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LEVITZKI RADICAL OF NOBUSAWA Γ-SEMIRING

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Abstract: In this paper we obtain the Levitzki radical of both sided $\Gamma-$semiring (Nobusawa $\Gamma-$ semiring or Γ_N-semiring). Relation between the Levitzki radicals of a Γ_N-semiring S and that of the matrix semiring $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$ is obtained.

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1. Introduction

Let S be a nonempty set and '+' and '.' be two binary operations on S, called addition and multiplication respectively. If $(S,+)$ is a commutative semigroup, $(S,.)$ is a semigroup , $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$ for all $a,b,c \in S$, there exists an element $0 \in$S such that $a + 0 = a$ for all $a \in$S with $a.0 = 0.a = 0$ for all $a \in$S then we call $(S,+,.)$ a semiring. M.K.Rao[7] introduced the notion of $\Gamma-$semiring which was one sided. We introduced the notion of both sided $\Gamma-$semiring which we call a Nobusawa $\Gamma-$semiring or Γ_N-semiring[8]. Suppose (S,+) and $(\Gamma,+)$ are two additive commutative monoids such that for all $a,b,c \in S$ and $\alpha,\beta \in \Gamma$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$, $(a+b)\alpha c = a\alpha c + b\alpha c$, $(a\alpha b)\beta c = a\alpha(b\beta c)$, $a0b = 0 = 0\alpha b$. If S is $\Gamma-$semiring, Γ is S-semiring such that for all $a,b,c \in S$, $\alpha,\beta,\gamma \in \Gamma$,

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$(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$, $(\alpha a\beta)b\gamma = \alpha(a\beta b)\gamma = \alpha a(\beta b\gamma)$ and for all s_1, $s_2 \in S$, $s_1\alpha s_2 = s_1\beta s_2$ implies $\alpha = \beta$, then S is called a Nobusawa Γ-semiring or simply Γ_N-semiring.

Dutta and Sardar[4] introduced the notion of operator semirings of a Γ-semiring. The operator semirings of a Γ−semiring turned out to be a very effective tool in transferring the notion of semirings to Γ−semirings. To make operator semiring effective, four mappings viz, $(\ )^+$,$(\ )^{+'}$, $(\ )^*$, $(\ )^{*'}$ play crucial role. These mappings arise for any Γ−semiring and for their operator semirings. These mappings establish association between power sets of S and the power sets of operator semirings L and R. In [8], we get some more mappings viz, ${}^+(\ )$, ${}^{+'}(\ )$, ${}^*(\ )$, ${}^{*'}(\ )$, S( ), Γ( ) for any Nobusawa Γ−semiring. In [2], we define $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix} := \left\{ \begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} : r \in R, l \in L, \gamma \in \Gamma, s \in S \right\}$ for a Γ_N−semiring S and the right operator semiring R, left operator semiring L. With respect to the addition and multiplication defined as follows:

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & \gamma_1 + \gamma_2 \\ s_1 + s_2 & l_1 + l_2 \end{pmatrix}$$

and

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + [\gamma_1, s_2] & r_1\gamma_2 + \gamma_1 l_2 \\ s_1 r_2 + l_1 s_2 & [s_1, \gamma_2] + l_1 l_2 \end{pmatrix},$$

$\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$ forms a semiring.

Olson et al [6] studied a lot on radicals of semiring. Dutta et al studied the Levitzki radical[3] of one sided Γ−semiring using its operator semirings[4]. In this paper we investigate the Levitzki radical of a Nobusawa Γ−semiring S and that of the matrix semiring$\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$. Let L and R be the left operator semiring and right operator semiring of a Nobusawa Γ−semiring S respectively. If there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in$ L $(\sum_{j=1}^{n}[\gamma_j, x_j] \in R)$ such that $\sum_{i=1}^{m} e_i\delta_i a = a$ $(\sum_{j=1}^{n} a\gamma_j x_j = a)$ for all $a \in S$ then S is said to have the left unity $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively right unity $\sum_{j=1}^{n}[\gamma_j, x_j]$). Readers are referred to [4, 5, 8, 9] for the requisite preliminaries on semirings and Γ−semiring.

Through out this paper S denotes a Nobusawa Γ−semiring with unities if not mentioned.

2. Levitzki Radical in Γ_N-Semiring

An ideal I of a semiring R is said to be locally nilpotent if for any finite set $F \subseteq I$ there exist a positive integer m such that $F^m = \{0\}$. Let S be a Nobusawa $\Gamma-$semiring with unities. An ideal P of S is said to be locally nilpotent if for any finite set $F \subseteq P$ and any finite set $\Delta \subseteq \Gamma$ there exist a positive integer n such that $(F\Delta)^{n-1}F = \{0\}$. An ideal Φ of Γ is said to be locally nilpotent if for any finite set $\Delta \subseteq \Phi$ and any finite set $A \subseteq$S there exist a positive integer m such that $(\Delta A)^{m-1}\Delta = \{\theta\}$, θ is the zero element of Γ.

