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ON THE DIOPHANTINE EQUATION $3^x + 3^{2s}n^y = z^{2t}$ WHERE n, s, t ARE NON-NEGATIVE INTEGERS AND $n \equiv 5 \pmod{20}$

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Abstract: In this paper, let n, s, t be any non-negative integers where $n \equiv 5 \pmod{20}$. We show that all non-negative integer solution (x, y, z) of the Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$ are the following:

$$(x, y, z) = \begin{cases} (1+2s, 0, 2(3^s)) & ; \quad t = 1 \\ \text{No solution} & ; \quad \text{otherwise.} \end{cases}$$

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1. Introduction

There are many mathematicians searched or studied all non-negative integer solutions (x, y, z) of the Diophantine equations $a^x + b^y = z^2$ where a, b are positive integers and x, y, z are non-negative integers.

In 2013, Rabago [4] showed that the non-negative integer solutions of the

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two Diophantine equations $3^x+19^y=z^2$ and $3^x+91^y=z^2$ are in $\{(1,0,2),(4,1,10)\}$ and $\{(1,0,2),(2,1,10)\}$, respectively.

In 2012-2013, Sroysang [5, 6] solved the two Diophantine equations $3^x+5^y=z^2$, $3^x+17^y=z^2$ and found that both Diophantine equations have exactly one non-negative integer solution, namely, $(x,y,z)=(1,0,2)$.

Later in 2014, Sroysang [7, 8] also studied $3^x+85^y=z^2$ and $3^x+45^y=z^2$ and he showed that there is the same non-negative integer solution $(x,y,z)=(1,0,2)$. Now all of the non-negative integer solutions of the Diophantine equation $3^x+n^y=z^2$ where n is a non-negative integer are open problems.

Inspired by all references, we want to find all possible non-negative integer solutions of the Diophantine equation $3^x+3^{2s}n^y=z^2$ when $n\equiv 5 \pmod{20}$, which is a generalization of the Diophantine equations $3^x+5^y=z^2$, $3^x+45^y=z^2$, $3^x+85^y=z^2$ when we set $s=0$ and $n=5,45,85$, respectively.

2. Preliminaries

In this paper, let n be a positive integer and $n\equiv 5 \pmod{20}$. It is clear that $n\equiv 1 \pmod 4$ and $n\equiv 0 \pmod 5$.

Lemma 1. *[5] $(1,2)$ is a unique non-negative integer solution (x,z) for the Diophantine equation $3^x+1=z^2$ where x and z are non-negative integers.*

Let p be an odd prime and a be a positive integer where $gcd(a,p)=1$. If the quadratic congruence $x^2\equiv a \pmod p$ has a solution, then a is said to be *a quadratic residue of p*. Otherwise, a is called *a quadratic nonresidue of p*. In 1798 Adrien-Marie Legendre introduced *the Legendre symbol* $\left(\frac{a}{p}\right)$ which is defined by

$$\left(\frac{a}{p}\right)=\begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p\\ -1 & \text{if } a \text{ is a quadratic non-residue of } p.\end{cases}$$

In the present paper, we need the following well-known facts about the Legendre symbols.

Theorem 2. *[2] If p is an odd prime, then*

$$\left(\frac{2}{p}\right)=\begin{cases} 1 & \text{if } p\equiv 1 \pmod 8 \text{ or } p\equiv 7 \pmod 8\\ -1 & \text{if } p\equiv 3 \pmod 8 \text{ or } p\equiv 5 \pmod 8.\end{cases}$$

Theorem 3. *[2] If $p\neq 3$ is an odd prime, then*

$$\left(\frac{3}{p}\right)=\begin{cases} 1 & \text{if } p\equiv \pm 1 \pmod{12}\\ -1 & \text{if } p\equiv \pm 5 \pmod{12}\ .\end{cases}$$

3. Main Results

First we consider the non-negative integer solutions of the Diophantine equation of the form $3^x + n^y = z^2$.

Theorem 4. $(1, 0, 2)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $3^x + n^y = z^2$ *where* x, y, z *are non-negative integers.*

Proof. Note that z is an even integer. Then $z^2 \equiv 0 \pmod 4$.
Case $y = 0$. By Lemma 1, we have $(x, y, z) = (1, 0, 2)$ is a unique non-negative integer solution of the Diophantine equation $3^x + n^y = z^2$.
Case $y \geq 1$. Suppose that there exists a non-negative integer solution for the Diophantine equation $3^x + n^y = z^2$. Since $n \equiv 5 \pmod{20}$, $n \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 5$. Then we get $0 \equiv z^2 = 3^x + n^y \equiv 3^x + 1 \pmod 4$. Thus $3^x \equiv -1 \pmod 4$. This implies that x is odd. Consequently, $3^x \equiv 2 \pmod 5$ or $3^x \equiv 3 \pmod 5$. Since $n^y \equiv 0 \pmod 5$, we obtain $z^2 = 3^x + n^y \equiv 2$ or $3 \pmod 5$. That is $\left(\frac{2}{5}\right) = 1$ and $\left(\frac{3}{5}\right) = 1$. This is a contradiction to Theorem 2 and Theorem 3, respectively. In this case, there is no non-negative integer solution.
It is easy to verify that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + n^y = z^2$. This completes the proof. □

The following examples, which are the main theorems of Sroysang [5, 7, 8], are easy to verify.

