LEVITZKI RADICAL OF NOBUSAWA Γ-SEMIRING

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Abstract: In this paper we obtain the Levitzki radical of both sided Γ-semiring (Nobusawa Γ-semiring or Γ_N-semiring). Relation between the Levitzki radicals of a Γ_N-semiring S and that of the matrix semiring \( \left( \begin{array}{cc} R & \Gamma \\ S & L \end{array} \right) \) is obtained.

AMS Subject Classification: 16Y60, 16Y99, 20N10
Key Words: Γ_N-semiring, operator semiring, locally nilpotent ideal, Levitzki radical, matrix semiring

1. Introduction

Let S be a nonempty set and ‘+’ and ‘.’ be two binary operations on S, called addition and multiplication respectively. If \((S, +)\) is a commutative semigroup, \((S, .)\) is a semigroup, \(a.(b + c) = a.b + a.c\) and \((b + c).a = b.a + c.a\) for all \(a, b, c \in S\), there exists an element \(0 \in S\) such that \(a + 0 = a\) for all \(a \in S\) with \(a.0 = 0.0 = 0\) for all \(a \in S\) then we call \((S, +, .)\) a semiring. M.K.Rao\[7\] introduced the notion of Γ-semiring which was one sided. We introduced the notion of both sided Γ-semiring which we call a Nobusawa Γ-semiring or Γ_N-semiring\[8\]. Suppose \((S, +)\) and \((\Gamma, +)\) are two additive commutative monoids such that for all \(a, b, c \in S\) and \(\alpha, \beta, \gamma \in \Gamma\), 
\[a(\alpha + \beta)c = a\alpha c + a\beta c,\]
\[a\alpha(b + c) = aab + aac,\]
\[(a + b)\alpha c = aac + bac,\]
\[(aab)\beta c = a\alpha (b\beta c),\]
\[a0b = 0 = 0ab.\]
If \(S\) is Γ-semiring, Γ is \(S\)-semiring such that for all \(a, b, c \in S\), \(\alpha, \beta, \gamma \in \Gamma\),
\[(a_0a)b_0c = a(0ab_0)c = a\alpha(b_0c), (\alpha a_0b_0)\gamma = \alpha(a_0b_0)\gamma = a\alpha(\beta b_0\gamma)\] and for all \(s_1, s_2 \in S, s_1 \alpha s_2 = s_1 \beta s_2\) implies \(\alpha = \beta\), then \(S\) is called a Nobusawa \(\Gamma\)-semiring or simply \(\Gamma\)-semiring.

Dutta and Sardar\[4\] introduced the notion of operator semirings of a \(\Gamma\)-semiring. The operator semirings of a \(\Gamma\)-semiring turned out to be a very effective tool in transferring the notion of semirings to \(\Gamma\)-semirings. To make operator semiring effective, four mappings viz, \((\ )^+, (\ )^{'+}, (\ )^*, (\ )^{'+}\) play crucial role. These mappings arise for any \(\Gamma\)-semiring and for their operator semirings.

These mappings establish association between power sets of \(S\) and the power sets of operator semirings \(L\) and \(R\). In \[8\], we get some more mappings viz, \((\ )^+, (\ )^+, (\ )^*, (\ )^*, (\ )^\Gamma(\ )\) for any Nobusawa \(\Gamma\)-semiring. In \[2\], we define \((R \Gamma S L) := \{(r \gamma s l) : r \in R, l \in L, \gamma \in \Gamma, s \in S\}\) for a \(\Gamma\)-semiring \(S\) and the right operator semiring \(R\), left operator semiring \(L\). With respect to the addition and multiplication defined as follows:

\[
\begin{pmatrix}
r_1 & \gamma_1 \\
s_1 & l_1
\end{pmatrix}
\begin{pmatrix}
r_2 & \gamma_2 \\
s_2 & l_2
\end{pmatrix} =
\begin{pmatrix}
r_1 + r_2 & \gamma_1 + \gamma_2 \\
\gamma_1 + \gamma_2 & s_1 + s_2 & l_1 + l_2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
r_1 & \gamma_1 \\
s_1 & l_1
\end{pmatrix}
\begin{pmatrix}
r_2 & \gamma_2 \\
s_2 & l_2
\end{pmatrix} =
\begin{pmatrix}
r_1 r_2 + [\gamma_1, s_2] & r_1 \gamma_2 + \gamma_1 l_2 \\
\gamma_1 + \gamma_2 & r_1 r_2 + [\gamma_1, s_2] & \gamma_1 + \gamma_2 & s_1 + s_2 & l_1 + l_2
\end{pmatrix},
\]

\((R \Gamma S L)\) forms a semiring.

