ALMOST CONTRA REGULAR GENERALIZED
\textit{b}-CONTINUOUS FUNCTIONS

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Abstract: In this paper, the authors introduce a new class of functions called almost contra regular generalized \textit{b}-continuous function (briefly almost contra \textit{rgb}-continuous) in topological spaces. Some characterizations and several properties concerning almost contra \textit{rgb}-continuous functions are obtained.

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1. Introduction

In 2002, Jafari and Noiri introduced and studied a new form of functions called contra-precontinuous functions. The purpose of this paper is to introduce and study almost contra \textit{rgb}-continuous functions via the concept of \textit{rgb}-closed sets. Also, properties of almost contra \textit{rgb}-continuity are discussed. Moreover, we obtain basic properties and preservation theorems of almost contra \textit{rgb}-continuous functions and relationships between almost contra \textit{rgb}-continuity and \textit{rgb}-regular graphs.

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Throughout this paper \((X, \tau)\) and \((Y, \sigma)\) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let \(A \subseteq X\), the closure of \(A\) and interior of \(A\) will be denoted by \(cl(A)\) and \(int(A)\) respectively, union of all \(rgb\)-open sets \(X\) contained in \(A\) is called \(rgb\)-interior of \(A\) and it is denoted by \(rgb – int(A)\), the intersection of all \(rgb\)-closed sets of \(X\) containing \(A\) is called \(rgb\)-closure of \(A\) and it is denoted by \(rgb – cl(A)\).

2. Preliminaries

**Definition 1.** Let \(A\) subset \(A\) of a topological space \((X, \tau)\), is called

1. a **pre-open set** [24] if \(A \subseteq int(cl(A))\).
2. a **semi-open set**[18] if \(A \subseteq cl(int(A))\).
3. a **\(b\)-open set** [5] if \(A \subseteq cl(int(A)) \cup int(cl(A))\).
4. a **generalized \(b\)- closed set** (briefly \(gb\)- closed) [2] if \(bcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
5. a **generalized \(\alpha b\)- closed set** (briefly \(g\alpha b\)- closed) [33] if \(bcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\)-open in \(X\).
6. a **regular generalized \(b\)- closed set** (briefly \(rgb\)-closed set)[21] if \(bcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).

**Definition 2.** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is called

1. almost contra continuous[31] if \(f^{-1}(V)\) is closed in \((X, \tau)\) for every regular-open set \(V\) of \((Y, \sigma)\).
2. almost contra \(b\)-continuous [3] if \(f^{-1}(V)\) is \(b\)-closed in \((X, \tau)\) for every regular open set \(V\) of \((Y, \sigma)\).
3. almost contra pre-continuous [15] if \(f^{-1}(V)\) is pre-closed in \((X, \tau)\) for every regular open set \(V\) of \((Y, \sigma)\).
4. a contra semi-continuous [14] if \(f^{-1}(V)\) is semi-closed in \((X, \tau)\) for every regular open set \(V\) of \((Y, \sigma)\).
5. a contra \(gb\)-continuous [2] if \(f^{-1}(V)\) is \(gb\)-closed in \((X, \tau)\) for every regular open set \(V\) of \((Y, \sigma)\).
3. Almost Contra Regular Generalized $b$-ConTinuOus Functions

In this section, we introduce Almost contra regular generalized $b$-continuous functions and investigate some of their properties.

**Definition 3.** A function $f : (X, \tau) \to (Y, \sigma)$ is called almost contra regular generalized $b$-continuous if $f^{-1}(V)$ is rgb-closed in $(X, \tau)$ for every regular open set $V$ in $(Y, \sigma)$.

**Example 4.** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Clearly $f$ is almost contra rgb-continuous.

**Theorem 5.** If $f : X \to Y$ is contra rgb-continuous then it is almost contra rgb-continuous.

**Proof.** Obvious, because every regular open set is open set. □

**Remark 6.** Converse of the above theorem need be true in general as seen from the following example.

**Example 7.** Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varnothing, \{b\}, \{a, b\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then $f$ is almost contra rgb-continuous function but not contra rgb-continuous, because for the open set $\{b\}$ in $Y$ and $f^{-1}(b) = \{a\}$ is not rgb-closed in $X$.

**Theorem 8.** (i) Every almost contra pre-continuous function is almost contra rgb-continuous function.

(ii) Every almost contra semi-continuous function is almost contra rgb-continuous function.

