ON UNIQUENESS MORAWETZ PROBLEM
FOR THE CHAPLYGIN EQUATION

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Abstract: For the equation

$$Lz = K(y)z_{xx} + z_{yy} = 0,$$

where $yK(y) > 0$ for $y \neq 0$ in $D$, bounded by a Jordan (non-selfintersecting) "elliptic" arc $\Gamma$ (for $> 0$) with endpoints $A(0, 0)$ and $B(l, 0)$, $l > 0$, and for $y < 0$ by a characteristic $\gamma_1$ through $A$ which meets the characteristic $\gamma_2$ through $B$ at the points $C$, the uniqueness of the Morawetz problem is proved without assuming that $\Gamma$ is monotone.

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1. Introduction

Consider the equation

$$Lz = K(y)z_{xx} + z_{yy} = 0,$$  \hspace{1cm} (1)

in an open domain $D$, where $yK(y) > 0$ for $y \neq 0$ and the domain $D$ is bounded by curves: a piecewise smooth curve $\Gamma$ in the half-plane $y > 0$, which intersects the line $y = 0$ at the points $A(0, 0)$ and $B(l, 0)$, $l > 0$; in $y < 0$, $D$ is bounded by two characteristics $\gamma_1$ and $\gamma_2$ of (1) issuing from $A$ and $B$ and meeting at
the point $C$:

$$
\gamma_1 : \xi = x + \int_0^y \sqrt{\gamma(t) \, dt} = 0,
$$

$$
\gamma_2 : \eta = x - \int_0^y \sqrt{\gamma(t) \, dt} = l,
$$

where $\gamma(y) \in C[y_c, 0] \cap C^2[y_c, 0]$, $y_c$ – the ordinate of point $C$. Let $D_+$ be subdomain of $D$ with $y > 0$ and $D_-$ be subdomain of $D$ with $y < 0$.

In this paper using a variation of the energy-integral method (abc method) we obtain sufficient conditions for the uniqueness of solution of Morawetz problem for the Chaplygin equation. It arises in the study of transonic flow, and the proof of uniqueness in this case leads to a proof that continuous transonic flows past smooth profiles do not exist in general [1].

The Morawetz problem. Find function $z(x, y)$ satisfying the following conditions:

$$
Lz(x, y) \equiv 0, (x, y) \in D_- \cup D_+;
$$

$$
z(x, y) \in C(D) \cap C^1(D \cup \Gamma) \cap C^2(D_- \cup D_+);
$$

$$
\delta_s[z] |_{\Gamma} = Kz_x \frac{dy}{ds} - z_y \frac{dx}{ds} = \varphi(s), \quad 0 \leq s \leq l;
$$

$$
\delta_x[z] |_{\gamma_1} = \sqrt{-K} z_x - z_y = \psi(x), \quad 0 \leq x \leq \frac{l}{2},
$$

where $\varphi$ and $\psi$ are given functions.

The question uniqueness of solution of Morawetz problem for equation of mixed type has been dealt with in the literature by many authors. For an extensive bibliography we refer the reader to [2], [3], [4].

2. Reducing Morawetz Problem to the Tricomi Problem

To every solution $z(x, y)$ of (1) there corresponds a function defined by the integral

$$
v(x, y) = \int_{(0,0)}^{(x,y)} -z_y dx + K(y) z_x dy, \quad (2)
$$

which is independent of the path of integration and the function $v(x, y)$ is to satisfy the equation

$$
L_0 v = v_{xx} + \left( \frac{v_y}{K(y)} \right)_y = 0. \quad (3)
$$
Then the Morawetz problem is transformed into the analogue Tricomi problem for equation (3)

\[ L_0 v(x, y) \equiv 0, (x, y) \in D_\pm \cup D_+; \tag{4} \]

\[ v(x, y) \in C(D) \cap C^2(D_\pm \cup D_+); \tag{5} \]

\[ \lim_{y \to 0^-} \frac{v_y}{K(y)} = \lim_{y \to 0^+} \frac{v_y}{K(y)}; \tag{6} \]

\[ v|_\Gamma = \varphi_0(s), \quad 0 \leq s \leq L; \tag{7} \]

\[ v|_{\gamma_1} = \psi_0(x), \quad 0 \leq x \leq \frac{l}{2}. \tag{8} \]

**Definition 1.** We call a function \( v(x, y) \) quazi-regular solution of (3) if the following hold:

i) \( v(x, y) \) satisfies (5);

ii) we can to applicate Green’s theorem to the integrals

\[ \int \int_D vL_0v dx dy, \int \int_D v_x L_0v dx dy, \int \int_D v_y L_0v dx dy; \]

iii) the boundary integrals which arise exist in the sense that: the limits taken over corresponding interior curves exist as these interior curves approach the boundary.