Olson et al \[6\] studied a lot on radicals of semiring. Dutta et al studied the Levitzki radical\[3\] of one sided \(\Gamma\)-semiring using its operator semirings\[4\]. In this paper we investigate the Levitzki radical of a Nobusawa \(\Gamma\)-semiring \(S\) and that of the matrix semiring \((R \Gamma S L)\). Let \(L\) and \(R\) be the left operator semiring and right operator semiring of a Nobusawa \(\Gamma\)-semiring \(S\) respectively.

If there exists an element \(\sum_{i=1}^{m} [e_i, \delta_i] \in L\) \((\sum_{j=1}^{n} [\gamma_j, x_j] \in R)\) such that \(\sum_{i=1}^{m} e_i \delta_i a = a\)

\((\sum_{j=1}^{n} a \gamma_j x_j = a)\) for all \(a \in S\) then \(S\) is said to have the left unity \(\sum_{i=1}^{m} [e_i, \delta_i]\)

(respectively right unity \(\sum_{j=1}^{n} [\gamma_j, x_j]\)). Readers are referred to \[4, 5, 8, 9\] for the requisite preliminaries on semirings and \(\Gamma\)-semirings.

Throughout this paper \(S\) denotes a Nobusawa \(\Gamma\)-semiring with unities if not mentioned.
2. Levitzki Radical in $\Gamma_N$-Semiring

An ideal $I$ of a semiring $R$ is said to be locally nilpotent if for any finite set $F \subseteq I$ there exist a positive integer $m$ such that $F^m = \{0\}$. Let $S$ be a Nobusawa $\Gamma$–semiring with unities. An ideal $P$ of $S$ is said to be locally nilpotent if for any finite set $F \subseteq P$ and any finite set $\Delta \subseteq \Gamma$ there exist a positive integer $n$ such that $(F\Delta)^{n-1}F = \{0\}$. An ideal $\Phi$ of $\Gamma$ is said to be locally nilpotent if for any finite set $\Delta \subseteq \Phi$ and any finite set $A \subseteq S$ there exist a positive integer $m$ such that $(\Delta A)^{m-1}\Delta = \{\theta\}$, $\theta$ is the zero element of $\Gamma$.

**Property 1.** Let $\sum_{i=1}^{m} [e_i, \delta_i]$ be the left unity and $\Phi$ be a right ideal of the $S$-semiring $\Gamma$ then $^+\Phi = \{\sum_{i=1}^{m} [e_i, \alpha_i] : \alpha_i \in \Phi\}$.

**Proof.** If $\alpha_1, \alpha_2, \ldots, \alpha_m \in \Phi$ then for any $\xi \in \Gamma$, $\alpha_i e_i \xi \in \Phi(i = 1, 2, \ldots, m)$. Then $[e_i, \alpha_i] \in ^+ \Phi$ for all $i = 1, 2, \ldots, m$. This implies that $\sum_{i=1}^{m} [e_i, \alpha_i] \in ^+ \Phi$.

Conversely: Let $\sum_{k=1}^{t} [x_k, \mu_k] \in ^+ \Phi$.

Then

$$\sum_{k=1}^{t} [x_k, \mu_k] = \sum_{i=1}^{m} [e_i, \delta_i] \sum_{k=1}^{t} [x_k, \mu_k] = \sum_{ik} [e_i, \delta_i x_k \mu_k] = \sum_{i=1}^{m} [e_i, \sum_{k=1}^{t} \delta_i x_k \mu_k].$$

Since $\sum_{k=1}^{t} \delta_i x_k \mu_k \in \Phi$ for all $i = 1, 2, \ldots, m$ the result follows. $\square$

The following proposition can be proved in an analogous manner.

