SOME APPLICATION OF PRIME IDEALS OF AG-RING

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Abstract: In this paper we define c-prime, 3-prime and weakly prime ideal of AG-ring which we will study relation of c-prime, 3-prime, weakly prime ideal and prime ideal.

Key Words: c-prime, 3-prime, weakly prime ideal

1. Introduction

M.A. Kazim and MD. Naseeruddin [2, Proposition 2.1] asserted that, in every LA-semigroups $G$ a medial law hold

$$(a\cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall a, b, c, d \in G.$$ 

Q. Mushtaq and M. Khan [4, p.322] asserted that, in every LA-semigroups $G$ with left identity

$$(a \cdot b) \cdot (c \cdot d) = (d \cdot c) \cdot (b \cdot a), \quad \forall a, b, c, d \in G.$$ 

Further M. Khan, Faisal, and V. Amjid [3], asserted that, if a LA-semigroup $G$ with left identity the following law holds

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall a, b, c \in G.$$ 

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M. Sarwar (Kamran) [5, p.112] defined LA-group as the following; a groupoid $G$ is called a left almost group, abbreviated as LA-group, if $(i)$ there exists $e \in G$ such that $ea = a$ for all $a \in G$, $(ii)$ for every $a \in G$ there exists $a' \in G$ such that, $a'a = e$, $(iii)$ $(ab)c = (cb)a$ for every $a, b, c \in G$.

S.M. Yusuf in [6, p.211] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set $R$ with two binary operations “$+$” and “$\cdot$” is called a left almost ring, if $\langle R, + \rangle$ is an LA-group, $\langle R, \cdot \rangle$ is an LA-semigroup and distributive laws of “$\cdot$” over “$+$” holds. T. Shah and I. Rehman [6, p.211] asserted that a commutative ring $\langle R, +, \cdot \rangle$, we can always obtain an LA-ring $\langle R, \oplus, \cdot \rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $\langle R, +, \cdot \rangle$ is said to be LA-integral domain if $a \cdot b = 0$, $a, b \in R$, then $a = 0$ or $b = 0$. Let $\langle R, +, \cdot \rangle$ be an LA-ring and $S$ be a non-empty subset of $R$ and $S$ is itself and LA-ring under the binary operation induced by $R$, the $S$ is called an LA-subring of $R$, then $S$ is called an LA-subring of $\langle R, +, \cdot \rangle$. If $S$ is an LA-subring of an LA-ring $\langle R, +, \cdot \rangle$, then $S$ is called a left ideal of $R$ if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner. An ideal $I$ of $R$ is called prime if $AB \in I$ implies $A \in I$ or $B \in I$.

In this note we prefer to called left almost rings (LA-rings) as Abel-Grassmann rings (abbreviated as an “AG-rings”).

In [1] An ideal $I$ of $N$ is called c-prime if $a, b \in N$ and $ab \in I$ implies $a \in I$ or $b \in I$. $R$ is called c-prime nearring if $\{0\}$ is a c-prime ideal of $R$.

An ideal $I$ of $R$ is called 3-prime if $a, b \in N$ and $anb \in I$ for all $n \in N$ implies $a \in I$ or $b \in I$.

The notions of c-ideal, 3-prime ideal and prime ideal coincide in rings.

In [7] A proper ideal $I$ of an ring $R$ to be weakly prime if $0 \neq AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$ for any ideals $A, B$ of $R$.

The following implications are well known in rings:

(1) c-prime ideal $\Rightarrow$ 3-prime ideal $\Rightarrow$ prime ideal;

(2) prime ideal $\Rightarrow$ weakly prime ideal.

2. Main Results

In this paper, we define c-prime and 3-prime of AG-ring.

**Definition 2.1.** An ideal $I$ of an AG-ring $R$ is called c-prime if $a, b \in R$ and $ab \in I$ implies $a \in I$ or $b \in I$. 
**Definition 2.2.** An ideal $I$ of an AG-ring $R$ is called 3-prime if $a, b \in R$ and $arb \in I$ for all $r \in R$ implies $a \in I$ or $b \in I$.

The following lemmas and theorem we will study relation of c-prime, 3-prime and prime ideals.

**Lemma 2.3.** Every c-prime ideal is a 3-prime ideal

**Proof.** Suppose that $I$ is a c-prime ideal of AG-ring $R$, let $a, b \in I$ and $arb \in I$ for all $r \in R$. Since $I$ is a c-prime ideal we have $a \in I$ or $b \in I$. Then $I$ is a 3-prime ideal of $R$. \hfill \Box

**Lemma 2.4.** Every 3-prime ideal is a prime ideal

**Proof.** Suppose that $I$ is a 3-prime ideal of AG-ring $R$, let $a, b \in I$ and $ab \in I$. Since $I$ is a 3-prime ideal we have $a \in I$ or $b \in I$. Then $I$ is a prime ideal of $R$. \hfill \Box

**Lemma 2.5.** Every c-prime ideal is a prime ideal

**Proof.** Suppose that $I$ is a c-prime ideal of AG-ring $R$, let $a, b \in I$ and $ab \in I$. Since $I$ is a c-prime ideal we have $a \in I$ or $b \in I$. Then $I$ is a prime ideal of $R$. \hfill \Box

In [6, p.221]. studied if $I$ is a prime ideal in AG-ring $R$ if and only if $R/I$ is an AG-integral domain. The following theorems are application by lemmas 2.4 and 2.5

**Theorem 2.6.** Let $R$ be an AG-ring. Then $I$ is a 3-prime ideal in $R$ if and only if $R/I$ is an AG-integral domain.

