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GEOMETRICAL QUATERNIONIC COUPLING FOR THREE DIMENSIONAL WAVE EQUATIONS

M.F. Borges[1,2 §], J.A.P.F. Marão[2], F.S. Costa[3], J. Coelho[3]

[1]UNESP - São Paulo State University
S.J. Rio Preto Campus
15054-000, São José do Rio Preto, BRAZIL

[2]Department of Mathematics
Federal University of Maranhão - São Luís-MA
65085-580, Maranhão, BRAZIL

[3]Department of Mathematics and Informatic
State University of Maranhão - São Luís-MA
65000-001, Maranhão, BRAZIL

Abstract: The present work has the scope to show the relationship between four three-dimensional waves. This fact will be made in the form of coupling, using for it the Cauchy-Riemann conditions for quaternionic functions [1], through certain Laplace's equation in [2]. The coupling will relate those functions that determine the wave as well as their respective propagation speeds.

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1. Introduction

The study of the three-dimensional wave patterns in major physical problems

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§Correspondence author

is such that the coupling of these waves can bring physical insights not yet detected. Let us consider that waves propagate with different velocities v_1, v_2, v_3 and v_4, and wave function given by F_1, F_2, F_3 and F_4.

To fix ideas, it will be considered a quaternionic function denoted by $F(q)$ with $q = t + xi + yj + zk$, and given by $F(q) = F_1 + F_2 i + F_3 j + F_4 k$ where the functions F_1, F_2, F_3 and F_4, are functions of the variables t, x, y and z. The following theorem establishes the Cauchy-Riemann conditions for quaternionic functions:

Theorem 1. *For any pair pontis a and b and any path joining them simply conect subdomain of the four-dimmensional space, the integral $\int_a^b f dq$ is independent form the given path if and only if there is a function $F = F_1 + F_2 i + F_3 j + F_4 k$ such that $\int_a^b f dq = F(a) - F(b)$, and satisfying the following relations:*

$$\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial x} = \frac{\partial F_3}{\partial y} = \frac{\partial F_4}{\partial z}, \tag{1}$$

$$\frac{\partial F_2}{\partial t} = -\frac{\partial F_1}{\partial x} = -\frac{\partial F_3}{\partial z} = \frac{\partial F_4}{\partial y}, \tag{2}$$

$$\frac{\partial F_3}{\partial t} = -\frac{\partial F_1}{\partial y} = -\frac{\partial F_2}{\partial z} = \frac{\partial F_4}{\partial x}, \tag{3}$$

$$\frac{\partial F_4}{\partial t} = \frac{\partial F_1}{\partial z} = -\frac{\partial F_2}{\partial y} = -\frac{\partial F_3}{\partial x}. \tag{4}$$

Proof. The proof of this theorem can be analyzed in greater detail in [1]. □

What will be done now is the derivation of each of the above relations on each one of the variables of the problem, t, x, y and z. Then:

$$\begin{aligned}
\frac{\partial^2 F_1}{\partial t^2} &= \frac{\partial^2 F_2}{\partial t \partial x} = \frac{\partial^2 F_3}{\partial t \partial y} = \frac{\partial^2 F_4}{\partial t \partial z}, \\
\frac{\partial^2 F_1}{\partial t \partial x} &= \frac{\partial^2 F_2}{\partial x^2} = \frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_4}{\partial x \partial z}, \\
\frac{\partial^2 F_1}{\partial y \partial t} &= \frac{\partial^2 F_2}{\partial y \partial x} = \frac{\partial^2 F_3}{\partial y^2} = \frac{\partial^2 F_4}{\partial z \partial y}, \\
\frac{\partial^2 F_1}{\partial t \partial z} &= \frac{\partial^2 F_2}{\partial z \partial x} = \frac{\partial^2 F_3}{\partial z \partial y} = \frac{\partial^2 F_4}{\partial z^2},
\end{aligned} \tag{5}$$

