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ON UNIQUENESS MORAWETZ PROBLEM FOR THE CHAPLYGIN EQUATION

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Abstract: For the equation

$$Lz = K(y)z_{xx} + z_{yy} = 0,$$

where $yK(y) > 0$ for $y \neq 0$) in D, bounded by a Jordan (non-selfintersecting) "elliptic" arc Γ (for > 0) with endpoints $A(0,0)$and $B(l,0)$, $l > 0$, and for $y < 0$ by a characteristic γ_1 through A which meets the characteristic γ_2 through B at the points C, the uniqueness of the Morawetz problem is proved without assuming that Γ is monotone.

AMS Subject Classification: 35M12
Key Words: Chaplygin equation, Morawetz problem, equation of mixed type

1. Introduction

Consider the equation

$$Lz = K(y)z_{xx} + z_{yy} = 0, \tag{1}$$

in an open domain$D$, where $yK(y) > 0$ for $y \neq 0$ and the domain D is bounded by curves: a piecewise smooth curve Γ in the half-plane $y > 0$, which intersects the line $y = 0$ at the points $A(0,0)$ and $B(l,0)$, $l > 0$; in $y < 0$, D is bounded by two charasteristics γ_1 and γ_2 of (1) issuing from A and B and meeting at

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the point C:

$$\gamma_1 : \xi = x + \int_0^y \sqrt{-K(t)}dt = 0,$$

$$\gamma_2 : \eta = x - \int_0^y \sqrt{-K(t)}dt = l,$$

where $K(y) \in C[y_c, 0] \bigcap C^2[y_c, 0)$, $\quad y_c$ – the ordinate of point C. Let D_+ be subdomain of D with $y > 0$ and D_- be subdomain of D with $y < 0$.

In this paper using a variation of the energy-integral method (abc method) we obtain sufficient conditions for the uniqueness of solution of Morawetz problem for the Chaplygin equation. It arises in the study of transonic flow, and the proof of uniqueness in this case leads to a proof that continuous transonic flows past smooth profilies do not exist in general [1].

The Morawetz problem. Find function $z(x, y)$ satisfying the following conditions:

$$Lz(x, y) \equiv 0, (x, y) \in D_- \cup D_+;$$

$$z(x, y) \in C(\overline{D}) \cap C^1(D \cup \Gamma) \cap C^2(D_- \cup D_+);$$

$$\delta_s[z]|_\Gamma = K z_x \frac{dy}{ds} - z_y \frac{dx}{ds} = \varphi(s), \quad 0 \le s \le l\ ;$$

$$\delta_x[z]|_{\gamma_1} = \sqrt{-K} z_x - z_y = \psi(x), \quad 0 \le x \le \frac{l}{2},$$

where φ and ψ are given functions.

The question uniqueness of solution of Morawetz problem for equation of mixed type has been dealt with in the literature by many authors. For an extensive bibliography we refer the reader to [2], [3], [4].

2. Reducing Morawetz Problem to the Tricomi Problem

To every solution $z(x, y)$ of (1) there corresponds a function defined by the integral

$$v(x, y) = \int_{(0,0)}^{(x,y)} -z_y dx + K(y) z_x dy, \tag{2}$$

which is independent of the path of integration and the function $v(x, y)$ is to satisfy the eqution

$$L_0 v = v_{xx} + \left(\frac{v_y}{K(y)}\right)_y = 0. \tag{3}$$

Then the Morawetz problem is transformed into the analogue Tricomi problem for equation (3)

$$L_0 v(x,y) \equiv 0, (x,y) \in D_- \cup D_+; \tag{4}$$

$$v(x,y) \in C(\overline{D}) \cap C^2(D_- \cup D_+); \tag{5}$$

$$\lim_{y\to 0-0} \frac{v_y}{K(y)} = \lim_{y\to 0+0} \frac{v_y}{K(y)}; \tag{6}$$

$$v|_\Gamma = \varphi_0(s), \quad 0 \le s \le L\ ; \tag{7}$$

$$v|_{\gamma_1} = \psi_0(x), \quad 0 \le x \le \frac{l}{2}. \tag{8}$$

Definition 1. We call a function $v(x,y)$ quazi-regular solution of (3) if the following hold:

i) $v(x,y)$ satisfies (5);

ii) we can to applicate Green's theorem to the integrals

$$\int\int_D vL_0 v dxdy, \quad \int\int_D v_x L_0 v dxdy, \quad \int\int_D v_y L_0 v dxdy;$$

iii) the boundary integrals which arise exist in the sense that: the limits taken over corresponding interior curves exist as these interior curves approach the boundary.