Property 1. *Let $\sum_{i=1}^{m}[e_i, \delta_i]$ be the left unity and Φ be a right ideal of the S-semiring Γ then ${}^{+'}\Phi = \{\sum_{i=1}^{m}[e_i, \alpha_i] : \alpha_i \in \Phi\}$.*

Proof. If $\alpha_1, \alpha_2, ...\alpha_m \in \Phi$ then for any $\xi \in \Gamma$, $\alpha_i e_i \xi \in \Phi(i = 1, 2, ...m)$. Then $[e_i, \alpha_i] \in^{+'} \Phi$ for all $i = 1, 2, ...m$. This implies that $\sum_{i=1}^{m}[e_i, \alpha_i] \in^{+'} \Phi$.

Conversely: Let $\sum_{k=1}^{t}[x_k, \mu_k] \in^{+'} \Phi$.

Then

$$\sum_{k=1}^{t}[x_k, \mu_k] = \sum_{i=1}^{m}[e_i, \delta_i]\sum_{k=1}^{t}[x_k, \mu_k] = \sum_{ik}[e_i, \delta_i x_k \mu_k] = \sum_{i=1}^{m}[e_i, \sum_{k=1}^{t}\delta_i x_k \mu_k].$$

Since $\sum_{k=1}^{t}\delta_i x_k \mu_k \in \Phi$ for all $i = 1, 2, ...m$ the result follows. □

The following proposition can be proved in an analogous manner.

Property 2. *Let $\sum_{j=1}^{n}[\gamma_j, f_j]$ be the right unity and Φ be a left ideal of the S-semiring Γ then ${}^{*'}\Phi = \{\sum_{j=1}^{n}[\lambda_j, f_j] : \lambda_j \in \Phi\}$.*

Property 3. *If Φ is a locally nilpotent ideal of the S-semiring Γ then ${}^{*'}\Phi({}^{+'}\Phi)$ is a locally nilpotent ideal of R(respectively of L).*

Proof. Let Φ be a locally nilpotent ideal of the S-semiring Γ. Then by the Note under the Theorem 4.11 of [8], ${}^{*'}\Phi$ is an ideal of R. Now by the Proposition 2, ${}^{*'}\Phi = \{\sum_{j=1}^{n}[\lambda_j, f_j] : \lambda_j \in \Phi, j = 1, 2, ...n\}$ where $\sum_{j=1}^{n}[\gamma_j, f_j]$ be the right unity of S. Let $[\Delta, F]$ be a finite subset of ${}^{*'}\Phi$. Then Δ is a finite subset of Φ and F is a finite subset of S. Consequently, $\Delta F\Gamma \subseteq \Phi$ which implies that $\Delta F\Delta \subseteq \Phi$. Now $\Delta F\Delta$ is a finite subset of Φ. Since Φ is locally nilpotent, there exists a positive integer n such that $((\Delta F\Delta)F)^{n-1}(\Delta F\Delta) = \{\theta\}$, θ is the zero element of Γ. This implies that $(\Delta F)^{2n-1}\Delta = \{\theta\}$. Hence $[\Delta, F]^{2n}$=$[(\Delta F)^{2n-1}\Delta$, F]=$[\theta$, F] =$\{0_R\}$. Hence ${}^{*'}\Phi$ is a locally nilpotent ideal of R. Analogously the proof for the left operator semiring L can be done. □

Property 4. *If P is a locally nilpotent ideal of R(L) then* $^*P$*(respectively* ^+P*) is a locally nilpotent ideal of S-semiring* Γ.

Proof. Let P be a locally nilpotent ideal of the R. Then *P is an ideal of the S-semiring Γ. Let Δ be a finite subset of P^* and F be a finite subset of S. Then $\{[\delta, f] : \delta \in \Delta, f \in F\}$ is a finite subset of P. Let the subset be denoted by $[(\Delta, F)]$. Since P is locally nilpotent, there exists a positive integer n such that $[(\Delta, F)]^n = \{0_R\}$. i.e., every product of n elements of P of the form $[\delta, f]$ where $\delta \in \Delta$, $f \in F$ is zero. Now $[\Delta, F]^n$ consists of finite sums where each summand is the product of n elements of the type $[\delta, f]$ where $\delta \in \Delta$, $f \in F$. So $[\Delta, F]^n = \{0_R\}$. Hence $(\Delta F)^n\Delta$=$(\Delta F\Delta F....\Delta F)\Delta$=$([\Delta, F][\Delta, F]......[\Delta, F])\Delta$=$[\Delta, F]^n\Delta = \{\theta\}$. Hence $^*P$ is a locally nilpotent ideal of S-semiring Γ.Analogously the proof for the left operator semiring L can be done. □

Now by the above two propositions and the Theorem 4.15 of [8], we have the following proposition.