Example 5. [5] $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers.

Example 6. [7] $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 85^y = z^2$ where x, y and z are non-negative integers.

Example 7. [8] $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 45^y = z^2$ where x, y and z are non-negative integers.

Next we want to find the non-negative integer solutions of the Diophantine equation of the form $3^x + 3^{2s}n^y = z^2$ when s is a non-negative integer. The following lemmas are needed.

Lemma 8. *Let* m *be a positive integer with* $m \equiv 1 \pmod 4$. *The Diophantine equation* $1 + mn^y = z^2$ *has no non-negative integer solution where* y, z

are non-negative integers.

Proof. Suppose that there exists a non-negative integer solution y, z such that $1 + mn^y = z^2$. Note that $mn^y \equiv 1 \pmod 4$, this implies that z is even. So $0 \equiv z^2 = 1 + mn^y \equiv 2 \pmod 4$. This is a contradiction. Thus the Diophantine equation $1 + mn^y = z^2$ has no non-negative integer solution. □

Lemma 9. *Let m be an integer such that $m \geq 2$. The Diophantine equation $3 + 3^m n^y = z^2$ has no non-negative integer solution (y, z).*

Proof. Suppose that there are non-negative integers y, z such that $3 + 3^m n^y = z^2$. Since $m \geq 2$, so we can let $z = 3^k r$ for some integers $r, k \geq 1$ and $\gcd(3, r) = 1$. The Diophantine equation $3 + 3^m n^y = z^2$ can be rewritten as $1 + 3^{m-1} n^y = 3^{2k-1} r^2$. Thus $3(3^{2k-2} r^2 - 3^{m-2}) = 1$. This implies that $3|1$. This is a contradiction. Thus $3 + 3^m n^y = z^2$ has no non-negative integer solution. □

The result of the Diophantine equation $3^x + 3^{2s} n^y = z^2$, when s is a non-negative integer, is as follows:

Theorem 10. *Let s be a non-negative integer. Then $(1 + 2s, 0, 2(3^s))$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 3^{2s} n^y = z^2$.*

Proof. We prove by induction on s.
Let $P(s)$: The Diophantine equation $3^x + 3^{2s} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 2(3^s))$.
By Theorem 4, $P(0)$ is true.

Suppose that $P(k)$ is true, that is the Diophantine equation $3^x + 3^{2k} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2k, 0, 2(3^k))$.

Consider the Diophantine equation $3^x + 3^{2(k+1)} n^y = z^2$ into the following cases:

Case $x = 0$. Since $3^{2(k+1)} \equiv 1 \pmod 4$ and by Lemma 8, we obtain that $1 + 3^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x = 1$. By Lemma 9, $3 + 3^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x \geq 2$. Note that $3^x + 3^{2(k+1)} n^y = z^2$ can be written as $3^{x-2} + 3^{2k} n^y = \left(\frac{z}{3}\right)^2$ and $x - 2, \frac{z}{3}$ are non-negative integers. Let $u = x - 2$ and $v = \frac{z}{3}$. By assumption $P(k)$ is true, we obtain that $3^u + 3^{2k} n^y = v^2$ has a unique non-negative integer solution $(u, y, v) = (1 + 2k, 0, 2(3^k))$. That is $u = 1 + 2k$ and

$v = 2(3^k)$. Thus $(x, y, z) = (1 + 2(k + 1), 0, 2(3^{(k+1)}))$ is a unique non-negative integer solution of $3^x + 3^{2(k+1)}n^y = z^2$. Therefore $P(k + 1)$ is true.

By Mathematical Induction, the Diophantine equation $3^x + 3^{2s}n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 2(3^s))$. □

As a consequence of Theorem 10, we obtain:

Corollary 11. *Let s, t be non-negative integers such that $t \geq 2$. The Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$ has no non-negative integer solution (x, y, z).*

Proof. Suppose that (x, y, z) is a non-negative integer solution of the Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$. Thus (x, y, z^t) is a non-negative integer solution of the Diophantine equation $3^x + 3^{2s}n^y = z^2$. By Theorem 10, we have $(x, y, z^t) = (1 + 2s, 0, 2(3^s))$. This implies that $z^t = 2(3^s)$. This is a contradiction. Thus the Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$ has no non-negative integer solution (x, y, z). □

The Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$, when s, t are non-negative integers, is a generalization of the Diophantine equation $3^x + n^y = z^2$. It easy to verify that the Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$ has no non-negative integer solution when $t = 0$. Theorem 10 and Corollary 11 give the following result.