(iii) Every almost contra rgb-continuous function is almost contra $gb$-continuous function.

(iv) Every almost contra rgb-continuous function is almost contra $gab$-continuous function.
(v) Every almost contra $b$-continuous function is almost contra $rgb$-continuous function.

**Remark 9.** Converse of the above statements is not true as shown in the following example.

**Example 10.** (i) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varnothing, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \varnothing, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly $f$ is almost contra $rgb$-continuous but $f$ is not contra pre-continuous. Because $f^{-1}(\{c\}) = \{b\}$ is not pre-closed in $(X, \tau)$ where $\{c\}$ is regular-open in $(Y, \sigma)$.

(ii) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varnothing, \{a\}\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Clearly $f$ is almost contra $rgb$-continuous but $f$ is not almost contra semi-continuous. Because $f^{-1}(\{c\}) = \{a\}$ is not semi-closed in $(X, \tau)$ where $\{c\}$ is regular-open in $(Y, \sigma)$.

(iii) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varnothing, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Clearly $f$ is almost contra $gb$-continuous but $f$ is not contra $rgb$-continuous. Because $f^{-1}(\{b\}) = \{b\}$ is not $rgb$-closed in $(X, \tau)$ where $\{b\}$ is regular-open in $(Y, \sigma)$.

(iv) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Clearly $f$ is almost contra $gb$-continuous but $f$ is not contra $rgb$-continuous. Because $f^{-1}(\{b\}) = \{b\}$ is not $rgb$-closed in $(X, \tau)$ where $\{b\}$ is regular-open in $(Y, \sigma)$.

(v) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varnothing, \{b\}, \{b, c\}\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Clearly $f$ is almost contra $rgb$-continuous but $f$ is not contra $b$-continuous. Because $f^{-1}(\{a\}) = \{b\}$ is not $b$-closed in $(X, \tau)$ where $\{a\}$ is regular-open in $(Y, \sigma)$.

**Theorem 11.** The following are equivalent for a function $f : X \rightarrow Y$,

1. $f$ is almost contra $rgb$-continuous.
2. for every regular closed set $F$ of $Y$, $f^{-1}(F)$ is $rgb$-open set of $X$. 
(3) for each $x \in X$ and each regular closed set $F$ of $Y$ containing $f(x)$, there exists rgb-open $U$ containing $x$ such that $f(U) \subseteq F$.

(4) for each $x \in X$ and each regular open set $V$ of $Y$ not containing $f(x)$, there exists rgb-closed set $K$ not containing $x$ such that $f^{-1}(V) \subseteq K$.

Proof. (1)⇒(2): Let $F$ be a regular closed set in $Y$, then $YF$ is a regular open set in $Y$. By (1), $f^{-1}(Y - F) = X - f^{-1}(F)$ is rgb-closed set in $X$. This implies $f^{-1}(F)$ is rgb-open set in $X$. Therefore, (2) holds.

(2)⇒(1): Let $G$ be a regular open set of $Y$. Then $YG$ is a regular closed set in $Y$. By (2), $f^{-1}(Y - G)$ is rgb-open set in $X$. This implies $X - f^{-1}(G)$ is rgb-open set in $X$, which implies $f^{-1}(G)$ is rgb-closed set in $X$. Therefore, (1) hold.

(2)⇒(3): Let $F$ be a regular closed set in $Y$ containing $f(x)$, which implies $x \in f^{-1}(F)$. By (2), $f^{-1}(F)$ is rgb-open in $X$ containing $x$. Set $U = f^{-1}(F)$, which implies $U$ is rgb-open in $X$ containing $x$ and $f(U) = f(f^{-1}(F)) \subseteq F$. Therefore (3) holds.

(3)⇒(2): Let $F$ be a regular closed set in $Y$ containing $f(x)$, which implies $x \in f^{-1}(F)$. From (3), there exists rgb-open $Ux$ in $X$ containing $x$ such that $f(Ux) \subseteq F$. That is $Ux \subseteq f^{-1}(F)$. Thus $f^{-1}(F) = \cup\{Ux : x \in f^{-1}(F)\}$, which is union of rgb-open sets. Therefore, $f^{-1}(F)$ is rgb-open set of $X$.