$$
\begin{aligned}
\frac{\partial^2 F_2}{\partial t^2} &= -\frac{\partial^2 F_1}{\partial t\partial x} = -\frac{\partial^2 F_3}{\partial t\partial z} = \frac{\partial^2 F_4}{\partial t\partial y},\\
\frac{\partial^2 F_2}{\partial t\partial x} &= -\frac{\partial^2 F_1}{\partial x^2} = -\frac{\partial^2 F_3}{\partial x\partial z} = \frac{\partial^2 F_4}{\partial y\partial x},\\
\frac{\partial^2 F_2}{\partial y\partial t} &= -\frac{\partial^2 F_1}{\partial y\partial x} = -\frac{\partial^2 F_3}{\partial y\partial z} = \frac{\partial^2 F_4}{\partial y^2},\\
\frac{\partial^2 F_2}{\partial z\partial t} &= -\frac{\partial^2 F_1}{\partial z\partial x} = -\frac{\partial^2 F_3}{\partial z^2} = \frac{\partial^2 F_4}{\partial z\partial y},
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\frac{\partial^2 F_3}{\partial t^2} &= -\frac{\partial^2 F_1}{\partial t\partial y} = -\frac{\partial^2 F_2}{\partial t\partial z} = \frac{\partial^2 F_4}{\partial t\partial x},\\
\frac{\partial^2 F_3}{\partial t\partial x} &= -\frac{\partial^2 F_1}{\partial x\partial y} = -\frac{\partial^2 F_2}{\partial x\partial z} = \frac{\partial^2 F_4}{\partial x^2},\\
\frac{\partial^2 F_3}{\partial y\partial t} &= -\frac{\partial^2 F_1}{\partial y^2} = -\frac{\partial^2 F_2}{\partial z\partial y} = \frac{\partial^2 F_4}{\partial y\partial x},\\
\frac{\partial^2 F_3}{\partial t\partial z} &= -\frac{\partial^2 F_1}{\partial z\partial y} = -\frac{\partial^2 F_2}{\partial z^2} = \frac{\partial^2 F_4}{\partial z\partial x},
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{\partial^2 F_4}{\partial t^2} &= \frac{\partial^2 F_1}{\partial t\partial z} = -\frac{\partial^2 F_2}{\partial t\partial y} = -\frac{\partial^2 F_3}{\partial t\partial x},\\
\frac{\partial^2 F_4}{\partial t\partial x} &= \frac{\partial^2 F_1}{\partial x\partial z} = -\frac{\partial^2 F_2}{\partial x\partial y} = -\frac{\partial^2 F_3}{\partial x^2},\\
\frac{\partial^2 F_4}{\partial y\partial t} &= \frac{\partial^2 F_1}{\partial y\partial z} = -\frac{\partial^2 F_2}{\partial y^2} = -\frac{\partial^2 F_3}{\partial y\partial x},\\
\frac{\partial^2 F_4}{\partial t\partial z} &= \frac{\partial^2 F_1}{\partial z^2} = -\frac{\partial^2 F_2}{\partial z\partial y} = -\frac{\partial^2 F_3}{\partial z\partial x}.
\end{aligned} \tag{8}
$$

Thus, the following equations are obtained:

$$\frac{\partial^2 F_1}{\partial^2 t^2} + \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} = 0 \tag{9}$$

$$\frac{\partial^2 F_2}{\partial t^2} + \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2} = 0 \tag{10}$$

$$\frac{\partial^2 F_3}{\partial t^2} + \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} = 0 \tag{11}$$

and

$$\frac{\partial^2 F_4}{\partial t^2} + \frac{\partial^2 F_4}{\partial x^2} + \frac{\partial^2 F_4}{\partial y^2} + \frac{\partial^2 F_4}{\partial z^2} = 0 \tag{12}$$

2. Wave Equations

The wave equations presents in Mathematical Physics are in the format:

$$\frac{\partial^2 u(t,x)}{\partial t^2} = c^2 \frac{\partial^2 u(t,x)}{\partial x^2},$$

for the one-dimensional case. The three-dimensional case, is written as:

$$\frac{\partial^2 u(t,x,y,z)}{\partial t^2} = c^2 \left(\frac{\partial^2 u(t,x,y,z)}{\partial x^2} + \frac{\partial^2 u(t,x,y,z)}{\partial y^2} + \frac{\partial^2 u(t,x,y,z)}{\partial z^2} \right), \quad (13)$$

where c is the speed of wave propagation, or

$$-\frac{1}{c^2}\frac{\partial^2 u(t,x,y,z)}{\partial t^2} + \frac{\partial^2 u(t,x,y,z)}{\partial x^2} + \frac{\partial^2 u(t,x,y,z)}{\partial y^2} + \frac{\partial^2 u(t,x,y,z)}{\partial z^2} = 0. \quad (14)$$