Let n, s, t be any non-negative integers where $n \equiv 5 \pmod{20}$. All non-negative integer solutions (x, y, z) of the Diophantine equation $3^x + 3^{2s}n^y = z^{2t}$ are the following:

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 2(3^s)) & ; \quad t = 1 \\ \text{No solution} & ; \quad \text{otherwise.} \end{cases}$$

Using Theorem 10, it is easy to verify the following example.

Example 12. For $n = 5, s = 1, 2, 3, 4, ...$

s	$3^x + 3^{2s}5^y = z^2$	solution (x, y, z)
$s = 1$	$3^x + (9)5^y = z^2$	$(3, 0, 6)$
$s = 2$	$3^x + (81)5^y = z^2$	$(5, 0, 18)$
$s = 3$	$3^x + (729)5^y = z^2$	$(7, 0, 54)$
$s = 4$	$3^x + (6561)5^y = z^2$	$(9, 0, 162)$
$\vdots$	$\vdots$	$\vdots$

For $n = 25, s = 0, 1, 2, 3, 4, ...$

s	$3^x + 3^{2s}25^y = z^2$	solution (x, y, z)
$s = 0$	$3^x + 25^y = z^2$	$(1, 0, 2)$
$s = 1$	$3^x + (9)25^y = z^2$	$(3, 0, 6)$
$s = 2$	$3^x + (81)25^y = z^2$	$(5, 0, 18)$
$s = 3$	$3^x + (729)25^y = z^2$	$(7, 0, 54)$
$s = 4$	$3^x + (6561)25^y = z^2$	$(9, 0, 162)$
$\vdots$	$\vdots$	$\vdots$

For $n = 45, s = 1, 2, 3, 4, ...$

s	$3^x + 3^{2s}45^y = z^2$	solution (x, y, z)
$s = 1$	$3^x + (9)45^y = z^2$	$(3, 0, 6)$
$s = 2$	$3^x + (81)45^y = z^2$	$(5, 0, 18)$
$s = 3$	$3^x + (729)45^y = z^2$	$(7, 0, 54)$
$s = 4$	$3^x + (6561)45^y = z^2$	$(9, 0, 162)$
$\vdots$	$\vdots$	$\vdots$

For $n = 65, s = 0, 1, 2, 3, 4...$

s	$3^x + 3^{2s}65^y = z^2$	solution (x, y, z)
$s = 0$	$3^x + 65^y = z^2$	$(1, 0, 2)$
$s = 1$	$3^x + (9)65^y = z^2$	$(3, 0, 6)$
$s = 2$	$3^x + (81)65^y = z^2$	$(5, 0, 18)$
$s = 3$	$3^x + (729)65^y = z^2$	$(7, 0, 54)$
$s = 4$	$3^x + (6561)65^y = z^2$	$(9, 0, 162)$
$\vdots$	$\vdots$	$\vdots$

For $n = 85, s = 1, 2, 3, 4, ...$

s	$3^x + 3^{2s}85^y = z^2$	solution (x, y, z)
$s = 1$	$3^x + (9)85^y = z^2$	$(3, 0, 6)$
$s = 2$	$3^x + (81)85^y = z^2$	$(5, 0, 18)$
$s = 3$	$3^x + (729)85^y = z^2$	$(7, 0, 54)$
$s = 4$	$3^x + (6561)85^y = z^2$	$(9, 0, 162)$
$\vdots$	$\vdots$	$\vdots$

For $n = 105, s = 0, 1, 2, 3, 4, ...$

s	$3^x + 3^{2s}105^y = z^2$	solution (x, y, z)
$s = 0$	$3^x + 105^y = z^2$	$(1, 0, 2)$
$s = 1$	$3^x + (9)105^y = z^2$	$(3, 0, 6)$
$s = 2$	$3^x + (81)105^y = z^2$	$(5, 0, 18)$
$s = 3$	$3^x + (729)105^y = z^2$	$(7, 0, 54)$
$s = 4$	$3^x + (6561)105^y = z^2$	$(9, 0, 162)$
$\vdots$	$\vdots$	$\vdots$

Moreover, for any $n \equiv 5 \pmod{20}$ we obtain the non-negative integer solutions of the Diophantine equation of the form $3^x + 3^{2s}n^y = z^2$ in a similar way.

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