**Proof.** $(\Rightarrow)$ Let $I$ is a 3-prime ideal in $R$. By Lemma 2.4 then $I$ is a prime ideal. Thus $R/I$ is an AG-integral domain.

$(\Leftarrow)$ Assume that $R/I$ is an AG-integral domain with $arb \in I$ for all $r \in R$. Then $I + arb = I$ so $(I + a)r(I + b) = I$. Since $R/I$ is an AG-integral domain we have $I + a = I$ or $I + b = I$. Then $a \in I$ or $b \in I$. Thus $P$ is a 3-prime ideal of $R$. \hfill \Box

**Theorem 2.7.** Let $R$ be an AG-ring. Then $I$ is a c-prime ideal in $R$ if and only if $R/I$ is an AG-integral domain.
Proof. \((\Rightarrow)\) Let \(I\) is a c-prime ideal in \(R\). By Lemma 2.5 then \(I\) is a prime ideal. Thus \(R/I\) is an AG-integral domain.

\((\Leftarrow)\) Assume that \(R/I\) is an AG-integral domain with \(ab \in I\) for all \(a, b \in R\). Then \(I + ab = I\) so \((I + a)(I + b) = I\). Since \(R/I\) is an AG-integral domain we have \(I + a = I\) or \(I + b = I\). Then \(a \in I\) or \(b \in I\). Thus \(P\) is a c-prime ideal of \(R\). \(\square\)

The next following we define of weakly prime ideal.

**Definition 2.8.** A proper ideal \(I\) of an AG-ring \(R\) to be weakly prime if \(0 \neq AB \subseteq I\) implies either \(A \subseteq I\) or \(B \subseteq I\) for any ideals \(A, B\) of \(R\).

Clearly every prime ideal is weakly prime and \(\{0\}\) is always weakly prime ideal of \(R\). The following theorem we will study properties

**Theorem 2.9.** If \(I\) is weakly prime but not prime, then \(I^2 = 0\).

Proof. Since \(I\) is weakly prime (but not prime), there exist ideals \(A \not\subseteq I\) and \(B \not\subseteq P\) but \(0 = AB \subseteq I\). Since \(I \subseteq A + I\) and \(I \subseteq B + I\). But if \(I^2 \neq 0\), by distributive laws “·” over “+” of AG-ring we have

\[
0 \neq I^2 = II \subseteq (A + I)(B + I) = [(A + I)B] + [(A + I)P] = AB + IB + AI + II \subseteq I
\]

which implies \((A + I) \subseteq I\) and \((B + I) \subseteq I\), since \(I\) is a weakly prime; that is \(A \subseteq I\) or \(B \subseteq I\), a contradiction. Hence, \(I^2 = 0\). \(\square\)

If \(R^2 = 0\), then it is evident that every ideal of \(R\) is weakly prime. In particular, if an ideal \(I\) of an AG-ring \(R\) is weakly prime but not a prime ideal, then every ideal of \(I\) as an AG-ring is weakly prime by Theorem 2.9.

**Theorem 2.10.** Every ideal of an AG-ring \(R\) is weakly prime if and only if for any ideals \(A\) and \(B\) of \(R\), \(AB = A\), \(AB = B\), or \(AB = 0\).

Proof. Suppose that every ideal of \(R\) is weakly prime. Let \(A, B\) be ideals of \(R\). Then \(AB\) is a left ideals of \(R\), if \(AB \neq R\), then by hypothesis, \(AB\) is weakly prime. We are consider two situation, that is \(AB = 0\) or \(AB \neq 0\). If \(0 \neq AB \subseteq AB\), then by Definition 2.8, we have \(A \subseteq AB\) or \(B \subseteq AB\). Since \(A\) and \(B\) are ideals of \(R\), we have \(AB \subseteq A\) and \(AB \subseteq B\). Therefore \(A = AB\) or \(B = AB\). If \(AB = R\), then we have \(A = B = R\) whence \(R^2 = R\).
Conversely, let $K$ be any proper ideal of $R$ and suppose that $0 \neq AB \subseteq K$ for ideals $A$ and $B$ of $R$. Then we have either $A = AB \subseteq K$ or $B = AB \subseteq K$.

**Corollary 2.11.** Let $R$ be an AG-ring and every ideal of $R$ is weakly prime. Then for any ideal $I$ of $R$, either $I^2 = I$ or $I^2 = 0$.

The following theorem we will study relation of c-prime, 3-prime, weakly prime ideals.

**Theorem 2.12.** Every c-prime ideal is a weakly prime ideal

*Proof.* Suppose that $I$ is a c-prime ideal of AG-ring $R$. By Lemma 2.3 we have $I$ is a prime ideal. Since every prime ideal is weakly prime ideal we have $I$ is a weakly prime ideal of $R$.

**Theorem 2.13.** Every 3-prime ideal is a weakly prime ideal

*Proof.* Suppose that $I$ is a 3-prime ideal of AG-ring $R$. By Lemma 2.4 we have $I$ is a prime ideal. Since every prime ideal is weakly prime ideal we have $I$ is a weakly prime ideal of $R$.

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**References**