3. Coupling Equations

Considering the equations:

$$-\frac{1}{c_1^2}\frac{\partial^2 F_1^{\Box}(t,x,y,z)}{\partial t^{\Box 2}} + \frac{\partial^2 F_1^{\Box}(t,x,y,z)}{\partial x^2} + \frac{\partial^2 F_1^{\Box}(t,x,y,z)}{\partial y^2} + \frac{\partial^2 F_1^{\Box}(t,x,y,z)}{\partial z^2} = 0, \quad (15)$$

$$-\frac{1}{c_2^2}\frac{\partial^2 F_2^{\Box}(t,x,y,z)}{\partial t^{\Box 2}} + \frac{\partial^2 F_2^{\Box}(t,x,y,z)}{\partial x^2} + \frac{\partial^2 F_2^{\Box}(t,x,y,z)}{\partial y^2} + \frac{\partial^2 F_2^{\Box}(t,x,y,z)}{\partial z^2} = 0, \quad (16)$$

$$-\frac{1}{c_3^2}\frac{\partial^2 F_3^{\Box}(t,x,y,z)}{\partial t^{\Box 2}} + \frac{\partial^2 F_3^{\Box}(t,x,y,z)}{\partial x^2} + \frac{\partial^2 F_3^{\Box}(t,x,y,z)}{\partial y^2} + \frac{\partial^2 F_3^{\Box}(t,x,y,z)}{\partial z^2} = 0, \quad (17)$$

and

$$-\frac{1}{c_4^2}\frac{\partial^2 F'_4(t,x,y,z)}{\partial t'^2}+\frac{\partial^2 F'_4(t,x,y,z)}{\partial x^2}+\frac{\partial^2 F'_4(t,x,y,z)}{\partial y^2}+\frac{\partial^2 F'_4(t,x,y,z)}{\partial z^2}=0. \quad (18)$$

Assuming that the set of equations described in (9), (10), (11) and (12) have no physical sense, that will be reached by considering the transformation below:

$$\frac{\partial F_i}{\partial t^2}=-\frac{1}{c_i^2}\frac{\partial F'_i}{\partial t'^2}, \quad i=1,2,3,4, \quad (19)$$

where F_i is function of variables t', x, y and z. The transformation (19) makes the set of equations (5), (6), (7) and (8) to be rewritten as follows:

$$\begin{aligned}
-\frac{1}{c_1^2}\frac{\partial^2 F'_1}{\partial t'^2}&=\frac{\partial^2 F'_2}{\partial t'\partial x}=\frac{\partial^2 F'_3}{\partial t'\partial y}=\frac{\partial^2 F'_4}{\partial t'\partial z},\\
\frac{\partial^2 F'_1}{\partial t'\partial x}&=\frac{\partial^2 F'_2}{\partial x^2}=\frac{\partial^2 F'_3}{\partial x\partial y}=\frac{\partial^2 F'_4}{\partial x\partial z},\\
\frac{\partial^2 F'_1}{\partial y\partial t'}&=\frac{\partial^2 F'_2}{\partial y\partial x}=\frac{\partial^2 F'_3}{\partial y^2}=\frac{\partial^2 F_4}{\partial z\partial y},\\
\frac{\partial^2 F'_1}{\partial t'\partial z}&=\frac{\partial^2 F'_2}{\partial z\partial x}=\frac{\partial^2 F'_3}{\partial z\partial y}=\frac{\partial^2 F'_4}{\partial z^2},
\end{aligned} \quad (20)$$

