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MATCHING AND EDGE COVERING NUMBER ON STRONG PRODUCT OF COMPLETE BIPARTITE GRAPHS

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Abstract: Let $\alpha'(G)$ and $\beta'(G)$ be the matching and edge covering number , respectively. The strong product $G_1 \boxtimes G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \boxtimes G_2) = \{(u_1, v_1)(u_2, v_2) | [u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)]$ or $[u_1 = u_2$ and $v_1v_2 \in E(G_2)]$ or $[u_1u_2 \in E(G_1)$ and $v_1 = v_2]\}$. In this paper, we determined generalization of matching number and edge covering number on strong product of complete bipartite graphs and any simple graph.

AMS Subject Classification: 05C69, 05C70, 05C76
Key Words: strong product, edge covering number, matching number

1. Introduction

In this paper, graphs must be simple graphs. Let G_1 and G_2 be graphs. The strong product of graph G_1 and G_2, denote by $G_1 \boxtimes G_2$, is the graph with $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \boxtimes G_2) = \{(u_1, v_1)(u_2, v_2) | [u_1u_2 \in E(G_1)$

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and $v_1v_2 \in E(G_2)$] or [$u_1 = u_2$ and $v_1v_2 \in E(G_2)$] or [$u_1u_2 \in E(G_1)$ and $v_1 = v_2$]}. There are some properties about strong product of graph. We recall these here.

Proposition 1. Let $H = G_1 \boxtimes G_2 = (V(H), E(H))$ then:

(i) $|V(H)| = |V(G_1)||V(G_2)|$;

(ii) $|E(H)| = 2|E(G_1)||E(G_2)| + |V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|$;

(iii) for every $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u) + d_{G_2}(v)$.

Theorem 2. *Let G_1 and G_2 be connected g graphs, The graph $H = G_1 \boxtimes G_2$ is connected if and only if G_1 or G_2 contains an odd cycle.*

Next we get that general form of graph of strong product of $K_{m,n}$ and a simple graph.

Proposition 3. Let G be a connected graph with order p, $V(G) = \{v_j/j = 1, 2, ..., p\}$ and $V(K_{m,n}) = \{u_i/i = 1, 2, ..., m + n\}$, the graph of

$$K_{m,n} \boxtimes G \cong \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=1}^{m+n} R_i \cup \bigcup_{j=1}^{p} S_j; H_i = \bigcup_{j=m+1}^{m+n} H_{ij},$$

where

$$V(H_{ij}) = W_i \cup W_j, \quad W_i = \{(u_i, v_1), (u_i, v_2), ..., (u_i, v_p)\}, \quad i < j$$

and

$$E(H_{ij}) = \{(u_i, v)(u_j, w)/vw \in E(G)\},$$

$V(R_i) = W_i$ and $E(R_i) = \{(u_i, v)(u_i, w)/vw \in E(G)\}$,

$$V(S_j) = \{(u_1, v_j), (u_2, v_j), ..., (u_{m+n}, v_j)\}$$

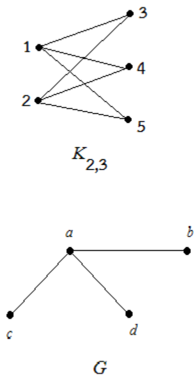
and

$$E(S_j) = \{(u, v_j)(w, v_j)/uw \in E(K_{m,n})\}.$$

Moreover, if G has no odd cycle then each H_{ij} has exactly two connected components isomorphic to G.

Example

Next, we give the definitions about some graph parameters. A subset of the edge set E of G is said to be matching or an independent edge set of G, if no two distinct edges in M have a common vertex. A matching M is maximum matching in G if there is no matching M' of G with $|M'| > |M|$. The cardinality



$$K_{2,3} \boxtimes G \cong \bigcup_{i=1}^{2} H_i \cup \bigcup_{i=1}^{5} R_i \cup \bigcup_{j=1}^{4} S_j$$
$$= [H_{13} \cup H_{14} \cup H_{15}] \cup [H_{23} \cup H_{24} \cup H_{25}] \cup \bigcup_{i=1}^{5} R_i \cup \bigcup_{j=1}^{4} S_j$$

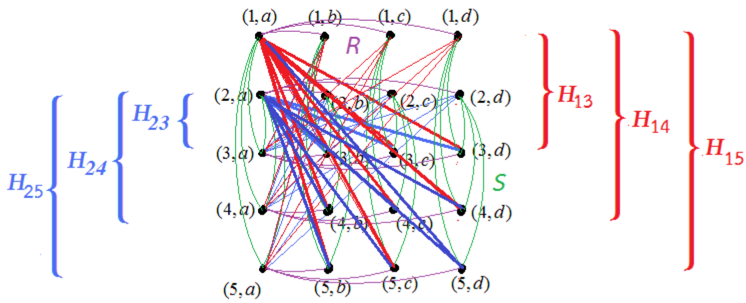


Figure 1: The graph of $K_{2,3} \boxtimes G$

of maximum matching of G is called the matching number of G, denoted by $\alpha'(G)$.