(3)⇒(4): Let $V$ be a regular open set in $Y$ not containing $f(x)$. Then $YV$ is a regular closed set in $Y$ containing $f(x)$. From (3), there exists a rgb-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq YV$. This implies $U \subseteq f^{-1}(Y - V) = Xf^{-1}(V)$. Hence, $f^{-1}(V) \subseteq XU$. Set $K = XU$, then $K$ is rgb-closed set not containing $x$ in $X$ such that $f^{-1}(V) \subseteq K$.

(4)⇒(3): Let $F$ be a regular closed set in $Y$ containing $f(x)$. Then $YF$ is a regular open set in $Y$ not containing $f(x)$. From (4), there exists rgb-closed set $K$ in $X$ not containing $x$ such that $f^{-1}(Y - F) \subseteq K$. This implies $X - f^{-1}(F) \subseteq K$. Hence, $X - K \subseteq f^{-1}(F)$, that is $f(X - K) \subseteq F$. Set $U = XK$, then $U$ is rgb-open set containing $x$ in $X$ such that $f(U) \subseteq F$. 

\[\square\]

**Theorem 12.** The following are equivalent for a function $f : X \to Y$,

(1) $f$ is almost contra rgb-continuous.

(2) $f^{-1}(\text{Int}(\text{Cl}(G)))$ is rgb-closed set in $X$ for every open subset $G$ of $Y$.

(3) $f^{-1}(\text{Cl}(\text{Int}(F)))$ is rgb-open set in $X$ for every closed subset $F$ of $Y$.  

Proof. (1)⇒(2): Let \( G \) be an open set in \( Y \). Then \( \text{Int}(\text{Cl}(G)) \) is regular open set in \( Y \). By (1), \( f^{-1}(\text{Int}(\text{Cl}(G))) \in \text{rgb} - C(X) \).

(2)⇒(1): Proof is obvious.

(1)⇒(3): Let \( F \) be a closed set in \( Y \). Then \( \text{Cl}(\text{Int}(G)) \) is regular closed set in \( Y \). By (1), \( f^{-1}(\text{Cl}(\text{Int}(G))) \in \text{rgb} - \text{O}(X) \).

(3)⇒(1): Proof is obvious.

Definition 13. A function \( f : X \to Y \) is said to be R-map if \( f^{-1}(V) \) is regular open in \( X \) for each regular open set \( V \) of \( Y \).

Definition 14. A function \( f : X \to Y \) is said to be perfectly continuous if \( f^{-1}(V) \) is clopen in \( X \) for each open set \( V \) of \( Y \).

Theorem 15. For two functions \( f : X \to Y \) and \( g : Y \to Z \), let \( g \circ f : X \to Z \) is a composition function. Then, the following properties hold.

(1) If \( f \) is almost contra rgb-continuous and \( g \) is an R-map, then \( g \circ f \) is almost contra rgb-continuous.

(2) If \( f \) is almost contra rgb-continuous and \( g \) is perfectly continuous, then \( g \circ f \) is contra rgb-continuous.

(3) If \( f \) is contra rgb-continuous and \( g \) is almost continuous, then \( g \circ f \) is almost contra rgb-continuous.

Proof. (1) Let \( V \) be any regular open set in \( Z \). Since \( g \) is an R-map, \( g^{-1}(V) \) is regular open in \( Y \). Since \( f \) is almost contra rgb-continuous, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is rgb-closed set in \( X \). Therefore \( g \circ f \) is almost contra rgb-continuous.

(2) Let \( V \) be any regular open set in \( Z \). Since \( g \) is perfectly continuous, \( g^{-1}(V) \) is clopen in \( Y \). Since \( f \) is almost contra rgb-continuous, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is rgb-open and rgb-closed set in \( X \). Therefore \( g \circ f \) is continuous and contra rgb-continuous.

(3) Let \( V \) be any regular open set in \( Z \). Since \( g \) is almost continuous, \( g^{-1}(V) \) is open in \( Y \). Since \( f \) is almost contra rgb-continuous, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is rgb-closed set in \( X \). Therefore \( g \circ f \) is almost contra rgb-continuous.
**Theorem 16.** Let $f : X \to Y$ is a contra $rgb$-continuous and $g : Y \to Z$ is $rgb$-continuous. If $Y$ is $Trgb$-space, then $g \circ f : X \to Z$ is an almost contra $rgb$-continuous.

