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EXISTENCE OF WEAK POSITIVE SOLUTION FOR CLASS OF $(p_1, p_{2,...,}p_n)$-LAPLACIAN ELLIPTIC SYSTEM WITH DIFFERENT WEIGHTS

Rafik Guefaifia[1], Kamel Akrout[2], Tahar Bouali[3 §]
[1,2,3]Department of Mathematics
University Badji Mokhtar
Annaba, 23000, ALGERIA

Abstract: Consider the system

$$\begin{cases} -\Delta_{P_1,p_1} u_1 = \lambda a_1(x)\, u_1^{\alpha_{11}} u_2^{\alpha_{12}} ... u_n^{\alpha_{1n}}, & \text{in } \Omega, \\ -\Delta_{P_2,p_2} u_2 = \lambda a_2(x)\, u_1^{\alpha_{21}} u_2^{\alpha_{22}} ... u_n^{\alpha_{2n}}, & \text{in } \Omega, \\ \vdots \\ -\Delta_{Pn,p_n} u_n = \lambda a_n(x)\, u_1^{\alpha_{n1}} u_2^{\alpha_{n2}} ... u_n^{\alpha_{nn}}, & \text{in } \Omega, \\ u_i = 0, \quad \text{on } \partial\Omega, \quad i = 1, 2, ...n, \end{cases}$$

where Δ_{R_i,r_i} with $r_i > 1$ and $R_i = R_i(x)$ is a weights functions, denotes the weighted r_i -Laplacian defined by $\Delta_{R_i,r_i} u_i = \operatorname{div}\left(R_i(x)\, |\nabla u_i|^{r_i - 2} \nabla u_i\right), i = 1, 2, ...n, \lambda$ is a positive parameter, $a_i(x)$, $i = 1, 2, ...n$, are a weights functions,and Ω is a bounded domain in $\mathbb{R}^N$ $(N > 1)$ with smooth boundary $\partial\Omega$. We prove the existence of a large positive solutions for λ large, we use the method of sub–supersolutions to establish our results.

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[§]Correspondence author

1. Introduction

In this paper, we are concerned with the existence of positive weak solutions for the nonlinear elliptic system

$$\begin{cases} -\Delta_{P_1,p_1} u_1 = \lambda a_1(x)\, u_1^{\alpha_{11}} u_2^{\alpha_{12}} ... u_n^{\alpha_{1n}}, & \text{in } \Omega, \\ -\Delta_{P_2,p_2} u_2 = \lambda a_2(x)\, u_1^{\alpha_{21}} u_2^{\alpha_{22}} ... u_n^{\alpha_{2n}}, & \text{in } \Omega, \\ \quad .... \\ -\Delta_{Pn,p_n} u_n = \lambda a_n(x)\, u_1^{\alpha_{n1}} u_2^{\alpha_{n2}} ... u_n^{\alpha_{nn}}, & \text{in } \Omega, \\ u_i = 0, \quad \text{on } \partial\Omega, \quad i = 1,2,...n, \end{cases} \tag{1.1}$$

where Δ_{R_i,r_i} with $r_i > 1$ and $R_i = R_i(x)$ is a weights functions, denotes the weighted r_i -Laplacian defined by $\Delta_{R_i,r_i} u_i = \text{div}\left(R_i(x)\,|\nabla u_i|^{r_i - 2}\,\nabla u_i\right), i = 1,2,...n, \lambda$ is a positive parameter, $a_i(x)$, $i = 1,2,...n,$ are a weights functions,and Ω is a bounded domain in $\mathbb{R}^N$ $(N > 1)$ with smooth boundary $\partial\Omega$. In addition, we assume that $1 < p_i < N$.

Problems involving the p -Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [4]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

S.A. Khafagy [1] studied the existence and nonexistence of positive solution for the p-Laplacian system

$$\begin{cases} -\Delta_{P,p} u = \lambda a(x)\, v^{\beta} & \text{in } \Omega, \\ -\Delta_{Q,q} v = \lambda b(x)\, u^{\alpha} & \text{in } \Omega, \\ u = v = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

In this paper, we shall prove that if $\sum_{j=1}^{n} \alpha_{1j} < p_1 - 1, \sum_{j=1}^{n} \alpha_{2j} < p_2 - 1, ..., \sum_{j=1}^{n} \alpha_{nj} < p_n - 1.$ (1.1) admits a positive weak solution for each $\lambda > 0$ .Our approach is based on the method of sub- and supersolutions, see [3] .

This paper is organized as follows:

In Section 2, we introduce some technical results and notations, which are established in [5]. In Section 3, we prove the existence of a positive weak solutions for system (1.1) by using the method of sub–supersolutions.