**Property 2.** Let $\sum_{j=1}^{n} [\gamma_j, f_j]$ be the right unity and $\Phi$ be a left ideal of the $S$-semiring $\Gamma$ then $^*\Phi = \{\sum_{j=1}^{n} [\lambda_j, f_j] : \lambda_j \in \Phi\}$.

**Property 3.** If $\Phi$ is a locally nilpotent ideal of the $S$-semiring $\Gamma$ then $^*\Phi (^+\Phi) \Phi$ is a locally nilpotent ideal of $R$ (respectively of $L$).
Proof. Let $\Phi$ be a locally nilpotent ideal of the $S$-semiring $\Gamma$. Then by the Note under the Theorem 4.11 of \[8\], $^*\Phi$ is an ideal of $R$. Now by the Proposition 2, $^*\Phi = \{ \sum_{j=1}^{n} [\lambda_{j}, f_{j}] : \lambda_{j} \in \Phi, j = 1, 2, \ldots, n \}$ where $\sum_{j=1}^{n} [\gamma_{j}, f_{j}]$ be the right unity of $S$. Let $[\Delta, F]$ be a finite subset of $^*\Phi$. Then $\Delta$ is a finite subset of $\Phi$ and $F$ is a finite subset of $S$. Consequently, $\Delta F \Gamma \subseteq \Phi$ which implies that $\Delta F \Delta \subseteq \Phi$. Now $\Delta F \Delta$ is a finite subset of $\Phi$. Since $\Phi$ is locally nilpotent, there exists a positive integer $n$ such that $(\Delta F \Delta)^{n-1}(\Delta F \Delta) = \{ \theta \}$, $\theta$ is the zero element of $\Gamma$. This implies that $(\Delta F)^{2n-1}\Delta = \{ \theta \}$. Hence $[\Delta, F]^{2n} = ([\Delta F])^{2n-1}\Delta, [\theta, F] = [0_{R}]$. Hence $^*\Phi$ is a locally nilpotent ideal of $R$. Analogously the proof for the left operator semiring $L$ can be done.

Property 4. If $P$ is a locally nilpotent ideal of $R(L)$ then $^*P$ (respectively $+P$) is a locally nilpotent ideal of $S$-semiring $\Gamma$.

Proof. Let $P$ be a locally nilpotent ideal of the $R$. Then $^*P$ is an ideal of the $S$-semiring $\Gamma$. Let $\Delta$ be a finite subset of $P^*$ and $F$ be a finite subset of $S$. Then $\{[\delta, f] : \delta \in \Delta, f \in F \}$ is a finite subset of $P$. Let the subset be denoted by $[(\Delta, F)]$. Since $P$ is locally nilpotent, there exists a positive integer $n$ such that $[(\Delta, F)]^{n} = \{0_{R}\}$. i.e., every product of $n$ elements of $P$ of the form $[\delta, f]$ where $\delta \in \Delta, f \in F$ is zero. Now $[\Delta, F]^{n}$ consists of finite sums where each summand is the product of $n$ elements of the type $[\delta, f]$ where $\delta \in \Delta, f \in F$. So $[\Delta, F]^{n} = \{0_{R}\}$. Hence $(\Delta F)^{n}\Delta = (\Delta F \Delta F \ldots \Delta F) \Delta = ([\Delta, F][\Delta, F] \ldots [\Delta, F]) \Delta = [\Delta, F]^{n} \Delta = \{\theta\}$. Hence $^*P$ is a locally nilpotent ideal of $S$-semiring $\Gamma$. Analogously the proof for the left operator semiring $L$ can be done.

Now by the above two propositions and the Theorem 4.15 of [8], we have the following proposition.

Property 5. $\Gamma$ is locally nilpotent if and only if $R(L)$ is locally nilpotent.

Now by above result and Proposition 3.6 of [3], we have the following proposition.

Property 6. $\Gamma$ is locally nilpotent if and only if $S$ is locally nilpotent.

Property 7. Sum of two locally nilpotent ideals of the $S$-semiring $\Gamma$ is locally nilpotent.