**Proof.** Let $V$ be any regular open and hence open set in $Z$. Since $g$ is $rgb$-continuous $g^{-1}(V)$ is $rgb$-open in $Y$ and $Y$ is $Trgb$-space implies $g^{-1}(V)$ open in $Y$. Since $f$ is contra $rgb$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $rgb$-closed set in $X$. Therefore, $g \circ f$ is an almost contra $rgb$-continuous.  

**Theorem 17.** If $f : X \to Y$ is surjective strongly $rgb$-open (or strongly $rgb$-closed) and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is an almost contra $rgb$-continuous, then $g$ is an almost contra $rgb$-continuous.

**Proof.** Let $V$ be any regular closed (resp. regular open) set in $Z$. Since $g \circ f$ is an almost contra $rgb$-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $rgb$-open (resp. $rgb$-closed) in $X$. Since $f$ is surjective and strongly $rgb$-open (or strongly $rgb$-closed), $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is $rgb$-open (or $rgb$-closed). Therefore $g$ is an almost contra $rgb$-continuous.

**Definition 18.** A function $f : X \to Y$ is called weakly $rgb$-continuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in rgb - O(X;x)$ such that $f(U) \subset cl(V)$.

**Theorem 19.** If a function $f : X \to Y$ is an almost contra $rgb$-continuous, then $f$ is weakly $rgb$-continuous function.

**Proof.** Let $x \in X$ and $V$ be an open set in $Y$ containing $f(x)$. Then $cl(V)$ is regular closed in $Y$ containing $f(x)$. Since $f$ is an almost contra $rgb$-continuous function by Theorem 3.10 (2), $f^{-1}(cl(V))$ is $rgb$-open set in $X$ containing $x$. Set $U = f^{-1}(cl(V))$, then $f(U) \subset f(f^{-1}(Cl(V))) \subset cl(V)$. This shows that $f$ is almost weakly $rgb$-continuous function.

**Definition 20.** A space $X$ is called locally $rgb$-indiscrete if every $rgb$-open set is closed in $X$.

**Theorem 21.** If a function $f : X \to Y$ is almost contra $rgb$-continuous and $X$ is locally $rgb$-indiscrete space, then $f$ is almost continuous.
Proof. Let $U$ be a regular open set in $Y$. Since $f$ is almost contra $rgb$-continuous $f^{-1}(U)$ is $rgb$-closed set in $X$ and $X$ is locally $rgb$-indiscrete space, which implies $f^{-1}(U)$ is an open set in $X$. Therefore $f$ is almost continuous.

Lemma 22. Let $A$ and $X_0$ be subsets of a space $X$. If $A \in rgb - O(X)$ and $X_0 \in \tau^\alpha$, then $A \cap X_0 \in rgb - O(X_0)$.

Theorem 23. If $f : X \rightarrow Y$ is almost contra $rgb$-continuous and $X_0 \in \tau^\alpha$ then the restriction $f/X_0 : X_0 \rightarrow Y$ is almost contra $rgb$-continuous.

Proof. Let $V$ be any regular open set of $Y$. By theorem, we have $f^{-1}(V) \in rgb - O(X)$ and hence $(f/X_0)^{-1}(V) = f^{-1}(V) \cap X_0 \in rgb - O(X_0)$. By lemma 3.20, it follows that $f/X_0$ is almost contra $rgb$-continuous.

Theorem 24. If $f : X \rightarrow \prod Y_\lambda$ is almost contra $rgb$-continuous, then $P_\lambda \circ f : X \rightarrow Y_\lambda$ is almost contra $rgb$-continuous for each $\lambda \in \nabla$, where $P_\lambda$ is the projection of $\prod Y_\lambda$ onto $Y_\lambda$.

Proof. Let $Y_\lambda$ be any regular open set of $Y$. Since $P_\lambda$ is continuous open, it is an $R$-map and hence $(P_\lambda)^{-1} \in RO(\prod Y_\lambda)$. By theorem, $f^{-1}(P_\lambda^{-1}(V)) = (P_\lambda \circ f)^{-1} \in rgbO(X)$. Hence $P_\lambda \circ f$ is almost contra $rgb$-continuous.

4. $rgb$-Regular Graphs and Strongly Contra $rgb$-Closed Graphs

Definition 25. A graph $G_f$ of a function $f : X \rightarrow Y$ is said to be $rgb$-regular (strongly contra $rgb$-closed) if for each $(x, y) \in (X \times Y)\backslash G_f$, there exist a $rgb$-closed set $U$ in $X$ containing $x$ and $V \in R - O(Y)$ such that $(U \times V) \cap G_f = \varphi$.