Property 5. Γ *is locally nilpotent if and only if R(L) is locally nilpotent.*

Now by above result and Proposition 3.6 of [3], we have the following proposition.

Property 6. Γ *is locally nilpotent if and only if S is locally nilpotent.*

Property 7. *Sum of two locally nilpotent ideals of the S-semiring* Γ *is locally nilpotent.*

Proof. This follows from the Proposition 3.8 of [3]. □

Now by using the Propositions 3, 4, 7 and Theorem 4.15 of [8], we have the following theorem.

Theorem 8. *There exists an inclusion preserving bijection between the locally nilpotent ideals of the S-semiring* Γ *and the right operator semiring R (the left operator semiring L) via the mapping* $\Phi \rightarrow^{*} \Phi$*(respectively* $\Phi \rightarrow^{+} \Phi$*).*

Levtzki radical of a semiring S is denoted by $\mathcal{L}$(S) and defined as the sum of all locally nilpotent ideals of S. Let S be a Nobusawa $\Gamma-$semiring, then the Levtzki radical of S is the sum of all locally nilpotent ideals of S of Γ is denoted by $\mathcal{L}(\Gamma)$ and defined as the sum of all locally nilpotent ideals of Γ.

Theorem 9. *Let S be a* Γ_N-*semiring and R and L be respectively the right and left operator semiring of S. Then (i)* ${}^{*'}\mathcal{L}(\Gamma) = \mathcal{L}(R)$ *and* $\mathcal{L}(\Gamma) =^{*} \mathcal{L}(R)$ *and (ii)* ${}^{+'}\mathcal{L}(\Gamma) = \mathcal{L}(L)$ *and* $\mathcal{L}(\Gamma) =^{+} \mathcal{L}(L)$*.*

Proof. (i) Since R is a semiring, $\mathcal{L}$(R) is a locally nilpotent ideal of S(cf. Lemma 4 [1]). Then by the Proposition 4, ${}^{*}\mathcal{L}$(R) is a locally nilpotent ideal of Γ. Hence ${}^{*}\mathcal{L}(R) \subseteq \mathcal{L}(\Gamma)$(since $\mathcal{L}(\Gamma)$ is the sum of all locally nilpotent ideal of Γ). Now let Δ be a locally nilpotent ideal of Γ. Then by Proposition 3, ${}^{*'}\Delta$ is a locally nilpotent ideal of R. Consequently, ${}^{*'}\Delta \subseteq \mathcal{L}(R)$(since $\mathcal{L}$(R) is the sum of all locally nilpotent ideal of R) which implies that ${}^{*}({}^{*'}\Delta) \subseteq {}^{*}(\mathcal{L}(R))$ i.e., $\Delta \subseteq {}^{*}(\mathcal{L}(R))$ (Theorem 4.15 [8]). Thus every locally nilpotent ideal of Γ is contained in ${}^{*}\mathcal{L}$(R) which implies that $\mathcal{L}(\Gamma) \subseteq^{*} \mathcal{L}(R)$. Hence $\mathcal{L}(\Gamma) =^{*} \mathcal{L}(R)$. Then $\mathcal{L}(\Gamma)$ is a locally nilpotent ideal of Γ and ${}^{*'}\mathcal{L}(\Gamma) =^{*'} ({}^{*}\mathcal{L}(R)) = \mathcal{L}(R)$ (Theorem 4.15 [8]). Similarly we can prove (ii). □

Property 10. *If I is a locally nilpotent ideal of the* $\Gamma-$*semiring S then* $\Gamma(I)$ *is a locally nilpotent ideal of S-semiring* Γ*.*

Proof. Let I be a locally nilpotent ideal of the $\Gamma-$semiring S. Then by Proposition 3.6 of [3], $I^{*'}$ is a locally nilpotent ideal of R. Consequently, by Proposition 4, ${}^{*}(I^{*'})$ is a locally nilpotent ideal of the S-semiring Γ. Now $I^{*'}=[\Gamma, I]$ and ${}^{*}(I^{*'})=I^{*'}\Gamma=[\Gamma, I]\Gamma=\Gamma I \Gamma=\Gamma(I)$. i.e., $\Gamma(I) =^{*} (I^{*'})$. Hence $\Gamma(I)$ is a locally nilpotent ideal of S-semiring Γ. □

Analogously we can prove the following proposition.