**3. Theorem of Uniqueness**

We introduce Francl’s function

\[ F(y) = 2 \left( \frac{K}{K'} \right)' + 1. \]

The following statement is a more general result than Theorem 6, given in [3].

**Theorem 2.** If 1) \( K(y) \in C^2[y_c, 0], K(0) = 0, K'(y) \neq 0 \) for \( y < 0, F(0) > 0; 2) \) there is constant \( d > 0 \) such that \( F(y) > -d[-K(y)]^\alpha \) in \( D_-; 3) v(x, y) \) quazi-regular solution of (3) in \( D, 4) v|_{\Gamma \cup \gamma_1} = 0, \) then \( v(x, y) \equiv 0 \) in \( D.\)
Proof. Consider the area integral \( I \) over domain \( D \)

\[
I = \int \int_D (av + bv_x + cv_y) \left( v_{xx} + \left( \frac{v_y}{K(y)} \right)_y \right) dx dy,
\]

(9)

where \( a(x, y), b(x, y), c(x, y) \) are given functions. By (3), the integral \( I \) vanishes. We shall show that over \( D \) integral \( I \) can be made non-positive by proper choice of functions \( a(x, y), b(x, y) \) and \( c(x, y) \).

Consider identities

\[
\begin{align*}
    cv_y \left( \frac{v_y}{K(y)} \right)_y &= \frac{1}{2} c \left( \frac{v_y^2}{K(y)} \right)_y + \frac{1}{2} \frac{cK'(y)v_y^2}{K^2(y)} - \frac{1}{2} \frac{c_yv_y^2}{K(y)}, \\
    bv_x \left( \frac{v_y}{K(y)} \right)_y &= \left( bv_x \frac{v_y}{K(y)} \right)_y - \frac{1}{2} \left( b \frac{v_y^2}{K(y)} \right)_x + \frac{1}{2} \frac{b_xv_y^2}{K(y)} - \frac{b_yv_xv_y}{K(y)}, \\
    av \left( \frac{v_y}{K(y)} \right)_y &= \left( av \frac{v_y}{K(y)} \right)_y - \frac{1}{2} \left( ay \frac{v^2}{K(y)} \right)_y + \frac{1}{2} \frac{a_yv^2}{K(y)} + \frac{1}{2} \frac{a_yv^2K'(y)}{K^2(y)} - \frac{av_y}{K(y)}, \\
    avv_{xx} &= (avv_x)_x - \frac{1}{2} (a_xv^2)_x + \frac{1}{2} a_{xx}v^2 - av_x^2, \\
    bv_xv_{xx} &= \frac{1}{2} (bv^2)_x - \frac{1}{2} b_xv^2, \\
    cv_yv_{xx} &= (cv_yv_x)_x - \frac{1}{2} (cv^2)_y + \frac{1}{2} c_yv^2 - c_xv_xv_y.
\end{align*}
\]

Then employing above identities and applying Greene theorem into relation (9) we get:

\[
0 = \int \int_D \left[ \frac{1}{2} \left( a_{xx} + \left( \frac{a_y}{K(y)} \right)_y \right) v^2 - a \left( v^2_x + \frac{v^2_y}{K(y)} \right) - \frac{1}{2} b_x \left( v^2_x - \frac{v^2_y}{K(y)} \right) - b_y \frac{v_xv_y}{K(y)} + \frac{1}{2} c_yv^2 - c_xv_xv_y - \frac{1}{2} \frac{c_yv^2}{K(y)} + \frac{1}{2} \frac{c_yv^2K'(y)}{K^2(y)} \right] dx dy + \\
+ \int_{\Gamma+\gamma_1+\gamma_2} \left[ -av \frac{v_y}{K(y)} + \frac{1}{2} a_y \frac{v^2}{K(y)} - bv_x \frac{v_y}{K(y)} + \frac{1}{2} \left( cv^2_x - \frac{v^2_y}{K(y)} \right) \right] dx +
\]
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\[ \int \left[ avv_x - \frac{1}{2} a_x v^2 + cv_x v_y + \frac{1}{2} b \left( \frac{v_x^2}{K(y)} - \frac{v_y^2}{K(y)} \right) \right] dy = J_1 + J_2. \]

Choose \( b = c \equiv 0 \) in \( D_+ \).