2. Technical Results

Now, we introduce some technical results [5] concerning the degenerated homogeneous eigenvalue problem

$$\begin{cases} -\Delta_{R,r} u = -\text{div}\left(R(x)\,|\nabla u|^{r-2}\,\nabla u\right) = \lambda S(x)\,|u|^{r-2}\,u \text{ in } \Omega, \\ u = 0, \quad \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $R(x)$ and $S(x)$ are measurable functions satisfying

$$\frac{\upsilon(x)}{c} < R(x) < c\upsilon(x), \tag{2.2}$$

for a.e. $x \in \Omega$ with some constant $c \geq 1$, where $\upsilon(x)$ is a weight function in Ω satisfying the conditions

$$\upsilon \in L^1_{lok}(\Omega), \upsilon^{\frac{-1}{r-1}} \in L^1_{lok}(\Omega), \upsilon^{-s} \in L^1(\Omega) \quad \text{with } s \in \left(\frac{N}{r}, \infty\right) \cap [\frac{1}{r-1}, \infty), \tag{2.3}$$

and

$$0 \leq S(x) \in L^{\frac{k}{k-r}} \text{ for } a.e. x \in \Omega, \tag{2.4}$$

with some k satisfies $r < k < r_s^{\star}$ where $r_s^{\star} = \frac{Nr_s}{N-r_s}$ with $r_s = \frac{rs}{s+1} < r < r_s^{\star}$ and $meas\{x \in \Omega : S(x) > 0\} > 0$ .Examples of functions satisfying (2.2) and (2.3) are mentioned in [5] .

Lemma 2.1. *There exists the first eigenvalue* $\lambda_1^{(r)} > 0$ *and precisely one corresponding eigenfunction* $\phi_1^{(r)} \geq 0$ a.e. *in* Ω ($\phi_1^{(r)}$ *not identical to* 0) *of the eigenvalue problem* (2.1) . *Moreover, it is characterized by*

$$\lambda_1^{(r)} \int_\Omega S(x)\,u^r dx \leq \int_\Omega R(x)\,|\nabla u|^r\,dx \tag{2.5}$$

for all $u \in W_0^{1,p}(P,\Omega)$.

Lemma 2.2. *Let* $\phi_1^{(r)} \in W_0^{1,p}(P,\Omega)$, $\phi_1^{(r)} \geq 0$ a.e. *in* Ω , *be the eigenfunction corresponding to the first eigenvalue* $\lambda_1^{(r)} > 0$ *of the eigenvalue problem* (2.1) .*Then* $\phi_1^{(r)} \in L^{\infty}(\Omega)$.

Now, let us introduce the weighted Sobolev space $W_0^{1,p}(P,\Omega)$ which is the set of all real valued functions u defined in Ω for which (see [5])

$$\|u\|_{1,\upsilon,r} = \left[\int_\Omega |u|^r + \int_\Omega \upsilon(x)|\nabla u|^r\right]^{\frac{1}{r}} < \infty. \tag{2.6}$$

Since we are dealing with the Dirichlet problem, we introduce also the space $W_0^{1,r}(\upsilon,\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,r}(\upsilon,\Omega)$ with respect to the norm

$$\|u\|_{1,\upsilon,r} = \left[\int_\Omega \upsilon(x)|\nabla u|^r\right]^{\frac{1}{r}}, \tag{2.7}$$

which is equivalent to the norm given by (2.6) . Both spaces $W^{1,r}(\upsilon,\Omega)$ and $W_0^{1,r}(\upsilon,\Omega)$ are well defined reflexive Banach Spaces.

In this paper, we shall take $c=1$ in (2.2) i. e. $\upsilon(x)=R(x)$.

3. Existence Results

Throughout this paper, we let X be the Cartesian product of n spaces $W_0^{1,p_i}(P_i,\Omega)$ for $1\le i\le n$, i.e, $X=\prod_{i=1}^{n} W^{1,p_i}(P_i,\Omega)$. We give the definition of weak solution and sub-super solution of (1.1) as follows.