Property 11. *If* Φ *is a locally nilpotent ideal of the S-semiring* Γ *then* $S(\Phi)$ *is a locally nilpotent ideal of* $\Gamma-$*semiring S.*

Theorem 12. *There exists a one one correspondence between the locally nilpotent ideals of the S-semiring* Γ *and the* $\Gamma-$*semiring S via the mapping* $I \rightarrow \Gamma(I)$*.*

Proof. The theorem will be proved if we can prove $I = S(\Gamma(I))$, for an ideal I of S. Infact S has the unities and hence we can write $S(\Gamma(I)) = S({}^*(I^{*'})) = ({}^{*'}({}^*(I^{*'})))^*$. Then by the Theorem 4.15 of [8], we have $S(\Gamma(I)) = (I^{*'})^*$. Hence by Theorem 6.6 of [4], we have $S(\Gamma(I)) = I$. □

In view of the above theorem we can get the following theorem.

Theorem 13. $\mathcal{L}(\Gamma) = \Gamma(\mathcal{L}(S))$.

3. Main Results

For a Nobusawa $\Gamma-$semiring S, one can define a semiring

$$S_2 = \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix} = \left\{ \begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} : r \in R, l \in L, \gamma \in \Gamma, s \in S \right\},$$

with respect to the addition and multiplication defined as follows:

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & \gamma_1 + \gamma_2 \\ s_1 + s_2 & l_1 + l_2 \end{pmatrix}$$

and

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + [\gamma_1, s_2] & r_1\gamma_2 + \gamma_1 l_2 \\ s_1 r_2 + l_1 s_2 & [s_1, \gamma_2] + l_1 l_2 \end{pmatrix}.$$

If I be an ideal of S. Then $I_2 = \begin{pmatrix} I^{*'} & \Gamma(I) \\ I & I^{+'} \end{pmatrix}$ is an ideal of S_2. Moreover, if I and J are two distinct ideals of S then $I_2 \neq J_2$. Also every ideal of S_2 is of the form $I_2 = \begin{pmatrix} I^{*'} & \Gamma(I) \\ I & I^{+'} \end{pmatrix}$, for an ideal I of S.

Property 14. *I_2 is locally nilpotent if and only if I is locally nilpotent.*

Proof. Let I_2 be locally nilpotent and F be a finite subset of I and Δ be a finite subset of Γ. Now $\begin{pmatrix} [\Delta, F] & \theta \\ F & 0_L \end{pmatrix}$ is a finite subset of I_2. Consequently, there exists a positive integer n such that $\begin{pmatrix} [\Delta, F] & \theta \\ F & 0_L \end{pmatrix}^{n+1} = 0$. Then $(F\Delta)^n F = 0$. Hence I is locally nilpotent.

Conversely, let I be locally nilpotent and F_2 be a finite subset of I_2. Then we can find finite subsets $A \subseteq I$ and $\Phi \subseteq \Gamma$ such that $F_2 \subseteq \begin{pmatrix} [\Phi, A] & \Phi A \Phi \\ A & [A, \Phi] \end{pmatrix}$.

Since I is locally nilpotent there exists positive integer n such that $(A\Phi)^n A = 0$. Then $F_2^{n+1} \subseteq \begin{pmatrix} [\Phi, A] & \Phi A\Phi \\ A & [A,\Phi] \end{pmatrix}^{n+1} = 0$. Hence I_2 is locally nilpotent. □

Theorem 15. $\mathcal{L}(S_2) = \begin{pmatrix} \mathcal{L}(R) & \mathcal{L}(\Gamma) \\ \mathcal{L}(S) & \mathcal{L}(L) \end{pmatrix}$

Proof. $\mathcal{L}(S_2)$ is the sum of locally nilpotent ideals of S_2 which is of the form $\begin{pmatrix} A^{*'} & \Gamma(A) \\ A & A^{+'} \end{pmatrix}$ where A is a ideal of S. Then by the Proposition 14, A is a locally nilpotent ideal. Again by Proposition 3, $A^{*'}$ and $A^{+'}$ are locally nilpotent ideal, by Proposition 10, Γ(A) is locally nilpotent ideal. Then by the Theorem 3.10 and Note 3.11 of [3] and Theorem 4, we get $\mathcal{L}(S_2) = \begin{pmatrix} \mathcal{L}(R) & \mathcal{L}(\Gamma) \\ \mathcal{L}(S) & \mathcal{L}(L) \end{pmatrix}$. □

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