From \( v(x,y) = 0 \) on \( \Gamma \cup \gamma_1 \) and the fact that
\[ dx = -\sqrt{-K} dy \text{ (on } \gamma_1 \text{)}, \quad dx = \sqrt{-K} dy \text{ (on } \gamma_2 \text{)} \]
we get
\[ J_2 = \frac{1}{2} \int_{\gamma_1} \frac{1}{K(y)} \left( b - c\sqrt{-K} \right) (\sqrt{-K} v_x dx + v_y dy) - \]
\[ -\frac{1}{2} \int_{\gamma_2} \frac{1}{K(y)} \left( b + c\sqrt{-K} \right) \left( \sqrt{-K} v_x^2 + 2v_x v_y + \frac{1}{\sqrt{-K}} v_y^2 \right) dx - \]
\[ - \int_{\gamma_2} \frac{a}{\sqrt{-K}} v dv - \frac{1}{2} \int_{\gamma_2} \frac{1}{\sqrt{-K}} v^2 (a_x dx + a_y dy) = I_1 + I_2 + I_3. \]

Since \( v|_{\gamma_1} = 0 \) implies \( v_x dx + v_y dy|_{\gamma_1} = 0 \), then \( I_1 = 0 \).

\[ J_1 = - \int \int_{D_+} a \left( \frac{v_x^2}{K(y)} + \frac{v_y^2}{K(y)} \right) dxdy - \]
\[ -\frac{1}{2} \int \int_{D_-} \left[ (2a + b_x - c_y)v_x^2 + 2v_x v_y \left( \frac{b_y}{K(y)} + c_x \right) + \right. \]
\[ + \left. \left( \frac{2a}{K(y)} - \frac{b_x}{K(y)} + \frac{c_y}{K(y)} - \frac{cK'(y)}{K(y)^2} \right) v_y^2 \right] dxdy + \]
\[ + \frac{1}{2} \int \int \left( a_{xx} + \left( \frac{a_y}{K(y)} \right)_y \right) v^2 dxdy = I_4 + I_5 + I_6. \]

We must choose functions \( a(x,y) \), \( b(x,y) \) and \( c(x,y) \) so that all the integrals \( I_2, I_3, ..., I_6 \) are non-positive. If this occurs then \( v(x,y) \equiv 0 \) follows immediately.

Follow [3] for \( y < 0 \) choose
\[ c = \frac{4aK(y)}{K'(y)}, \quad b = -c\sqrt{-K(y)}. \] (10)

Obviously \( I_2 = 0 \). An integration by parts \( I_3 \) we get
\[ I_3 = \int_{\gamma_2} \frac{-1}{K(y)} \left( -\sqrt{-K} a_x - a_y + \frac{aK'}{4K} \right) v^2 dx. \]
The integrals \( I_3 \) and \( I_5 \) are non-positive if the following two conditions hold in \( D \):

\[-\sqrt{-K}a_x - a_y + \frac{aK'}{4K} \leq 0 \text{ for } y \leq 0, \tag{11}\]

\[(Kc_x + b_y)^2 \leq (2a + b_x - c_y)(2aK - Kb_x + (Kc)_y), \quad y \leq 0, \tag{12}\]

\[2a + b_x - c_y \geq 0, \quad y \leq 0. \tag{13}\]

Obviously, condition (12) holds for all \( a(x, y) \). If, now we substitute functions \( a(x, y) \), \( b(x, y) \) and \( c(x, y) \) into (13) we obtain

\[\sqrt{-K}a_x + a_y + a \frac{K'}{2K} F(y) \leq 0. \tag{14}\]

Consider two cases.

**Case 1.** If \( F(y) > 0 \) for \( y \leq 0 \) then choose \( a(x, y) = \text{const} \geq 0 \). In this case conditions (11) and (14) hold for \( y \leq 0 \) and all the integrals \( I_1, I_2, \ldots, I_6 \) are non-positive.

**Case 2.** If \( F(y) \) takes negative values when \( y < 0 \) then choose

\[a = \begin{cases} e^{-K(y)}^\alpha, & y \leq 0, \\ 1, & y \geq 0, \end{cases}\]

where \( \alpha, \beta - \text{positive constants} \).

Obviously \( I_4 \) is non-positive.

\( I_6 \leq 0 \) is equivalent to

\[F(y) - 2\alpha \beta [-K(y)]^\alpha - 2\alpha + 1 \leq 0. \tag{15}\]

Substituting \( a(x, y) \) in (11) and (14) we get

\[\frac{1}{4} - \alpha \beta [-K(y)]^\alpha \geq 0, \tag{16}\]

\[F(y) + 2\alpha \beta [-K(y)]^\alpha \geq 0. \tag{17}\]

If we choose \( \alpha, \beta \) so that

\[\alpha > \max \frac{F(y) + 1}{2}\]

\[\beta = \frac{1}{\alpha [-K(y_{\text{min}})]^\alpha},\]

then (15), (16), (17) hold. \qed
References