Definition 3.1. A pair of nonnegative functions $(\psi_1,\psi_2...,\psi_n),(z_1,z_2,...,z_n)$ in X are called a weak subsolution and supersolution of (1.1) if they satisfy $\psi_i\le z_i$ in Ω, for $i=1,2,...n$, and

$$\int_\Omega P_i(x)|\nabla\psi_i|^{p_i-2}\nabla\psi_i\nabla\omega_i dx \le \lambda\int_\Omega \prod_{j=1}^{n} a_i(x)\psi_j^{\alpha_{ij}}\omega_i dx$$

for $i=1,2,...n$ and

$$\int_\Omega P_i(x)|\nabla z_i|^{p_i-2}\nabla z_i\nabla\omega_i dx \ge \lambda\int_\Omega \prod_{j=1}^{n} a_i(x) z_j^{\alpha_{ij}}\omega_i dx$$

for $i=1,2,...n$ and for all $\omega_i(x)\in W_0^{1,p_i}(P_i,\Omega)$, with $\omega_i\ge 0$.

We shall obtain the existence of positive weak solution to system (1.1) by constructing a positive weak subsolution $(\psi_1,\psi_2...,\psi_n)$ and supersolution $(z_1,z_2,...,z_n)$. Our main result is formulated in the following theorem.

Theorem 3.2. *Suppose that* $\alpha_{ii} > 0, \alpha_{ij} \neq 0 \ (i \neq j)$, *and* $\sum_{j=1}^{n} \alpha_{1j} < p_1 - 1, \sum_{j=1}^{n} \alpha_{2j} < p_2 - 1, ..., \sum_{j=1}^{n} \alpha_{nj} < p_n - 1$, *Then system* (1.1) *has a positive weak solution for each* $\lambda > 0$.

Proof. Let $\lambda_1^{(i)}$ $(i = 1, 2, ...n)$ be the first eigenvalue of the problems,respectively

$$\begin{cases} -\Delta_{P_i,p_i} \phi_1^{(i)} = \lambda_1^{(i)} a_i(x) \left| \nabla \phi_1^{(i)} \right|^{p_i - 1} \phi_1^{(i)}, \text{in } \Omega \\ \phi_1^{(i)} = 0 \text{ on } \partial\Omega, i = 1, 2, ...n, \end{cases}$$

where $\phi_1^{(i)}$, denote the corresponding eigenfunctions, respectively, satisfying $\phi_1^{(i)} > 0$ in $\Omega, \nabla \phi_1^{(i)} > 0$ on $\partial\Omega$, and $\left\| \phi_1^{(i)} \right\|_{\infty} = 1$, for $i = 1, 2, ...n$.

We shall verify that

$$(\psi_1, \psi_2 ..., \psi_n) = \left(k \left(\frac{p_1 - 1}{p_1} \right) \left(\phi_1^{(1)} \right)^{\frac{p_1}{p_1 - 1}}, k \left(\frac{p_2 - 1}{p_2} \right) \left(\phi_1^{(2)} \right)^{\frac{p_2}{p_2 - 1}}, ..., k \left(\frac{p_n - 1}{p_n} \right) \left(\phi_1^{(n)} \right)^{\frac{p_n}{p_n - 1}} \right)$$

is a subsolution of (1.1), where $k > 0$ is small and specified later. Let $\omega_i(x) \in W_0^{1,p_i}(P_i, \Omega)$ with $\omega_i \geq 0$ $(i = 1, 2, ...n)$. A calculation shows that

$$\begin{aligned} \int_\Omega P_i(x) \left| \nabla \psi_i \right|^{p_i - 2} \nabla \psi_i \nabla \omega_i dx &= k^{p_i - 1} \int_\Omega P_i(x) \phi_1^{(i)} \left| \nabla \phi_1^{(i)} \right|^{p_i - 2} \nabla \phi_1^{(i)} \nabla \omega_i(x) dx \\ &= k^{p_i - 1} \left\{ \int_\Omega P_i(x) \left| \nabla \phi_1^{(i)} \right|^{p_i - 2} \nabla \phi_1^{(i)} \nabla \left(\phi_1^{(i)} \omega_i \right) dx \right. \\ &\quad \left. - \int_\Omega P_i(x) \left| \nabla \phi_1^{(i)} \right|^{p_i} \omega_i dx \right\} \\ &= k^{p_i - 1} \int_\Omega \left(\lambda_1^{(i)} a_i(x) \left(\phi_1^{(i)} \right)^{p_i} \right. \\ &\quad \left. - P_i(x) \left| \nabla \phi_1^{(i)} \right|^{p_i} \right) \omega_i dx, \end{aligned}$$

for $i = 1, 2, ...n$ .Since $\phi_1^{(i)} = 0$ and $\left| \nabla \phi_1^{(i)} \right| > 0$ on $\partial\Omega$, for $i = 1, 2, ...n$, there is $\delta > 0$ such that for $i = 1, 2, ...n$, we have