An edge of graph G is said to cover the two vertices incident with it, and an edge cover of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of a graph G is called the edge covering number of G, denoted by $\beta'(G)$.

By definitions of edge covering number and matching number , clearly that $\alpha'(K_{m,n}) = \min\{m, n\}$ and $\beta'(K_{m,n}) = \max\{m, n\}$.

2. Matching Number of the Graph of $K_{m,n} \boxtimes G$

We begin this section by giving the lemma 4 show character of maximum matching for each H_{ij}, R_i and S_j.

Lemma 4. Let $K_{m,n} \boxtimes G \cong \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=1}^{m+n} R_i \cup \bigcup_{j=1}^{p} S_j$; $H_i = \bigcup_{j=m+1}^{m+n} H_{ij}$. Then $\alpha'(H_{ij}) = 2\alpha'(R_i) = 2\alpha'(G)$ and $\alpha'(S_j) = \min\{m, n\}$.

Proof. Suppose G has no odd cycle, by proposition 3, we have thatH_{ij} contains 2 components isomorphic to G. Therefore H_{ij}=2G. So $\alpha'(H_{ij})$ =

$2\alpha'(G)$.

If G has odd cycle, then we have $d_{H_{\mathrm{ij}}}((u_i, v)) = d_{H_{\mathrm{ij}}}(u_j, v)) = d_G(v))$ for $(u_i, v) \in W_i$ and $(u_j, v) \in W_j$, $i < j$.

In the rest of the proof we will consider H_{ij}. We first note that by definition of tensor product of H_{ij}, that for all $v, x \in G$, (u_i, v) and (u_i, x) are not joined by an edge, and that because the vertices u_i, u_j are always joined by an edge, two vertices $(u_i, v), (u_j, w) \in H_{ij}$ are joined by an edge if and only if the two vertices are joined by an edge.

We now prove that $\alpha'(H_{ij}) = 2\alpha'(G)$.

(1) We first prove that if two vertices v, w are matched in G, then the two vertices $(u_i, v), (u_j, w)$ can also be matched in H_{ij} and the two vertices $(u_i, w), (u_j, v)$ can also be matched in H_{ij} . As stated above, for all vertices $v, w \in G$, edges $(u_i, v), (u_i, w)$ and $(u_j, v), (u_j, w)$ cannot exist. However, if v, w are matched vertices in G, then the pair $(u_i, v), (u_j, w)$ can be matched in H_{ij} and the pair $(u_j, v), (u_i, w)$ can also be matched in H_{ij} because edge vw exists in G. Therefore, the result follows.

(2) We next prove that if $v, w \in G$ cannot be matched in G, then the pair $(u_i, v), (u_j, w)$ cannot be matched in H_{ij} and the pair $(u_j, v), (u_i, w)$ cannot be matched in H_{ij}. The reason is that $v, w \in G$ cannot be matched only if the edge vw does not exist and therefore an edge does not exist between vertices $(u_i, v), (u_j, w) \in H_{ij}$ and between vertices $(u_j, v), (u_i, w) \in H_{ij}$. Therefore two vertices matched in G correspond to two possible matchings in and if two vertices cannot be matched in G then no vertices in the tensor product containing these two vertices can be matched.

Hence a matched set in G will correspond to two matched sets in H_{ij} and therefore $\alpha'(H_{ij}) = 2\alpha'(R_i) = 2\alpha'(G)$ [since $R_i \cong G$]. From $S_j \cong K_{m,n}$, we get $\alpha'(S_j) = \alpha'(K_{m,n}) = \min\{m, n\}$. □

Definition 5. Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are unsaturated by M is an M-augmenting path.

Theorem 6. *A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.*

Next, we establish theorem 5 for a matching number of $K_{m,n} \boxtimes G$.