Theorem 26. If $f : X \rightarrow Y$ is almost contra $rgb$-continuous and $Y$ is $T_2$, then $G_f$ is $rgb$-regular in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y)\backslash G_f$. It is obvious that $f(x) \neq y$. Since $Y$ is $T_2$, there exists $V, W \in RO(Y)$ such that $f(x)) \in V$, $Y \in W$ and $V \cap W = \varphi$. Since $f$ is almost contra $rgb$-continuous, $f^{-1}(V)$ is a $rgb$-closed set in $X$ containing $x$. If we take $U = f^{-1}(V)$, we have $f(U) \subset V$. Hence, $f(U) \cap W = \varphi$ and $G_f$ is $rgb$-regular.
Theorem 27. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then $f$ is almost rgb-continuous if and only if $g$ is almost rgb-continuous.

Proof. Necessary: Let $x \in X$ and $V \in \text{rgb} - O(Y)$ containing $f(x)$. Then, we have $g(x) = (x, f(x)) \in R - O(X \times Y)$. Since $f$ is almost rgb-continuous, there exists a rgb-open set $U$ of $X$ containing $x$ such that $g(U) \subset X \times Y$. Therefore, we obtain $f(U) \subset V$. Hence $f$ is almost rgb-continuous.

Sufficiency: Let $x \in X$ and $w$ be a regular open set of $X \times Y$ containing $g(x)$. There exists $U_1 \in \text{RO}(X, \tau)$ and $V \in \text{RO}(Y, \sigma)$ such that $(x, f(x)) \in (U_1 \times V) \subset W$. Since $f$ is almost rgb-continuous, there exists $U_2 \in \text{rgb} - O(X, \tau)$ such that $x \in U_2$ and $f(U_2) \subset V$. Set $U = U_1 \cap U_2$. We have $x \in Ux \in \text{rgb} - O(X, \tau)$ and $g(U) \subset (U_1 \times V) \subset W$. This shows that $g$ is almost rgb-continuous. 

Theorem 28. If a function $f : X \to Y$ be a almost rgb-continuous and almost continuous, then $f$ is regular set-connected.

Proof. Let $V \in \text{RO}(Y)$. Since $f$ is almost contra rgb-continuous and almost continuous, $f^{-1}(V)$ is rgb-closed and open. So $f^{-1}(V)$ is clopen. It turns out that $f$ is regular set-connected.

5. Connectedness

Definition 29. A Space $X$ is called rgb-connected if $X$ cannot be written as a disjoint union of two non-empty rgb-open sets.

Theorem 30. If $f : X \to Y$ is an almost contra rgb-continuous surjection and $X$ is rgb-connected, then $Y$ is connected.

Proof. Suppose that $Y$ is not a connected space. Then $Y$ can be written as $Y = U_0 \cup V_0$ such that $U_0$ and $V_0$ are disjoint non-empty open sets. Let $U = \text{int}(\text{cl}(U_0))$ and $V = \text{int}(\text{cl}(V_0))$. Then $U$ and $V$ are disjoint nonempty regular open sets such that $Y = U \cup V$. Since $f$ is almost contra rgb-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are rgb-open sets of $X$. We have $X = f^{-1}(U) \cup f^{-1}(V)$ such that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Since $f$ is surjective, this shows that $X$ is not rgb-connected. Hence $Y$ is connected.
**Theorem 31.** The almost contra rgb-continuous image of rgb-connected space is connected.

**Proof.** Let \( f : X \to Y \) be an almost contra rgb-continuous function of a rgb-connected space \( X \) onto a topological space \( Y \). Suppose that \( Y \) is not a connected space. There exist non-empty disjoint open sets \( V_1 \) and \( V_2 \) such that \( Y = V_1 \cup V_2 \). Therefore, \( V_1 \) and \( V_2 \) are clopen in \( Y \). Since \( f \) is almost contra rgb-continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are rgb-open in \( X \). Moreover, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are non-empty disjoint and \( X = f^{-1}(V_1) \cup f^{-1}(V_2) \). This shows that \( X \) is not rgb-connected. This is a contradiction and hence \( Y \) is connected. ∎

**Definition 32.** A topological space \( X \) is said to be rgb-ultra connected if every two non-empty rgb-closed subsets of \( X \) intersect.