Proof. This follows from the Proposition 3.8 of [3].

Now by using the Propositions 3, 4, 7 and Theorem 4.15 of [8], we have the following theorem.
Theorem 8. There exists an inclusion preserving bijection between the locally nilpotent ideals of the \( S \)-semiring \( \Gamma \) and the right operator semiring \( R \) (the left operator semiring \( L \)) via the mapping \( \Phi \to \Phi^* \) (respectively \( \Phi \to \Phi^+ \)).

Levitzki radical of a semiring \( S \) is denoted by \( \mathcal{L}(S) \) and defined as the sum of all locally nilpotent ideals of \( S \). Let \( S \) be a Nobusawa \( \Gamma \)-semiring, then the Levitzki radical of \( S \) is the sum of all locally nilpotent ideals of \( S \) of \( \Gamma \) is denoted by \( \mathcal{L}(\Gamma) \) and defined as the sum of all locally nilpotent ideals of \( \Gamma \).

Theorem 9. Let \( S \) be a \( \Gamma_N \)-semiring and \( R \) and \( L \) be respectively the right and left operator semiring of \( S \). Then (i) \( \Gamma \mathcal{L}(\Gamma) = \mathcal{L}(R) \) and \( \mathcal{L}(\Gamma) = \mathcal{L}^+(R) \) and (ii) \( \mathcal{L}(\Gamma) = \mathcal{L}(L) \) and \( \mathcal{L}(\Gamma) = \mathcal{L}^+(L) \).

Proof. (i) Since \( R \) is a semiring, \( \mathcal{L}(R) \) is a locally nilpotent ideal of \( S \) (cf. Lemma 4 [1]). Then by the Proposition 4, \( \mathcal{L}(R) \) is a locally nilpotent ideal of \( \Gamma \). Hence \( \mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma)(\text{since } \mathcal{L}(\Gamma) \text{ is the sum of all locally nilpotent ideal of } \Gamma) \). Now let \( \Delta \) be a locally nilpotent ideal of \( \Gamma \). Then by Proposition 3, \( \mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma) \) (since \( \mathcal{L}(\Gamma) \) is the sum of all locally nilpotent ideal of \( \Gamma \)).

Property 10. If \( I \) is a locally nilpotent ideal of the \( \Gamma \)-semiring \( S \) then \( \Gamma(I) \) is a locally nilpotent ideal of \( S \)-semiring \( \Gamma \).

Proof. Let \( I \) be a locally nilpotent ideal of the \( \Gamma \)-semiring \( S \). Then by Proposition 3.6 of [3], \( I^* \) is a locally nilpotent ideal of \( R \). Consequently, by Proposition 4, \( \mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma) \) (since \( \mathcal{L}(\Gamma) \) is the sum of all locally nilpotent ideal of \( R \)). Thus every locally nilpotent ideal of \( \Gamma \) is contained in \( \mathcal{L}(\Gamma) \). Hence \( \mathcal{L}(\Gamma) = \mathcal{L}(\Gamma) \).

Analogously we can prove (ii).

Property 11. If \( \Phi \) is a locally nilpotent ideal of the \( S \)-semiring \( \Gamma \) then \( \mathcal{S}(\Phi) \) is a locally nilpotent ideal of \( \Gamma \)-semiring \( S \).

Theorem 12. There exists a one one correspondence between the locally nilpotent ideals of the \( S \)-semiring \( \Gamma \) and the \( \Gamma \)-semiring \( S \) via the mapping \( I \to \Gamma(I) \).
Proof. The theorem will be proved if we can prove \( I = S(\Gamma(I)) \), for an ideal \( I \) of \( S \). Infact \( S \) has the unities and hence we can write \( S(\Gamma(I)) = (I^*)^* \). Then by the Theorem 4.15 of [8], we have \( S(\Gamma(I)) = (I^*)^* \). Hence by Theorem 6.6 of [4], we have \( S(\Gamma(I)) = I \).

In view of the above theorem we can get the following theorem.