Theorem 7. *Let G be a connected graph with order p, then*

$$\alpha'(K_{m,n} \boxtimes G) = \begin{cases} pm, & m = n \\ pn + (m-n)\alpha'(G), & m > n \\ pm + (n-m)\alpha'(G), & m < n \end{cases}$$

Proof. Let $V(K_{m,n}) = \{u_i/i = 1,2,...,m+n\}$ and $V(G) = \{v_j/j = 1,2,...,p\}$. Since $\alpha'(K_{m,n}) = \min\{m,n\}$. Let $\alpha'(G) = k$, assume that the maximum matching of $K_{m,n}, G$ be

$$M_1 = \begin{cases} \{u_1u_{m+1}, u_2u_{m+2}, ..., u_mu_{2m}\} & \text{where } m \leq n \\ \{u_1u_{m+1}, u_2u_{m+2}, ..., u_nu_{m+n}\} & \text{where } m > n \end{cases},$$

$M_2 = \{v_jv_{j+1}/j = 1,3,...,2k-1\}$ respectively.

Case 1. $m = n$. We haveM_1 is a maximum matching in $K_{m,n}$. Let $S_j^* = \{(u,v_j)(w,v_j)|uw \in M_1\}; j = 1,2,...,p$. Therefore a maximum matching in $K_{m,n} \boxtimes G$ is $\bigcup_{j=1}^{p} S_j^*$.

Hence $\alpha^{'}(K_{m,n} \boxtimes G) = |\bigcup_{j=1}^{p} S_j^*| = pm$ where $m = n$.

Case 2. $m > n$. We have the matching $\bigcup_{j=1}^{p} S_j^*$. But $\bigcup_{i=1}^{n-1} R_{m+i}$ is not matched yet. Let $R_{m+i}^* = \{(u_{m+i},v)(u_{m+i},w)|vw \in M_2\}; i = 1,2,...,n-1$. So we get $\bigcup_{j=1}^{p} S_j^* \cup \bigcup_{i=1}^{n-1} R_{m+i}^*$ is matching in $K_{m,n} \boxtimes G$).

So $\alpha^{'}(K_{m,n} \boxtimes G) \geq |\bigcup_{j=1}^{p} S_j^*| + |\bigcup_{i=1}^{n-1} R_{m+i}^*| = pn + (m-n)\alpha^{'}(G)$.

Suppose that $\alpha^{'}(K_{m,n} \boxtimes G) > pn + (m-n)\alpha^{'}(G)$, then there exist an augmenting path for $\bigcup_{j=1}^{p} S_j^* \cup \bigcup_{i=1}^{n-1} R_{m+i}^*$. That is not true because each edges in $K_{m,n} \boxtimes G$ incident with edge in $\bigcup_{j=1}^{p} S_j^* \cup \bigcup_{i=1}^{n-1} R_{m+i}^*$.

Hence $\alpha^{'}(K_{m,n} \boxtimes G) = pn + (m-n)\alpha^{'}(G)$ where $m > n$.

Case 3. $m < n$. In the same as case 2, we have $\alpha^{'}(K_{m,n} \boxtimes G) = |\bigcup_{j=1}^{p} S_j^*| + |\bigcup_{i=1}^{n-m} R_{2m+i}^*| = pm + (n-m)\alpha^{'}(G)$ where $m < n$. □

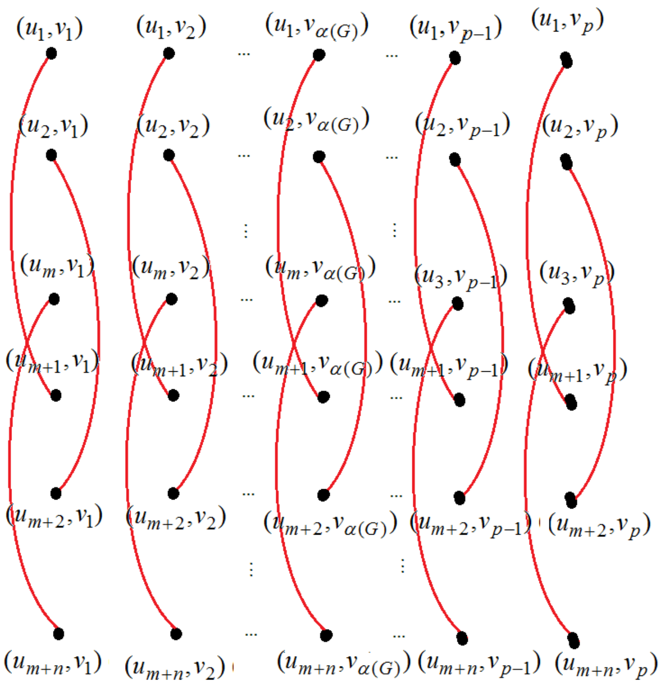


Figure 2: The case of $m = n$

3. Edge Covering Number of the Graph of$K_{m,n} \boxtimes G$

We begin this section by by giving the lemma 8 that shows a relation of matching number and edge covering number.