$$\begin{aligned}
-\frac{1}{c_2^2}\frac{\partial^2 F'_2}{\partial t'^2}&=-\frac{\partial^2 F'_1}{\partial t'\partial x}=-\frac{\partial^2 F'_3}{\partial t'\partial z}=\frac{\partial^2 F'_4}{\partial t'\partial y},\\
\frac{\partial^2 F'_2}{\partial t'\partial x}&=-\frac{\partial^2 F'_1}{\partial x^2}=-\frac{\partial^2 F'_3}{\partial x\partial z}=\frac{\partial^2 F'_4}{\partial y\partial x},\\
\frac{\partial^2 F'_2}{\partial y\partial t'}&=-\frac{\partial^2 F'_1}{\partial y\partial x}=-\frac{\partial^2 F'_3}{\partial y\partial z}=\frac{\partial^2 F'_4}{\partial y^2},\\
\frac{\partial^2 F'_2}{\partial z\partial t'}&=-\frac{\partial^2 F'_1}{\partial z\partial x}=-\frac{\partial^2 F'_3}{\partial z^2}=\frac{\partial^2 F'_4}{\partial z\partial y},
\end{aligned} \quad (21)$$

$$\begin{aligned}
-\frac{1}{c_3^2}\frac{\partial^2 F'_3}{\partial t'^2}&=-\frac{\partial^2 F'_1}{\partial t'\partial y}=-\frac{\partial^2 F'_2}{\partial t'\partial z}=\frac{\partial^2 F'_4}{\partial t'\partial x},\\
\frac{\partial^2 F'_3}{\partial t'\partial x}&=-\frac{\partial^2 F'_1}{\partial x\partial y}=-\frac{\partial^2 F'_2}{\partial x\partial z}=\frac{\partial^2 F'_4}{\partial x^2},\\
\frac{\partial^2 F'_3}{\partial y\partial t'}&=-\frac{\partial^2 F'_1}{\partial y^2}=-\frac{\partial^2 F'_2}{\partial z\partial y}=\frac{\partial^2 F'_4}{\partial y\partial x},\\
\frac{\partial^2 F'_3}{\partial t'\partial z}&=-\frac{\partial^2 F'_1}{\partial z\partial y}=-\frac{\partial^2 F'_2}{\partial z^2}=\frac{\partial^2 F'_4}{\partial z\partial x},
\end{aligned} \quad (22)$$

$$
\begin{aligned}
-\frac{1}{c_4^2}\frac{\partial^2 F_4'}{\partial t'^2} = \frac{\partial^2 F_1'}{\partial t'\partial z} = -\frac{\partial^2 F_2'}{\partial t'\partial y} = -\frac{\partial^2 F_3'}{\partial t'\partial x},\\
\frac{\partial^2 F_4'}{\partial t'\partial x} = \frac{\partial^2 F_1'}{\partial x\partial z} = -\frac{\partial^2 F_2'}{\partial x\partial y} = -\frac{\partial^2 F_3'}{\partial x^2},\\
\frac{\partial^2 F_4'}{\partial y\partial t'} = \frac{\partial^2 F_1'}{\partial y\partial z} = -\frac{\partial^2 F_2'}{\partial y^2} = -\frac{\partial^2 F_3'}{\partial y\partial x},\\
\frac{\partial^2 F_4'}{\partial t'\partial z} = \frac{\partial^2 F_1'}{\partial z^2} = -\frac{\partial^2 F_2'}{\partial z\partial y} = -\frac{\partial^2 F_3'}{\partial z\partial x}.
\end{aligned}
\tag{23}
$$

4. Concluding Remarks

Taking only the equations that depend on time in the sets of equations (20) - (23), follows that:

(i)

$$-\frac{1}{c_1^2}\frac{\partial^2 F_1'}{\partial t'^2} = \frac{\partial^2 F_2'}{\partial t\partial x};$$

$$-\frac{\partial F_4'}{\partial y\partial x} = -\frac{\partial^2 F_2'}{\partial t'\partial x};$$

$$\frac{\partial^2 F_3'}{\partial t'\partial y} = \frac{\partial^2 F_4'}{\partial t'\partial z};$$

$$-\frac{\partial F_2'}{\partial z\partial x} = \frac{\partial^2 F_4'}{\partial t'\partial z}$$

obtaining the equation:

$$-\frac{1}{c_1^2}\frac{\partial^2 F_1'}{\partial t'^2} - \frac{\partial^2 F_2'}{\partial t'\partial x} - \frac{\partial^2 F_3'}{\partial t'\partial y} + \frac{\partial^2 F_4'}{\partial t'\partial z} = 0. \tag{24}$$

Proceeding similarly, we have that:

(ii)

$$-\frac{1}{c_2^2}\frac{\partial^2 F_2'}{\partial t'^2} = -\frac{\partial^2 F_1'}{\partial t'\partial x};$$

$$\frac{\partial F_4'}{\partial x\partial z} = \frac{\partial^2 F_1'}{\partial t'\partial x};$$

$$-\frac{\partial^2 F_3'}{\partial t'\partial z} = \frac{\partial^2 F_4'}{\partial t'\partial y};$$

$$-\frac{\partial F_3'}{\partial y \partial x} = \frac{\partial^2 F_4'}{\partial t' \partial y};$$

where, adding plots and establishing equalities, we have that:

$$-\frac{1}{c_2^2}\frac{\partial^2 F_2'}{\partial t'^2} + \frac{\partial^2 F_1'}{\partial t' \partial x} - \frac{\partial^2 F_3'}{\partial t' \partial z} - \frac{\partial^2 F_4'}{\partial t' \partial y} = 0 \tag{25}$$

Following the set of equations (22). We obtain that:

(iii)

$$-\frac{1}{c_3^2}\frac{\partial^2 F_3'}{\partial t'^2} = -\frac{\partial^2 F_1'}{\partial t' \partial y};$$

$$\frac{\partial F_2'}{\partial y \partial x} = \frac{\partial^2 F_1'}{\partial t' \partial y};$$

$$-\frac{\partial^2 F_2'}{\partial t' \partial z} = \frac{\partial^2 F_4'}{\partial t' \partial x};$$

$$-\frac{\partial F_3'}{\partial x^2} = -\frac{\partial^2 F_4'}{\partial t' \partial x};$$

which gives us:

$$-\frac{1}{c_3^2}\frac{\partial^2 F_3'}{\partial t'^2} + \frac{\partial^2 F_1'}{\partial t' \partial y} - \frac{\partial^2 F_2'}{\partial t' \partial z} - \frac{\partial^2 F_4'}{\partial t' \partial x} = 0. \tag{26}$$

Finally, we have that:

(iv)

$$-\frac{1}{c_4^2}\frac{\partial^2 F_4'}{\partial t'^2} = -\frac{\partial^2 F_3'}{\partial t' \partial x};$$

$$-\frac{\partial^2 F_2'}{\partial x \partial z} = \frac{\partial^2 F_3'}{\partial t' \partial x};$$

$$\frac{\partial F_1'}{\partial t' \partial z} = -\frac{\partial F_2'}{\partial t' \partial y};$$

$$-\frac{\partial F_1'}{\partial y \partial x} = \frac{\partial F_2'}{\partial t' \partial y};$$

again adding and making the equalities, we have:

$$-\frac{1}{c_4^2}\frac{\partial^2 F_4'}{\partial t'^2} + \frac{\partial^2 F_3'}{\partial t' \partial x} + \frac{\partial^2 F_1'}{\partial t' \partial z} + \frac{\partial F_2'}{\partial t' \partial y} = 0 \tag{27}$$

The above equations determine when a coupling between the wave equations may be considered. This coupling is done at variable time.

Considering $c_1 = c_2 = c_3 = c_4 = c$ then the following general time depending coupling equation is obtained:

$$
\begin{aligned}
-\frac{1}{c^2}\Big(\frac{\partial^2 F_1'}{\partial t^2} + \frac{\partial^2 F_2'}{\partial t'^2} + \frac{\partial^2 F_3'}{\partial t'^2} + \frac{\partial^2 F_4'}{\partial t'^2}\Big) \\
= \frac{\partial}{\partial t'}\Big(\frac{\partial}{\partial x}(F_2' - F_1' + F_4' - F_3') + \frac{\partial}{\partial y}(F_3' - F_1' + F_4' - F_2') \\
+ \frac{\partial}{\partial z}(-F_4' + F_3' + F_2' - F_1')\Big).
\end{aligned} \tag{28}
$$

5. Conclusion

The work succeeded in establishing in a single equation the coupling between three-dimensional waves. The problem posed in this paper can be applied in the following areas of physics:

(i) Quantum Mechanics;

(ii) Electromagnetism (in the treatment of electromagnetic waves);

Therefore, we believe the formula (28) is suitable for coupling waves in space.

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