We recall that a topological space \( X \) is said to be hyper connected if every open set is dense.

**Theorem 33.** If \( X \) is rgb-ultra connected and \( f : X \to Y \) is an almost contra rgb-continuous surjection, then \( Y \) is hyper connected.

**Proof.** Suppose that \( Y \) is not hyperconnected. Then, there exists an open set \( V \) such that \( V \) is not dense in \( Y \). So, there exist non-empty regular open subsets \( B_1 = \text{int}(\text{cl}(V)) \) and \( B_2 = Y - \text{cl}(V) \) in \( Y \). Since \( f \) is almost contra rgb-continuous, \( f^{-1}(B_1) \) and \( f^{-1}(B_2) \) are disjoint rgb-closed. This is contrary to the rgb-ultra-connectedness of \( X \). Therefore, \( Y \) is hyperconnected. ∎

6. Separation Axioms

**Definition 34.** A topological space \( X \) is said to be rgb – \( T_1 \) space if for any pair of distinct points \( x \) and \( y \), there exist a rgb-open sets \( G \) and \( H \) such that \( x \in G, \ y \notin G \) and \( x \notin H, \ y \notin H \).

**Theorem 35.** If \( f : X \to Y \) is an almost contra rgb-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is rgb – \( T_1 \).

**Proof.** Suppose \( Y \) is weakly Hausdorff. For any distinct points \( x \) and \( y \) in \( X \), there exist \( V \) and \( W \) regular closed sets in \( Y \) such that \( f(x) \in V, \ f(y) \notin V \)
, \( f(y) \in W \) and \( f(x) \notin W \). Since \( f \) is almost contra rgb-continuous, \( f^{-1}(V) \) and \( f^{-1}(W) \) are rgb-open subsets of \( X \) such that \( x \in f^{-1}(V) \), \( y \notin f^{-1}(V) \), \( y \in f^{-1}(W) \) and \( x \notin f^{-1}(W) \). This shows that \( X \) is rgb\( -T_1 \).

\[ \square \]

**Corollary 36.** If \( f : X \to Y \) is a contra rgb-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is rgb\( -T_1 \).

**Definition 37.** A topological space \( X \) is called Ultra Hausdorff space, if for every pair of distinct points \( x \) and \( y \) in \( X \), there exist disjoint clopen sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \), respectively.

**Definition 38.** A topological space \( X \) is said to be rgb\( -T_2 \) space if for any pair of distinct points \( x \) and \( y \), there exist disjoint rgb-open sets \( G \) and \( H \) such that \( x \in G \) and \( y \in H \).

**Theorem 39.** If \( f : X \to Y \) is an almost contra rgb-continuous injective function from space \( X \) into a Ultra Hausdorff space \( Y \), then \( X \) is rgb\( -T_2 \).

**Proof.** Let \( x \) and \( y \) be any two distinct points in \( X \). Since \( f \) is an injective \( f(x) \neq f(y) \) and \( Y \) is Ultra Hausdorff space, there exist disjoint clopen sets \( U \) and \( V \) of \( Y \) containing \( f(x) \) and \( f(y) \) respectively. Then \( x \in f^{-1}(U) \) and \( y \in f^{-1}(V) \), where \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint rgb-open sets in \( X \). Therefore \( X \) is rgb\( -T_2 \).

\[ \square \]

**Definition 40.** A topological space \( X \) is called Ultra normal space, if each pair of disjoint closed sets can be separated by disjoint clopen sets.

**Definition 41.** A topological space \( X \) is said to be rgb-normal if each pair of disjoint closed sets can be separated by disjoint rgb-open sets.

**Theorem 42.** If \( f : X \to Y \) is an almost contra rgb-continuous closed injection and \( Y \) is ultra normal, then \( X \) is rgb-normal.

**Proof.** Let \( E \) and \( F \) be disjoint closed subsets of \( X \). Since \( f \) is closed and injective \( f(E) \) and \( f(F) \) are disjoint closed sets in \( Y \). Since \( Y \) is ultra normal there exists disjoint clopen sets \( U \) and \( V \) in \( Y \) such that \( f(E) \subset U \) and \( f(F) \subset V \). This implies \( E \subset f^{-1}(U) \) and \( F \subset f^{-1}(V) \). Since \( f \) is an almost contra rgb-continuous injection, \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint rgb-open sets.
in $X$. This shows $X$ is rgb-normal. 