**Theorem 13.** \( L(\Gamma) = \Gamma(L(S)) \).

### 3. Main Results

For a Nobusawa \( \Gamma \)-semiring \( S \), one can define a semiring

\[
S_2 = \begin{pmatrix}
R & \Gamma \\
S & L
\end{pmatrix} = \left\{ \begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} : r \in R, l \in L, \gamma \in \Gamma, s \in S \right\},
\]

with respect to the addition and multiplication defined as follows:

\[
\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & \gamma_1 + \gamma_2 \\ s_1 + s_2 & l_1 + l_2 \end{pmatrix}
\]

and

\[
\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + [\gamma_1, s_2] & r_1 \gamma_2 + \gamma_1 l_2 \\ s_1 r_2 + l_1 s_2 & [s_1, \gamma_2] + l_1 l_2 \end{pmatrix}.
\]

If \( I \) be an ideal of \( S \). Then \( I_2 = \begin{pmatrix} I^* & \Gamma(I) \\ I & I^+ \end{pmatrix} \) is an ideal of \( S_2 \). Moreover, if \( I \) and \( J \) are two distinct ideals of \( S \) then \( I_2 \neq J_2 \). Also every ideal of \( S_2 \) is of the form \( I_2 = \begin{pmatrix} I^* & \Gamma(I) \\ I & I^+ \end{pmatrix} \), for an ideal \( I \) of \( S \).

**Property 14.** \( I_2 \) is locally nilpotent if and only if \( I \) is locally nilpotent.

**Proof.** Let \( I_2 \) be locally nilpotent and \( F \) be a finite subset of \( I \) and \( \Delta \) be a finite subset of \( \Gamma \). Now \( \begin{pmatrix} [\Delta, F] & \theta \\ F & 0_L \end{pmatrix} \) is a finite subset of \( I_2 \). Consequently, there exists a positive integer \( n \) such that \( \begin{pmatrix} [\Delta, F] & \theta \\ F & 0_L \end{pmatrix}^{n+1} = 0 \). Then \( (F\Delta)^n F = 0 \). Hence \( I \) is locally nilpotent.

Conversely, let \( I \) be locally nilpotent and \( F_2 \) be a finite subset of \( I_2 \). Then we can find finite subsets \( A \subseteq I \) and \( \Phi \subseteq \Gamma \) such that \( F_2 \subseteq \begin{pmatrix} \Phi, A \\ A \end{pmatrix} \).
Since $I$ is locally nilpotent there exists positive integer $n$ such that $(A\Phi)^n A = 0$. Then $F_2^{n+1} \subseteq \left( \begin{array}{cc} [\Phi, A] & \Phi A \Phi \\ A & [A, \Phi] \end{array} \right)^{n+1} = 0$. Hence $I_2$ is locally nilpotent.

**Theorem 15.** \( \mathcal{L}(S_2) = \left( \begin{array}{cc} \mathcal{L}(R) & \mathcal{L}(\Gamma) \\ \mathcal{L}(S) & \mathcal{L}(L) \end{array} \right) \)

**Proof.** \( \mathcal{L}(S_2) \) is the sum of locally nilpotent ideals of $S_2$ which is of the form 
\[
\begin{pmatrix} A^\ast & \Gamma(A) \\ A & A^+ \end{pmatrix}
\]
where $A$ is a ideal of $S$. Then by the Proposition 14, $A$ is a locally nilpotent ideal. Again by Proposition 3, $A^\ast$ and $A^+$ are locally nilpotent ideal, by Proposition 10, $\Gamma(A)$ is locally nilpotent ideal. Then by the Theorem 3.10 and Note 3.11 of [3] and Theorem 4, we get 
\[
\mathcal{L}(S_2) = \left( \begin{array}{cc} \mathcal{L}(R) & \mathcal{L}(\Gamma) \\ \mathcal{L}(S) & \mathcal{L}(L) \end{array} \right).
\]

**Acknowledgments**

This paper is an output of the minor research product ‘The radicals of the matrix semiring \( \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix} \)’. The author is thankful to University Grant Commission of India for supporting the minor research project.

The author is also grateful to Dr. Sujit Kumar Sardar, Department of Mathematics, Jadavpur University, INDIA for his valuable suggestions.

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