$$\lambda_1^{(i)} a_i(x) \left(\phi_1^{(i)} \right)^{p_i} - P_i(x) \left| \nabla \phi_1^{(i)} \right|^{p_i} \leq 0, \quad x \in \overline{\Omega}_\delta,$$

with $\overline{\Omega}_\delta = \{ x \in \Omega \backslash d(x, \partial\Omega) \leq \delta \}$. Now on $\overline{\Omega}_\delta$ we have

$$k^{p_i-1}\left(\lambda_1^{(i)}a_i(x)\left(\phi_1^{(i)}\right)^{p_i}-P_i(x)\left|\nabla\phi_1^{(i)}\right|^{p_i}\right)\le 0\le\lambda\prod_{j=1}^{n}a_i(x)\psi_i^{\alpha_{ij}}$$
$$(i=1,2,...n).$$

Next, we note that $\phi_1^{(i)}(x)\ge\eta>0$ in $\Omega_0=\Omega\setminus\overline{\Omega}_\delta$ for some $\eta>0$, and $i=1,2,...n$. Since for $i=1,2,...n$. we have $\sum_{i=1}^{n}\alpha_{ij}<p_i-1$, then there is $k_0>0$ such that if $k\in(0,k_0)$ we have

$$\begin{aligned}
&k^{p_i-1}\lambda_1^{(i)}a_i(x)\left(\phi_1^{(i)}\right)^{p_i-\alpha_{ii}\left(\frac{p_i}{p_i-1}\right)}\\
&\quad\le\lambda a_i(x)k^{\sum_{j=1}^{n}\alpha_{ij}}\left(\prod_{j=1}^{n}\left(\frac{p_j}{p_j-1}\right)^{\alpha_{ij}}\right)\left(\eta^{\sum_{j=2}^{n}\frac{\alpha_{ij}p_j}{p_j-1}}\right)\\
&\quad\le\lambda a_i(x)\left(\prod_{j=1}^{n}\left(\frac{p_j}{p_j-1}\right)^{\alpha_{ij}}\right)\left(\prod_{j=2}^{n}\left(\phi_1^{(i)}\right)^{\frac{\alpha_{ij}p_j}{p_j-1}}\right),\quad x\in\Omega_0,
\end{aligned}$$

for $i=1,2,...n$ .Then in Ω_0

$$k^{p_i-1}\left(\lambda_1^{(i)}a_i(x)\left(\phi_1^{(i)}\right)^{p_i}-P_i(x)\left|\nabla\phi_1^{(i)}\right|^{p_i}\right)\le\lambda\prod_{j=1}^{n}a_i(x)\psi_j^{\alpha_{ij}}$$

for $i=1,2,...n$

Hence

$$\begin{aligned}
\int_\Omega P_i(x)|\nabla\psi_i|^{p_i-2}\nabla\psi_i\nabla\omega_i dx &=\int_{\overline{\Omega}_\delta}P_i(x)|\nabla\psi_i|^{p_i-2}\nabla\psi_i\nabla\omega_i dx\\
&\quad+\int_{\Omega_0}P_i(x)|\nabla\psi_i|^{p_i-2}\nabla\psi_i\nabla\omega_i dx\\
&=k^{p_i-1}\int_{\overline{\Omega}_\delta}\left(\lambda_1^{(i)}a_i(x)\left(\phi_1^{(i)}\right)^{p_i}-P_i(x)\left|\nabla\phi_1^{(i)}\right|^{p_i}\right)\omega_i dx\\
&\quad+\int_{\Omega_0}\left(\lambda_1^{(i)}a_i(x)\left(\phi_1^{(i)}\right)^{p_i}-P_i(x)\left|\nabla\phi_1^{(i)}\right|^{p_i}\right)\omega_i dx\\
&\le\lambda\int_{\overline{\Omega}_\delta}\prod_{j=1}^{n}a_i(x)\psi_j^{\alpha_{ij}}\omega_i dx+\int_{\Omega_0}\prod_{j=1}^{n}a_i(x)\psi_j^{\alpha_{ij}}\omega_i dx\\
&\le\lambda\int_\Omega\prod_{j=1}^{n}a_i(x)\psi_j^{\alpha_{ij}}\omega_i dx
\end{aligned}$$

for $i=1,2,...n, i.e. \left(\psi_1,\psi_2...,\psi_n\right)$ is a subsolution of (1.1).