3. Theorem of Uniqueness

We introduce Francl's function

$$F(y) = 2\left(\frac{K}{K'}\right)' + 1.$$

The following statement is a more general result than Theorem 6, given in [3].

Theorem 2. *If* 1) $K(y) \in C^2[y_c, 0)$, $K(0) = 0$, $K'(y) \neq 0$ *for* $y < 0$, $F(0) > 0$; 2) *there is constant* $d > 0$ *such that* $F(y) > -d[-K(y)]^\alpha$ *in* D_-; 3) $v(x,y)$ *quazi-regular solution of* (3) *in* D, 4) $v|_{\Gamma\cup\gamma_1} = 0$, *then* $v(x,y) \equiv 0$ *in* D.

Proof. Consider the area integral I over domain D

$$I = \int\int_D (av + bv_x + cv_y)\left(v_{xx} + \left(\frac{v_y}{K(y)}\right)_y\right) dxdy, \tag{9}$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$ are given functions. By (3), the integral I vanishes. We shall show that over D integral I can be made non-positive by proper choice of functions $a(x,y)$, $b(x,y)$ and $c(x,y)$.

Consider identities

$$cv_y\left(\frac{v_y}{K(y)}\right)_y = \frac{1}{2}c\left(\frac{v_y^2}{K(y)}\right)_y + \frac{1}{2}\frac{cK'(y)v_y^2}{K^2(y)} - \frac{1}{2}\frac{c_y v_y^2}{K(y)},$$

$$bv_x\left(\frac{v_y}{K(y)}\right)_y = \left(bv_x\frac{v_y}{K(y)}\right)_y - \frac{1}{2}\left(b\frac{v_y^2}{K(y)}\right)_x + \frac{1}{2}\frac{b_x v_y^2}{K(y)} - \frac{b_y v_x v_y}{K(y)},$$

$$av\left(\frac{v_y}{K(y)}\right)_y = \left(av\frac{v_y}{K(y)}\right)_y - \frac{1}{2}\left(a_y\frac{v^2}{K(y)}\right)_y +$$

$$+\frac{1}{2}\frac{a_{yy}v^2}{K(y)} - \frac{1}{2}\frac{a_y v^2 K'(y)}{K^2(y)} - \frac{av_y^2}{K(y)},$$

$$avv_{xx} = (avv_x)_x - \frac{1}{2}(a_x v^2)_x + \frac{1}{2}a_{xx}v^2 - av_x^2,$$

$$bv_x v_{xx} = \frac{1}{2}(bv_x^2)_x - \frac{1}{2}b_x v_x^2,$$

$$cv_y v_{xx} = (cv_y v_x)_x - \frac{1}{2}(cv_x^2)_y + \frac{1}{2}c_y v_x^2 - c_x v_x v_y.$$

Then employing above identities and applying Greene theorem into relation (9) we get:

$$0 = \int\int_D \left[\frac{1}{2}\left(a_{xx} + \left(\frac{a_y}{K(y)}\right)_y\right)v^2 - a\left(v_x^2 + \frac{v_y^2}{K(y)}\right) - \frac{1}{2}b_x\left(v_x^2 - \frac{v_y^2}{K(y)}\right) - \right.$$

$$\left. -b_y\frac{v_x v_y}{K(y)} + \frac{1}{2}c_y v_x^2 - c_x v_x v_y - \frac{1}{2}c_y\frac{v_y^2}{K(y)} + \frac{1}{2}cv_y^2\frac{K'(y)}{K^2(y)}\right] dxdy+$$

$$+\int_{\Gamma+\gamma_1+\gamma_2}\left[-av\frac{v_y}{K(y)} + \frac{1}{2}a_y\frac{v^2}{K(y)} - bv_x\frac{v_y}{K(y)} + \frac{1}{2}\left(cv_x^2 - \frac{v_y^2}{K(y)}\right)\right] dx+$$

$$+\left[avv_x - \frac{1}{2}a_x v^2 + cv_x v_y + \frac{1}{2}b\left(v_x^2 - \frac{v_y^2}{K(y)}\right)\right] dy = J_1 + J_2.$$

Choose $b = c \equiv 0$ in D_+.