Lemma 8. Let G be a simple graph with order n. Then $\beta'(G)+\alpha'(G) = n$

Next we establish theorem 9 for a edge covering number of $K_{m,n} \boxtimes G$.

Theorem 9. *Let G be a connected graph with order p, then*

$$\beta^{'}(K_{m,n} \boxtimes G) = \begin{cases} pm, & m = n \\ pn + (m-n)\beta^{'}(G), & m > n \\ pm + (n-m)\beta^{'}(G), & m < n \end{cases}$$

Proof. By theorem 7 and lemma 8, we can also show that $\alpha'(K_{m,n} \boxtimes G) + \beta'(K_{m,n} \boxtimes G) = (m+n)p$.

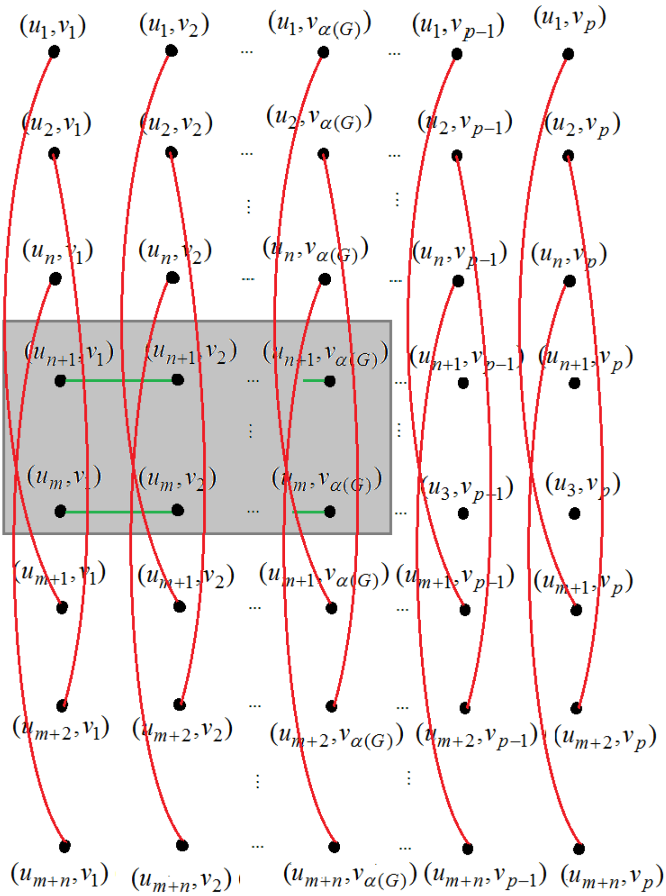


Figure 3: The case of $m > n$

Case 1. $m = n$.

$$\begin{aligned}\beta^{'}(K_{m,n} \boxtimes G) &= 2mp - pm\\ &= mp\end{aligned}$$

Hence $\beta^{'}(K_{m,n} \boxtimes G) = mp$, where $m = n$.

Case 2. $m > n$.

$$\begin{aligned}\beta^{'}(K_{m,n} \boxtimes G) &= (m + n)p - pn + (n - m)\alpha^{'}(G)\\ &= mp + (n - m)(p - \beta^{'}(G))\end{aligned}$$

$$= n(p - \beta^{'}(G)) + m\beta^{'}(G)$$
$$= np + (m - n)\beta^{'}(G)$$

Hence $\beta^{'}(K_{m,n} \boxtimes G) = np + (m - n)\beta^{'}(G)$, where $m > n$.

Case 3. $m < n$.

$$\begin{aligned}\beta^{'}(K_{m,n} \boxtimes G) &= (m + n)p - pm + (m - n)\alpha^{'}(G)\\ &= np + (m - n)(p - \beta^{'}(G))\\ &= m(p - \beta^{'}(G)) + n\beta^{'}(G)\\ &= mp + (n - m)\beta^{'}(G)\end{aligned}$$

Hence $\beta^{'}(K_{m,n} \boxtimes G) = mp + (n - m)\beta^{'}(G)$, where $m < n$. □

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