Next, let $\zeta_i(x)$ $(i=1,2,...n)$ be the positive solution of

$$\begin{cases} -\Delta_{P_i,p_i}\zeta_i = 1, \text{in } \Omega \\ \zeta_i = 0 \text{ on } \partial\Omega, i=1,2,...n, \end{cases}$$

For existence results of positive solutions for above boundary value problems see [9] . Let

$$\left(z_1,z_2,...,z_n\right) = \left(C_1\zeta_1(x), C_2\zeta_2(x),...,C_n\zeta_n(x)\right),$$

where $C_i>0$ are large numbers to be chosen later. We shall verify that $(z_1,z_2,...,z_n)$ is a supersolution of (1.1) . To this end,let $\omega_i(x)\in W_0^{1,p_i}(P_i,\Omega)$, with $\omega_i\geq 0$, for $i=1,2,...n$. Then we have

$$\begin{aligned}\int_\Omega P_i(x)\left|\nabla z_i\right|^{p_i-2}\nabla z_i\nabla\omega_i dx &= C_i^{p_i-1}\int_\Omega P_i(x)\left|\nabla\zeta_i\right|^{p_i-2}\nabla\zeta_i\nabla\omega_i dx \\ &= C_i^{p_i-1}\int_\Omega \omega_i dx,\end{aligned}$$

for $i=1,2,...n$. Let $l_i=\|\zeta_i\|_\infty$, $i=1,2,...n$. It is easy to prove that there exist positive large constants $C_1,C_2,...,C_n$ such that

$$\begin{aligned} C_1^{p_1-1-\alpha_{11}} &\geq \lambda\left(\prod_{j=2}^{n} C_j^{\alpha_{1j}}\right)\left(\prod_{j=1}^{n} l_j^{\alpha_{1j}}\right) \\ &\cdots \\ C_n^{p_n-1-\alpha_{nn}} &\geq \lambda\left(\prod_{j=1}^{n-1} C_j^{\alpha_{nj}}\right)\left(\prod_{j=1}^{n} l_j^{\alpha_{nj}}\right). \end{aligned}$$

Then for $i=1,2,...n$, we have

$$C_i^{p_i-1}\geq\lambda\left(\prod_{j=1}^{n}\left(C_il_i\right)^{\alpha_{ij}}\right)\geq\lambda\left(\prod_{j=1}^{n}\left(C_i\zeta_i\right)^{\alpha_{ij}}\right)=\lambda\prod_{j=1}^{n}z_i^{\alpha_{ij}}$$

and therefore

$$\int_\Omega P_i(x)\left|\nabla z_i\right|^{p_i-2}\nabla z_i\nabla\omega_i dx\geq\lambda\int_\Omega\prod_{j=1}^{n}a_i(x)\,z_i^{\alpha_{ij}}\omega_i dx$$

for $i=1,2,...n, i.e. \left(z_1,z_2,...,z_n\right)$ is a supersolution of (1.1) with $z_i\geq\psi_i$ in Ω for large $C_i, i=1,2,...n$. Thus, by the comparison principle, there exists a solution $(u_1,u_2,...,u_n)\in X$ of (1.1) with $\psi_i\leq u\leq z_i$, for $i=1,2,...n$.

This completes the proof of Theorem 1. □

References

[1] S.A. Khafagy, Existence and nonexistence of positive weak solutions for a class of (p,q) -Laplacian with different weights, *Int. J. Contemp. Math. Sciences*, **6**, No. 48 (2011), 2391-2400.

[2] C. Chen, On positive weak solutions for a class of quasilinear elliptic systems, *Nonlinear Analysis*, **62** (2005), 751-756, doi: 10.1016/j.na.2005.04.007.

[3] G. Astarita, G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics*, McGraw-Hill (1974).

[4] J. Ali, R. Shivaji, Existence results for classes of Laplacian systems with sign-changing weight, *Applied Mathematics Letters*, **20** (2007), 558–562.

[5] P. Drabek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equation with Degenerations And Singularities*, Walter de Gruyter, Bertin-New York (1997).

[6] D.D. Hai, R. Shivaji, An existence result on positive solutions for a class of p -Laplacian systems, *Nonlinear Analysis*, **56** (2004), 1007-1010.

[7] R. Dalmasso, Existence and uniqueness of positive solutions of semilinear elliptic systems, *Nonlinear Analysis*, **39** (2000), 559-568, doi: 10.1016/S0362-546X(98)00221-1.

[8] S. Haghaieghi, G.A. Afrouzi, Sub-super solutions for $(p-q)$ Laplacian systems, *Boundary Value Problems* (2011), 52.

[9] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York (1992).

[10] H.M. Serag, S.A. Khafagy, On a nonhomogeneous elliptic systems $n \times n$ involving p -Laplacian with different weights, *Journal of Advanced Research in Differential Equations*, **1**, No. 1 (2009), 47-62.