From $v(x,y) = 0$ on $\Gamma \cup \gamma_1$ and the fact that

$$dx = -\sqrt{-K}dy \ (on\ \gamma_1), \ \ dx = \sqrt{-K}dy \ (on\ \gamma_2)$$

we get

$$J_2 = \frac{1}{2}\int_{\gamma_1} \frac{1}{K(y)}(b - c\sqrt{-K})(\sqrt{-K}v_x dx + v_y dy) -$$

$$-\frac{1}{2}\int_{\gamma_2} \frac{1}{K}(b + c\sqrt{-K})(\sqrt{(}-K)v_x^2 + 2v_x v_y + \frac{1}{\sqrt{-K}}v_y^2)dx -$$

$$-\int_{\gamma_2} \frac{a}{\sqrt{-K}}vdv - \frac{1}{2}\frac{1}{\sqrt{-K}}v^2(a_x dx + a_y dy) = I_1 + I_2 + I_3.$$

Since $v|_{\gamma_1} = 0$ implies $v_x dx + v_y dy|_{\gamma_1} = 0$, then $I_1 = 0$.

$$J_1 = -\int\int_{D_+} a\left(v_x^2 + \frac{v_y^2}{K(y)}\right) dxdy -$$

$$-\frac{1}{2}\int\int_{D_-}\left[(2a + b_x - c_y)v_x^2 + 2v_x v_y\left(\frac{b_y}{K(y)} + c_x\right) + \right.$$

$$\left. + \left(\frac{2a}{K(y)} - \frac{b_x}{K(y)} + \frac{c_y}{K(y)} - \frac{cK'(y)}{K(y)^2}\right)v_y^2\right] dxdy +$$

$$+\frac{1}{2}\int\int\left(a_{xx} + \left(\frac{a_y}{K(y)}\right)_y\right)v^2 dxdy = I_4 + I_5 + I_6.$$

We must choose functions $a(x,y)$, $b(x,y)$ and $c(x,y)$ so that all the integrals I_2, I_3, ..., I_6 are non-positive. If this occurs then $v(x,y) \equiv 0$ follows immediately.

Follow [3] for $y < 0$ choose

$$c = \frac{4aK(y)}{K'(y)}, \qquad b = -c\sqrt{-K(y)}. \tag{10}$$

Obviously $I_2 = 0$. An integration by parts I_3we get

$$I_3 = \int_{\gamma_2} \frac{-1}{K(y)}\left(-\sqrt{-K}a_x - a_y + \frac{aK'}{4K}\right)v^2 dx.$$

The integrals I_3 and I_5 are non-positive if the following two conditions hold in D:

$$-\sqrt{-K}a_x - a_y + \frac{aK'}{4K} \leq 0 \ \ for \ \ y \leq 0, \tag{11}$$

$$(Kc_x + b_y)^2 \leq (2a + b_x - c_y)(2aK - Kb_x + (Kc)_y), \quad y \leq 0, \tag{12}$$

$$2a + b_x - c_y \geq 0 \ , \ \ y \leq 0. \tag{13}$$

Obviously, condition (12) holds for all $a(x, y)$. If, now we substitute functions $a(x, y)$, $b(x, y)$ and $c(x, y)$ into (13) we obtain

$$\sqrt{-K}a_x + a_y + a\frac{K'}{2K}F(y) \leq 0. \tag{14}$$

Consider two cases.

Case 1. If $F(y) > 0$ for $y \leq 0$ then choose $a(x, y) = const \geq 0$. In this case conditions (11) and (14) hold for $y \leq 0$ and all the integrals I_1, I_2, ..., I_6 are non-positive.

Case 2. If $F(y)$ takes negative values when $y < 0$ then choose

$$a = \begin{cases} e^{\beta[-K(y)]^\alpha}, & y \leq 0, \\ 1, \quad y \geq 0, \end{cases}$$

where α, β – positive constants.

Obviously I_4 is non-positive.

$I_6 \leq 0$ is equivalent to

$$F(y) - 2\alpha\beta[-K(y)]^\alpha - 2\alpha + 1 \leq 0. \tag{15}$$

Substituting $a(x, y)$in (11) and (14) we get

$$\frac{1}{4} - \alpha\beta[-K(y)]^\alpha \geq 0, \tag{16}$$

$$F(y) + 2\alpha\beta[-K(y)]^\alpha \geq 0. \tag{17}$$

If we choose α, β so that

$$\alpha > max\frac{F(y) + 1}{2}$$

$$\beta = \frac{1}{\alpha[-K(y_{min})]^\alpha},$$

then (15), (16), (17) hold. □

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