**Theorem 43.** If $f : X \to Y$ is an almost contra rgb-continuous and $Y$ is semi-regular, then $f$ is rgb-continuous.

*Proof.* Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. By definition of semi-regularity of $Y$, there exists a regular open set $G$ of $Y$ such that $f(x) \in G \subset V$. Since $f$ is almost contra rgb-continuous, there exists $U \in rgb-O(X, x)$ such that $f(U) \subset G$. Hence we have $f(U) \subset G \subset V$. This shows that $f$ is rgb-continuous function.

7. Compactness

**Definition 44.** A space $X$ is said to be:

1. rgb-compact if every rgb-open cover of $X$ has a finite subcover.
2. rgb-closed compact if every rgb-closed cover of $X$ has a finite subcover.
3. Nearly compact if every regular open cover of $X$ has a finite subcover.
4. Countably rgb-compact if every countable cover of $X$ by rgb-open sets has a finite subcover.
5. Countably rgb-closed compact if every countable cover of $X$ by rgb-closed sets has a finite sub cover.
6. Nearly countably compact if every countable cover of $X$ by regular open sets has a finite sub cover.
7. rgb-Lindelof if every rgb-open cover of $X$ has a countable sub cover.
8. rgb-Lindelof if every rgb-closed cover of $X$ has a countable sub cover.
9. Nearly Lindelof if every regular open cover of $X$ has a countable sub cover.
10. $S$-Lindelof if every cover of $X$ by regular closed sets has a countable sub cover.
11. Countably $S$-closed if every countable cover of $X$ by regular closed sets has a finite sub-cover.
(12) $S$-closed if every regular closed cover of $x$ has a finite sub cover.

**Theorem 45.** Let $f : X \to Y$ be an almost contra rgb-continuous surjection. Then, the following properties hold:

1. If $X$ is rgb-closed compact, then $Y$ is nearly compact.
2. If $X$ is countably rgb-closed compact, then $Y$ is nearly countably compact.
3. If $X$ is rgb-Lindelof, then $Y$ is nearly Lindelof.

**Proof.**

1. Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of $Y$. Since $f$ is almost contra rgb-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is rgb-open cover of $X$.

Since $X$ is rgb-closed compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I\}$. Since $f$ is surjective, $Y = \bigcup \{(V_\alpha) : \alpha \in I\}$ which is finite sub cover of $Y$, therefore $Y$ is nearly compact.

2. Let $\{V_\alpha : \alpha \in I\}$ be any countable regular open cover of $Y$. Since $f$ is almost contra rgb-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable rgb-open cover of $X$. Since $X$ is countably rgb-closed compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I\}$. Since $f$ is surjective, $Y = \bigcup \{(V_\alpha) : \alpha \in I\}$ is finite sub cover for $Y$. Hence $Y$ is nearly countably compact.

3. Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of $Y$. Since $f$ is almost contra rgb-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is rgb-closed cover of $X$. Since $X$ is rgb-Lindelof, there exists a countable subset $I_0$ of $I$ such that $X = \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \bigcup \{(V_\alpha) : \alpha \in I_0\}$ is finite sub cover for $Y$. Therefore, $Y$ is nearly Lindelof.

$\square$

**Theorem 46.** Let $f : X \to Y$ be an almost contra rgb-continuous surjection. Then, the following properties hold:

1. If $X$ is rgb-compact, then $Y$ is $S$-closed.
2. If $X$ is countably rgb-closed, then $Y$ is countably $S$-closed.
3. If $X$ is rgb-Lindelof, then $Y$ is $S$-Lindelof.

**Proof.**

1. Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of $Y$. Since $f$ is almost contra rgb-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is rgb-open cover of
X. Since X is rgb-compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ is finite sub cover for $Y$. Therefore, $Y$ is $S$-closed.

(2) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular closed cover of $Y$. Since f is almost contra rgb-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable rgb-open cover of $X$. Since $X$ is countably rgb-compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ is finite sub cover for $Y$. Hence, $Y$ is countably $S$-closed.

(3) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of $Y$. Since f is almost contra rgb-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is rgb-open cover of $X$. Since $X$ is rgb-Lindelof, there exists a countable sub-set $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ is finite sub cover for $Y$. Hence, $Y$ is $S$-Lindelof.

\[\square\]

References


